Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 235359, 14 pages
doi:10.1155/2012/235359

Research Article

On the Stability of a Parametric Additive Functional Equation in Quasi-Banach Spaces

A. Ebadian,[1] N. Ghobadipour,[2] and M. Eshaghi Gordji[3]

[1] *Department of Mathematics, Payame Noor University, P.O. BOX 19395-3697, Tehran, Iran*
[2] *Department of Mathematics, Urmia University, Urmia, Iran*
[3] *Department of Mathematics, Semnan University, P.O. Box 35195-363, Semnan, Iran*

Correspondence should be addressed to M. Eshaghi Gordji, madjid.eshaghi@gmail.com

Received 22 January 2012; Revised 16 February 2012; Accepted 16 February 2012

Academic Editor: Narcisa C. Apreutesei

We investigate the generalized Hyers-Ulam stability of the following functional equation $\sum_{i=1}^{m} f(x_i) = 1/2m[\sum_{i=1}^{m} f(mx_i + \sum_{j=1, j\neq i}^{m} x_j) + f(\sum_{i=1}^{m} x_i)]$ for a fixed positive integer m with $m \geq 2$ in quasi-Banach spaces.

1. Introduction

It is of interest to consider the concept of stability for a functional equation arising when we replace the functional equation by an inequality which acts as a perturbation of the equation.

The first stability problem was raised by Ulam [1] during his talk at the University of Wisconsin in 1940. The stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation? If the answer is affirmative, we would say that the equation is stable.

In 1941, Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \to E'$ be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta, \tag{1.1}$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \leq \delta, \tag{1.2}$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is linear. Aoki [3], Bourgin [4] considered the stability problem with unbounded Cauchy differences. In 1978, Rassias [5] provided a generalization of Hyers' theorem by proving the existence of unique linear mappings near approximate additive mappings. It was shown by Gajda [6], as well as by Rassias and Šemrl [7] that one cannot prove a stability theorem of the additive equation for a specific function. Găvruţa [8] obtained generalized result of Rassias' theorem which allows the Cauchy difference to be controlled by a general unbounded function. Isac and Rassias [9] generalized the Hyers' theorem by introducing a mapping $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ subject to the conditions:

(1) $\lim_{t\to\infty}\psi(t)/t = 0$,

(2) $\psi(ts) \le \psi(t)\psi(s)$; $s, t > 0$,

(3) $\psi(t) < t$; $t > 1$.

These stability results can be applied in stochastic analysis [10], financial, and actuarial mathematics, as well as in psychology and sociology.

In 1987 Gajda and Ger [11] showed that one can get analogous stability results for subadditive multifunctions. In 1978 Gruber [12] remarked that Ulam's problem is of particular interest in probability theory and in the case of functional equations of different types. We refer the readers to [2, 5–32] and references therein for more detailed results on the stability problems of various functional equations.

We recall some basic facts concerning quasi-Banach space. A quasi-norm is a real-valued function on X satisfying the following.

(1) $\|x\| \ge 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(3) There is a constant $K \ge 1$ such that $\|x + y\| \le K(\|x\| + \|y\|)$ for all $x, y \in X$. The pair $(X, \|\cdot\|)$ is called a quasi-normed space if $\|\cdot\|$ is a quasi-norm on X. A quasi-Banach space is a complete quasi-normed space. A quasi-norm $\|\cdot\|$ is called a p-norm $(0 < p \le 1)$ if

$$\|x + y\|^p \le \|x\|^p + \|y\|^p, \tag{1.3}$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a p-Banach space. Given a p-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz theorem (see [33]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms, henceforth we restrict our attention mainly to p-norms. In this paper, we consider the generalized Hyers-Ulam stability of the following functional equation:

$$\sum_{i=1}^{m} f\left(mx_i + \sum_{j=1, j\neq i}^{m} x_j\right) + f\left(\sum_{i=1}^{m} x_i\right) = 2m\sum_{i=1}^{m} f(x_i), \tag{1.4}$$

for a fixed positive integer m with $m \ge 2$ in quasi-Banach spaces.

Throughout this paper, assume that X is a quasi-normed space with quasi-norm $\|\cdot\|_X$ and that Y is a p-Banach space with p-norm $\|\cdot\|_Y$.

2. Stability of Functional Equation (1.4) in Quasi-Banach Spaces

For simplicity, we use the following abbreviation for a given mapping $f : X \to Y$:

$$Df(x_1, x_2, \ldots, x_m) = \sum_{i=1}^{m} f\left(mx_i + \sum_{j=1, j\neq i}^{m} x_j\right) + f\left(\sum_{i=1}^{m} x_i\right) - 2m\sum_{i=1}^{m} f(x_i), \tag{2.1}$$

for all $x_j \in Y$ $(1 \leq j \leq m)$.

We start our work with the following theorem which can be regard as a general solution of functional equation (1.4).

Theorem 2.1. *Let V and W be real vector spaces. A mapping $f : V \to W$ satisfies in* (1.4) *if and only if f is additive.*

Proof. Setting $x_j = 0$ in (1.4) $(1 \leq j \leq m)$, we obtain

$$(m+1)f(0) = 2m^2 f(0). \tag{2.2}$$

Since $m \geq 2$, we have

$$f(0) = 0. \tag{2.3}$$

Setting $x_1 = x, x_j = 0$ $(2 \leq j \leq m)$ in (1.4), we obtain

$$f(mx) = mf(x). \tag{2.4}$$

Putting $x_1 = x, x_2 = y, x_j = 0$ $(3 \leq j \leq m)$, we get

$$f(mx+y) + f(my+x) + (m-1)f(x+y) = 2m(f(x) + f(y)). \tag{2.5}$$

Putting $x_1 = x, x_j = y/(m-1)$ $(2 \leq j \leq m)$, we get

$$f(mx+y) + (m-1)f(2y+x) + f(x+y) = 2m\left(f(x) + (m-1)f\left(\frac{y}{m-1}\right)\right). \tag{2.6}$$

Let $x = 0$ in (2.6), we obtain

$$2m(m-1)f\left(\frac{y}{m-1}\right) = 2f(y) + (m-1)f(2y). \tag{2.7}$$

So, (2.6) turns to the following:

$$f(mx+y) - (m-1)f(2y+x) + (m-2)f(x+y) = (2m-2)f(y) - (m-1)f(2y). \tag{2.8}$$

From (2.5) and (2.8), we have

$$f(mx+y)+(m-1)f(2y+x)+f(x+y)=2mf(x)+2f(y)+(m-1)f(2y). \tag{2.9}$$

Replacing x by y and y by x in (2.8) and comparing it with (2.9), we get

$$\begin{aligned}(m-1)\left[f(2x+y)+f(2y+x)\right]-(m-3)f(x+y)\\ =2\left[f(x)+f(y)\right]+(m-1)\left[f(2x)+f(2y)\right].\end{aligned} \tag{2.10}$$

Letting $x=y$ in (2.5), (2.8), and (2.10), respectively, we obtain

$$\begin{gathered}2f((m+1)x)+(m-1)f(2x)=4mf(x),\\ f((m+1)x)+(m-1)f(3x)=(2m+2)f(x)+(m-2)f(2x),\\ f(3x)=f(2x)+f(x).\end{gathered} \tag{2.11}$$

From (2.11) we have

$$f(2x)=2f(x). \tag{2.12}$$

Replacing $f(2x)$ and $f(2y)$ by their equivalents by using (2.12) in (2.10), we get

$$(m-1)\left[f(2x+y)+f(2y+x)\right]-(m-3)f(x+y)=2mf(x+y). \tag{2.13}$$

Replacing y by $-x$ in (2.13), we get

$$f(x)=-f(x). \tag{2.14}$$

Replacing x by $x-y$ in (2.13), we get

$$(m-1)\left[f(2x-y)+f(x+y)\right]-(m-3)f(x)=2m\left(f(x-y)+f(y)\right). \tag{2.15}$$

Similarly, replacing y by $y-x$ in (2.13), we obtain

$$(m-1)\left[f(2y-x)+f(x+y)\right]-(m-3)f(y)=2m\left(f(x)+f(y-x)\right). \tag{2.16}$$

Replacing y by $-y$ and x by $-x$ in (2.15) and (2.16), respectively, we obtain

$$\begin{aligned}(m-1)\left[f(2x+y)+f(x-y)\right]-(m-3)f(x)&=2m\left(f(x+y)+f(-y)\right),\\ (m-1)\left[f(2y+x)+f(y-x)\right]-(m-3)f(y)&=2m\left(f(x+y)+f(-x)\right).\end{aligned} \tag{2.17}$$

Adding both sides of (2.17) and using (2.14), we get

$$(m-1)\left[f(2x+y)+f(2y+x)\right]+(m+3)\left[f(x)+f(y)\right]=4mf(x+y). \tag{2.18}$$

Comparing (2.18) and (2.13), we obtain

$$f(x+y)=f(x)+f(y), \tag{2.19}$$

for all $x, y \in V$. So, if a mapping f satisfying (1.4) it must be additive. Conversely, let $f : V \to W$ be additive, it is clear that f satisfying (1.4), and the proof is complete. □

Now, we investigate the generalized Hyers-Ulam stability of functional equation (1.4) in quasi-Banach spaces.

Theorem 2.2. *Let* $\phi : \underbrace{X\times\cdots\times X}_{m\text{-times}} \to [0,\infty)$ *be a function satisfying*

$$\Phi(x)=\sum_{i=1}^{\infty}\left(\frac{1}{m}\right)^{ip}\left(\phi\left(m^{i-1}x,0,\ldots,0\right)\right)^{p}<\infty, \tag{2.20}$$

for all $x \in X$, *and*

$$\lim_{n\to\infty}\frac{1}{m^n}\phi(m^n x_1,\ldots,m^n x_m)=0, \tag{2.21}$$

for all $x_j \in X$ $(1 \le j \le m)$. *Suppose that a function* $f : X \to Y$ *with* $f(0)=0$ *satisfies the inequality:*

$$\left\|Df(x_1,\ldots,x_m)\right\| \le \phi(x_1,\ldots,x_m), \tag{2.22}$$

for all $x_j \in X$ $(1 \le j \le m)$. *Then there exists a unique additive mapping* T *defined by*

$$T(x)=\lim_{n\to\infty}\frac{1}{m^n}f(m^n x), \tag{2.23}$$

for all $x \in X$ *and the mapping* $T : X \to Y$ *satisfies the inequality:*

$$\left\|f(x)-T(x)\right\| \le [\Phi(x)]^{1/p}, \tag{2.24}$$

for all $x \in X$.

Proof. Putting $x_1 = x$ and $x_j = 0$ $(2 \le j \le m)$ in (2.22) and using $f(0)=0$, we obtain

$$\left\|f(mx)-mf(x)\right\| \le \phi(x,0,\ldots,0), \tag{2.25}$$

for all $x \in X$. By a simple induction we can prove that

$$\left\| f(x) - \frac{1}{m^n} f(m^n x) \right\|^p \leq \sum_{i=1}^{n} \left(\frac{1}{m}\right)^{ip} \left(\phi\left(m^{i-1}x, 0, \ldots, 0\right)\right)^p, \tag{2.26}$$

for all $x \in X$ and $n \in \mathbb{N}$. Thus

$$\left\| \frac{1}{m^l} f\left(m^l x\right) - \frac{1}{m^{n+l}} f\left(m^{l+n} x\right) \right\|^p \leq \sum_{i=1+l}^{n} \left(\frac{1}{m}\right)^{ip} \left(\phi\left(m^{i-1}x, 0, \ldots, 0\right)\right)^p, \tag{2.27}$$

for all $x \in X$ and all $l \in \mathbb{N}$ $(l \leq n)$. (2.20) and (2.27) show the sequence $\{(1/m^n) f(m^n x)\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\{(1/m^n) f(m^n x)\}$ converges in Y for all $x \in X$. Hence we can define the mapping $T : X \to Y$ by

$$T(x) = \lim_{n \to \infty} \frac{1}{m^n} f(m^n x), \tag{2.28}$$

for all $x \in X$. Letting $n \to \infty$ in (2.26), we obtain (2.24). Now we show that the mapping T is additive. We conclude from (2.21), (2.22), and (2.28)

$$\begin{aligned} \|DT(x_1, x_2, \ldots, x_m)\| &= \lim_{n \to \infty} \frac{1}{m^n} \|Df(m^n x_1, \ldots, m^n x_m)\| \\ &\leq \lim_{n \to \infty} \frac{1}{m^n} \phi(m^n x_1, \ldots, m^n x_m) \\ &= 0, \end{aligned} \tag{2.29}$$

for all $x_j \in G$ $(1 \leq j \leq m)$. So

$$\sum_{i=1}^{m} T\left(m x_i + \sum_{j=1, j \neq i}^{m} x_j\right) + T\left(\sum_{i=1}^{m} x_i\right) = 2m \sum_{i=1}^{m} T(x_i). \tag{2.30}$$

Hence, by Theorem 2.1, the mapping $T : X \to Y$ is additive. Now we prove the uniqueness assertion of T, by this mean let $T' : X \to Y$ be another mapping satisfies (2.24). It follows from (2.24)

$$\begin{aligned}
\|T(x) - T'(x)\| &= m^{-j}\left\|T\left(m^j x\right) - T'\left(m^j x\right)\right\| \\
&\le m^{-j}\left\|T\left(m^j x\right) - f\left(m^j x\right)\right\| + \left\|f\left(m^j x\right) - T'\left(m^j x\right)\right\| \\
&\le 2m^{-j}\left(\Phi\left(m^j x\right)\right)^{1/p} \\
&= 2m^{-j}\left(\sum_{i=1}^{\infty}\left(\frac{1}{m}\right)^{ip}\left(\phi\left(m^{i-1}x, 0, \ldots, 0\right)\right)^p\right)^{1/p} \\
&= 2\sum_{i=1+j}^{\infty}\left(\left(\frac{1}{m}\right)^{ip}\left(\phi\left(m^{i-1}x, 0, \ldots, 0\right)\right)^p\right)^{1/p},
\end{aligned} \tag{2.31}$$

for all $x \in X$. The right-hand side tends to zero as $j \to \infty$, hence $T(x) = T'(x)$ for all $x \in X$. This show the uniqueness of T. □

Corollary 2.3. *Let θ, r_j $(1 \le j \le m)$ be nonnegative real numbers such that $0 < r_j < 1$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality:*

$$\|Df(x_1, \ldots, x_m)\|_Y \le \theta \sum_{j=1}^{m} \|x_j\|_X^{r_j}, \tag{2.32}$$

for all $x_j \in X$ $(1 \le j \le m)$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\|_Y \le \frac{\theta\|x\|^{r_1}}{m^{r_1}}\left\{\frac{m^{(1-r_1)p}}{m^{(1-r_1)p} - 1}\right\}^{1/p}, \tag{2.33}$$

for all $x \in X$.

Proof. This is a simple consequence of Theorem 2.2. □

The following corollary is Hyers-Ulam-type stability for the functional equation (1.4).

Corollary 2.4. *Let θ be nonnegative real number. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality:*

$$\|Df(x_1, \ldots, x_m)\|_Y \le \theta, \tag{2.34}$$

for all $x_j \in X$ $(1 \le j \le m)$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\|_Y \le \theta\left\{\frac{1}{m^p - 1}\right\}^{1/p}, \tag{2.35}$$

for all $x \in X$.

Proof. In Theorem 2.2, let

$$\phi(x_1, x_2, \ldots, x_m) := \theta, \tag{2.36}$$

for all $x_i \in X(1 \le i \le m)$. □

The following corollary is Isac-Rassias-type stability for the functional equation (1.4).

Corollary 2.5. *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a mapping such that*

$$\begin{gathered} \lim_{t\to\infty} \frac{\psi(t)}{t} = 0, \\ \psi(ts) \le \psi(t)\psi(s) \quad s, t > 0, \\ \psi(t) < t \quad t > 1. \end{gathered} \tag{2.37}$$

Let θ, r_j $(1 \le j \le m)$ be nonnegative real numbers. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|Df(x_1, \ldots, x_m)\|_Y \le \theta \sum_{j=1}^{m} \psi\left(\|x_j\|_X\right), \tag{2.38}$$

for all $x_j \in X$ $(1 \le j \le m)$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\|_Y \le k\theta\psi\left(m^{-1}\right)\psi(\|x\|), \tag{2.39}$$

for all $x \in X$, where $k = \psi(m)/(m - \psi(m))$.

Proof. The proof follows from Theorem 2.2 by taking

$$\phi(x_1, \ldots, x_m) := \theta \sum_{j=i}^{m} \psi\left(\|x_j\|_X\right), \tag{2.40}$$

for all $x_j \in X(1 \le j \le m)$. □

Remark 2.6. In Theorem 2.2, If we replace control function by $\theta\prod_{j=1}^{m}\|x_j\|^{r_j}$, then $T = f$. Therefore in this case, f is superstable.

Theorem 2.7. *Let $\phi : \underbrace{X \times \cdots \times X}_{m\text{-times}} \to [0, \infty)$ be a mapping such that*

$$\lim_{n\to\infty} m^n \phi\left(\frac{x_1}{m^n}, \ldots, \frac{x_m}{m^n}\right) = 0, \tag{2.41}$$

for all $x_j \in X$ $(1 \le j \le m)$ *and*

$$\Phi(x) = \sum_{i=0}^{\infty} m^{ip} \phi\left(\frac{x}{m^{i+1}}, 0, \ldots, 0\right)^p < \infty, \tag{2.42}$$

for all $x \in X$. *Suppose that a function* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality:*

$$\|Df(x_1, \ldots, x_m)\| \le \phi(x_1, \ldots, x_m), \tag{2.43}$$

for all $x_j \in X$ $(1 \le j \le m)$. *Then there exists a unique additive mapping* T *defined by*

$$T(x) = \lim_{n \to \infty} m^n f\left(\frac{x}{m^n}\right), \tag{2.44}$$

for all $x \in X$ *and the mapping* $T : X \to Y$ *satisfies the inequality:*

$$\|f(x) - T(x)\| \le [\Phi(x)]^{1/p}, \tag{2.45}$$

for all $x \in X$.

Proof. Putting $x_1 = x$ and $x_j = 0$ $(2 \le j \le m)$ in (2.43) we obtain

$$\|f(mx) - mf(x)\| \le \phi(x, 0, \ldots, 0), \tag{2.46}$$

for all $x \in X$. Replacing x by x/m^{n+1} in (2.46) and multiplying both sides of (2.46) to m^n, we get

$$\left\| m^n f\left(\frac{x}{m^n}\right) - m^{n+1} f\left(\frac{x}{m^{n+1}}\right) \right\| \le m^n \phi\left(\frac{x}{m^{n+1}}, 0, \ldots, 0\right), \tag{2.47}$$

for all $x \in X$ and all $n \in \mathbb{N} \cup \{0\}$. Since Y is a p-Banach space, we have

$$\begin{aligned} \left\| m^{n+1} f\left(\frac{x}{m^{n+1}}\right) - m^r f\left(\frac{x}{m^r}\right) \right\|^p &\le \sum_{i=r}^{n} \left\| m^{i+1} f\left(\frac{x}{m^{i+1}}\right) - m^i f\left(\frac{x}{m^i}\right) \right\| \\ &\le \sum_{i=r}^{n} m^{ip} \left(\phi\left(\frac{x}{m^{i+1}}, 0, \ldots, 0\right)\right)^p, \end{aligned} \tag{2.48}$$

for all $x \in X$ and all nonnegative n and r with $n \ge r$. Therefore, we have from (2.42) and (2.48) that the sequence $\{m^n f(x/m^n)\}$ is a Cauchy in Y for all $x \in X$. Because of Y is complete, the sequence $\{m^n f(x/m^n)\}$ converges for all $x \in X$. Hence, we can define the mapping $T : X \to Y$ by

$$T(x) = \lim_{n \to \infty} m^n f\left(\frac{x}{m^n}\right), \tag{2.49}$$

for all $x \in X$. Putting $r = 0$ and passing the limit $n \to \infty$ in (2.48), we obtain (2.45). Showing the additivity and uniqueness of T is similar to Theorem 2.2, and the proof is complete. □

Corollary 2.8. *Let θ, r_j $(1 \le j \le m)$ be nonnegative real numbers such that $r_j > 1$ $(1 \le j \le m)$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality:*

$$\|Df(x_1, \ldots, x_m)\|_Y \le \theta \sum_{j=1}^{m} \|x_j\|_X^{r_j}, \tag{2.50}$$

for all $x_j \in X$ $(1 \le j \le m)$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\|_Y \le \frac{\theta \|x\|^{r_1}}{m^{r_1}} \left\{ \frac{m^{(r_1-1)p}}{1 - m^{(r_1-1)p}} \right\}^{1/p}, \tag{2.51}$$

for all $x \in X$.

Proof. This is a simple consequence of Theorem 2.7. □

Remark 2.9. We can formulate similar statement to Corollaries 2.4 and 2.5 for Theorem 2.7. Moreover, In Theorem 2.7, If we replace control function by $\theta \prod_{j=1}^{m} \|x_j\|^{r_j}$, then $T = f$. Therefore in this case, f is superstable.

Now, we apply a fixed point method and prove the generalized Hyers-Ulam stability of functional equation (1.4).

We recall a fundamental result in fixed point theory.

Theorem 2.10 (see [34]). *Let $(\mathcal{X}, d)$ be a complete generalized metric space and let $J : \mathcal{X} \to \mathcal{X}$ be a strictly contractive mapping with Lipschits constant $L \in (0, 1)$. Then, for a given element $x \in \mathcal{X}$, exactly one of the following assertions is true:*
either

(1) *$d(J^n x, J^{n+1} x) = \infty$ for all $n \ge 0$ or*

(2) *there exists n_0 such that $d(J^n x, J^{n+1} x) < \infty$ for all $n \ge n_0$.*
Actually, if (a_2) holds, then the sequence $J^n x$ is a convergent to a fixed point x^* of J and

(3) *x^* is the unique fixed point of J in $\Lambda := \{y \in \mathcal{X}, d(J^{n_0} x, y) < \infty\}$;*

(4) *$d(y, x^*) \le d(y, Jy)/(1 - L)$ for all $y \in \Lambda$.*

Theorem 2.11. *Let $f : X \to Y$ with $f(0) = 0$ be a mapping for which there exists a function $\phi : \underbrace{X \times \cdots \times X}_{m\text{-times}} \to [0, \infty)$ such that*

$$\|Df(x_1, \ldots, x_m)\| \le \phi(x_1, \ldots, x_m), \tag{2.52}$$

$$\lim_{n \to \infty} m^n \phi\left(\frac{x_1}{m^n}, \ldots, \frac{x_m}{m^n}\right) = 0, \tag{2.53}$$

for all $x_j \in X$ $(1 \le j \le m)$. *If there exists an* $L < 1$ *such that* $\phi(x_1, \ldots, x_j) \le (L/m)\phi(mx_1, \ldots, mx_j)\backslash, (1 \le j \le m)$, *then there exists a unique additive mapping* $T : X \to Y$ *satisfying*

$$\|f(x) - T(x)\| \le \frac{L}{m - mL}\phi(x, 0, \ldots, 0), \tag{2.54}$$

for all $x \in X$.

Proof. Putting $x_1 = x$ and $x_j = 0$ $(2 \le j \le m)$ in (2.52) and using $f(0) = 0$, we obtain

$$\|f(mx) - mf(x)\| \le \phi(x, 0, \ldots, 0), \tag{2.55}$$

for all $x \in X$. Hence,

$$\left\|f(x) - mf\left(\frac{x}{m}\right)\right\| \le \phi\left(\frac{x}{m}, 0, \ldots, 0\right) \le \frac{L}{m}\phi(x, \ldots, 0), \tag{2.56}$$

for all $x \in X$.

Let $E := \{g : X \to Y\}$. We introduce a generalized metric on E as follows:

$$d(g, h) := \inf\{C \in \mathbb{R}^+ : \|g(x) - h(x)\| \le C\phi(x, 0, \ldots, 0)\ \forall x \in X\}. \tag{2.57}$$

It is easy to show that (E, d) is a generalized complete metric space.

Now we consider the mapping $J : E \to E$ defined by

$$(Jg)(x) = mg\left(\frac{x}{m}\right), \tag{2.58}$$

for all $g \in E$ and all $x \in X$. Let $g, h \in E$ and let $C \in \mathbb{R}^+$ ba an arbitrary constant with $d(g, h) \le C$. From the definition of d, we have

$$\|g(x) - h(x)\| \le C\phi(x, 0, \ldots, 0), \tag{2.59}$$

for all $x \in X$. By the assumption and last inequality, we have

$$\|(Jg)(x) - (Jh)(x)\| = m\left\|g\left(\frac{x}{m}\right) - h\left(\frac{x}{m}\right)\right\| \le mC\phi\left(\frac{x}{m}, 0, \ldots, 0\right), \tag{2.60}$$

for all $x \in X$. So $d(Jg, Jh) \le Ld(g, h)$ for all $g, h \in E$. It follows from (2.56) that $d(Jf, f) \le L/m$. Therefore, according to Theorem 2.10, the sequence $\{J^n f\}$ converges to a fixed point T of J, that is,

$$T : X \longrightarrow Y, \quad T(x) = \lim_{n\to\infty}(J^n f)(x) = \lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right), \tag{2.61}$$

and $T(mx) = mT(x)$ for all $x \in X$. Also T is the unique fixed point of J in the set $E_\phi = \{g \in E : d(f,g) < \infty\}$ and

$$d(T,f) \le \frac{1}{1-L} d(Jf,f) \le \frac{L}{m(1-L)}, \tag{2.62}$$

that is, inequality (2.54) holds true for all $x \in X$. It follows from the definition of T, and (2.52) and (2.53) that

$$\sum_{i=1}^{m} T\left(mx_i + \sum_{j=1, j\neq i}^{m} x_j\right) + T\left(\sum_{i=1}^{m} x_i\right) = 2m \sum_{i=1}^{m} T(x_i). \tag{2.63}$$

Hence, by Theorem 2.1, the mapping $T : X \to Y$ is additive. □

Corollary 2.12. *Let $r_j \in (1,\infty)$ $(1 \le j \le m)$ and θ be real numbers. Let $f : X \to Y$ with $f(0) = 0$ such that*

$$\|Df(x_1,\ldots,x_m)\| \le \theta \sum_{j=1}^{m} \|x_j\|^{r_j}, \tag{2.64}$$

for all $x_j \in X$ $(1 \le j \le m)$. Then there exists a unique additive mapping $T : X \to Y$ satisfies the inequality:

$$\|f(x) - T(x)\| \le \frac{\theta}{m^{r_1} - m} \|x\|, \tag{2.65}$$

for all $x \in X$.

Proof. Setting $\phi(x_1, x_2, \ldots, x_m) := \theta \sum_{j=1}^{m} \|x_j\|^{r_j}$ for all $x_j \in X$ $(1 \le j \le m)$ in Theorem 2.11. Then by $L = m^{1-r_1}$, we get the desired result. □

Remark 2.13. We can formulate similar statement to Theorem 2.11 in which we can define the sequence $T(x) := \lim_{n\to\infty}(1/m^n) f(m^n x)$ under suitable conditions on the function ϕ and then obtain similar result to Corollary 2.12.

Remark 2.14. We can formulate similar statements for stability of (1.4) on Banach spaces.

References

[1] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, NY, USA, 1960.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[3] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.

[4] D. G. Bourgin, "Classes of transformations and bordering transformations," *Bulletin of the American Mathematical Society*, vol. 57, pp. 223–237, 1951.

[5] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[6] Z. Gajda, "On stability of additive mappings," *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.

[7] T. M. Rassias and P. Šemrl, "On the behavior of mappings which do not satisfy Hyers-Ulam stability," *Proceedings of the American Mathematical Society*, vol. 114, no. 4, pp. 989–993, 1992.

[8] P. Găvruţa, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.

[9] G. Isac and T. M. Rassias, "Stability of ψ-additive mappings: applications to nonlinear analysis," *International Journal of Mathematics and Mathematical Sciences*, vol. 19, no. 2, pp. 219–228, 1996.

[10] M. Eshaghi Gordji and N. Ghobadipour, "Nearly generalized Jordan derivations," *Mathematica Slovaca*, vol. 61, no. 1, pp. 55–62, 2011.

[11] Z. Gajda and R. Ger, "Subadditive multifunctions and Hyers-Ulam stability, Volume 5," in *General Inequalities*, vol. 80 of *International Schriftenreihe Numerical Mathematics*, pp. 281–291, Birkhäuser, Basel, Switzerland, 1987.

[12] P. M. Gruber, "Stability of isometries," *Transactions of the American Mathematical Society*, vol. 245, pp. 263–277, 1978.

[13] A. Ebadian, N. Ghobadipour, and M. Eshaghi Gordji, "A fixed point method for perturbation of bimultipliers and Jordan bimultipliers in C^*-ternary algebras," *Journal of Mathematical Physics*, vol. 51, no. 10, Article ID 103508, 10 pages, 2010.

[14] A. Ebadian, N. Ghobadipour, M. B. Savadkouhi, and M. Eshaghi Gordji, "Stability of a mixed type cubic and quartic functional equation in non-Archimedean L-fuzzy normed spaces," *Thai Journal of Mathematics*, vol. 9, no. 2, pp. 225–241, 2011.

[15] A. Ebadian, N. Ghobadipour, T. M. Rassias, and M. Eshaghi Gordji, "Functional inequalities associated with Cauchy additive functional equation in non-Archimedean spaces," *Discrete Dynamics in Nature and Society*, Article ID 929824, 14 pages, 2011.

[16] A. Ebadian, N. Ghobadipour, T. M. Rassias, and I. Nikoufar, "Stability of generalized derivations on Hilbert C^*-modules associated to a pexiderized Cuachy-Jensen type functional equation,," *Acta Mathematica Scintia*. In press.

[17] A. Ebadian, A. Najati, and M. Eshaghi Gordji, "On approximate additive-quartic and quadratic-cubic functional equations in two variables on abelian groups," *Results in Mathematics*, vol. 58, no. 1-2, pp. 39–53, 2010.

[18] M. Eshaghi Gordji and M. Bavnd Savadkouhei, "Approximation of generalized homomorphisms in quasi-Banach algebras," *Analele Stiintifice ale Universitatii Ovidius Constanta*, vol. 17, no. 2, Article ID 203214, 2009.

[19] M. E. Gordji and N. Ghobadipour, "Stability of (α, β, γ)-derivations on Lie C^*-algebras," *International Journal of Geometric Methods in Modern Physics*, vol. 7, no. 7, pp. 1093–1102, 2010.

[20] M. Eshaghi Gordji, J. M. Rassias, and N. Ghobadipour, "Generalized Hyers-Ulam stability of generalized (n,k)-derivations," *Abstract and Applied Analysis*, vol. 2009, Article ID 437931, 8 pages, 2009.

[21] M. Eshaghi Gordji and N. Ghobadipour, "Approximately quartichomomorphisms on Banach algebras," *Word Applied Sciences Journal*. In press.

[22] G. Z. Eskandani, "On the Hyers-Ulam-Rassias stability of an additive functional equation in quasi-Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 1, pp. 405–409, 2008.

[23] P. Gavruta and L. Gavruta, "A new method for the generalized Hyers-Ulam-Rassias stability," *International Journal of Non-Linear Applications*, vol. 1, no. 2, pp. 11–18, 2010.

[24] N. Ghobadipour, A. Ebadian, T. M. Rassias, and M. Eshaghi Gordji, "A perturbation of double derivations on Banach algebras," *Communications in Mathematical Analysis*, vol. 11, no. 1, pp. 51–60, 2011.

[25] P. Malliavin, *Stochastic Analysis*, Springer, Berlin, Germany, 1997.

[26] C. G. Park, "Linear $*$-derivations on C^*-algebras," *Tamsui Oxford Journal of Mathematical Sciences*, vol. 23, no. 2, pp. 155–171, 2007.

[27] C. G. Park, "Homomorphisms between Lie JC^*-algebras and Cauchy-Rassias stability of Lie JC^*-algebra derivations," *Journal of Lie Theory*, vol. 15, no. 2, pp. 393–414, 2005.

[28] J. M. Rassias, "Solution of the Ulam stability problem for quartic mappings," *Glasnik Matematički*, vol. 34, no. 2, pp. 243–252, 1999.

[29] J. M. Rassias, "On a new approximation of approximately linear mappings by linear mappings," *Discussiones Mathematicae*, vol. 7, pp. 193–196, 1985.

[30] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Journal of Functional Analysis*, vol. 46, no. 1, pp. 126–130, 1982.
[31] T. M. Rassias, "Problem 16; 2, report of the 27th international symposium on functional equations," *Aequationes Mathematicae*, vol. 39, pp. 292–293, 1990.
[32] T. M. Rassias, Ed., *Functional Equations, Inequalities and Applications*, Kluwer Academic, Dordrecht, The Netherlands, 2003.
[33] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis, Volume 1*, vol. 48 of *American Mathematical Society Colloquium Publications*, American Mathematical Society, Providence, RI, USA, 2000.
[34] J. B. Diaz and B. Margolis, "A fixed point theorem of the alternative, for contractions on a generalized complete metric space," *Bulletin of the American Mathematical Society*, vol. 74, pp. 305–309, 1968.