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Research Article

On an Integral Transform of a Class of Analytic Functions

Sarika Verma, Sushma Gupta, and Sukhjit Singh

Department of Mathematics, Sant Longowal Institute of Engineering and Technology, Longowal 148106, India

Correspondence should be addressed to Sarika Verma, sarika.16984@gmail.com

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For $\alpha, \gamma \geq 0$ and $\beta < 1$, let $\mathcal{W}_\beta(\alpha,\gamma)$ denote the class of all normalized analytic functions f in the open unit disc $E = \{z : |z| < 1\}$ such that $\Re e^{i\phi}((1-\alpha+2\gamma)(f(z)/z) + (\alpha-2\gamma)f'(z) + \gamma z f''(z) - \beta) > 0$, $z \in E$ for some $\phi \in \mathbb{R}$. It is known (Noshiro (1934) and Warschawski (1935)) that functions in $\mathcal{W}_\beta(1,0)$ are close-to-convex and hence univalent for $0 \leq \beta < 1$. For $f \in \mathcal{W}_\beta(\alpha,\gamma)$, we consider the integral transform $F(z) = V_\lambda(f)(z) := \int_0^1 \lambda(t)(f(tz)/t)dt$, where λ is a nonnegative real-valued integrable function satisfying the condition $\int_0^1 \lambda(t)dt = 1$. The aim of present paper is, for given $\delta < 1$, to find sharp values of β such that (i) $V_\lambda(f) \in \mathcal{W}_\delta(1,0)$ whenever $f \in \mathcal{W}_\beta(\alpha,\gamma)$ and (ii) $V_\lambda(f) \in \mathcal{W}_\delta(\alpha,\gamma)$ whenever $f \in \mathcal{W}_\beta(\alpha,\gamma)$.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f defined in the open unit disc $E = \{z : |z| < 1\}$ with the normalizations $f(0) = f'(0) - 1 = 0$, and let S be the subclass of $\mathcal{A}$ consisting of functions univalent in E. For any two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ in $\mathcal{A}$, the Hadamard product (or convolution) of f and g is the function $f * g$ defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.1}$$

For $f \in \mathcal{A}$, Fournier and Ruscheweyh [1] introduced the integral operator

$$F(z) = V_\lambda(f)(z) := \int_0^1 \lambda(t) \frac{f(tz)}{t} dt, \tag{1.2}$$

where λ is a nonnegative real-valued integrable function satisfying the condition $\int_0^1 \lambda(t)dt = 1$. This operator contains some well-known operators such as Libera, Bernardi, and Komatu as its special cases. Fournier and Ruscheweyh [1] applied the famous duality theory to show that for a function f in the class

$$\mathcal{P}(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \mid \Re e^{i\phi}\left(f'(z) - \beta\right) > 0,\ z \in E \right\}, \tag{1.3}$$

the linear integral operator $V_\lambda(f)$ is univalent in E. Since then, this operator has been studied by a number of authors for various choices of $\lambda(t)$. In another remarkable paper, Barnard et al. in [2] obtained conditions such that $V_\lambda(f) \in \mathcal{P}_1(\beta)$ whenever f is in the class

$$\mathcal{P}_\gamma(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \mid \Re e^{i\phi}\left((1-\gamma)\frac{f(z)}{z} + \gamma f'(z) - \beta \right) > 0,\ z \in E \right\}, \tag{1.4}$$

with $\beta < 1$, $\gamma \geq 0$. Note that for $0 \leq \beta < 1$, functions in $\mathcal{P}_1(\beta) \equiv \mathcal{P}(\beta)$ satisfy the condition $\Re f'(z) > \beta$ in E and thus are close-to-convex in E. A domain D in $\mathbb{C}$ is close-to-convex if its compliment in $\mathbb{C}$ can be written as union of nonintersecting half lines.

In 2008, Ponnusamy and Rønning [3] discussed the univalence of $V_\lambda(f)$ for the functions in the class

$$\mathcal{R}_\gamma(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \mid \Re e^{i\phi}\left(f'(z) + \gamma z f''(z) - \beta\right) > 0,\ z \in E \right\}. \tag{1.5}$$

In a very recent paper, Ali et al. [4] studied the class

$$\begin{aligned} &\mathcal{W}_\beta(\alpha,\gamma) \\ &\quad := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \mid \Re e^{i\phi}\left((1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma z f''(z) - \beta \right) > 0,\ z \in E \right\}, \end{aligned} \tag{1.6}$$

where $\alpha, \gamma \geq 0$ and $\beta < 1$. In this paper, they obtained sufficient conditions so that the integral transform $V_\lambda(f)$ maps normalized analytic functions $f \in \mathcal{W}_\beta(\alpha,\gamma)$ into the class of starlike functions. It is evident that $\mathcal{W}_\beta(1,0) \equiv \mathcal{P}(\beta)$, $\mathcal{W}_\beta(\alpha,0) \equiv \mathcal{P}_\alpha(\beta)$ and $\mathcal{W}_\beta(1+2\gamma,\gamma) \equiv \mathcal{R}_\gamma(\beta)$.

In the present paper, we shall mainly tackle the following problems.

(1) For given $\delta < 1$, find sharp values of $\beta = \beta(\delta,\alpha)$ such that $V_\lambda(f) \in \mathcal{W}_\delta(1,0)$ whenever $f \in \mathcal{W}_\beta(\alpha,\gamma)$.

(2) For given $\delta < 1$, find sharp values of $\beta = \beta(\delta)$ such that $V_\lambda(f) \in \mathcal{W}_\delta(\alpha,\gamma)$ whenever $f \in \mathcal{W}_\beta(\alpha,\gamma)$.

To prove one of our results, we shall need the generalized hypergeometric function ${}_pF_q$, so we define it here.

Let α_j $(j = 1,2,\ldots,p)$ and β_j $(j = 1,2,\ldots,q)$ be complex numbers with $\beta_j \neq 0,-1,-2,\ldots$ $(j = 1,2,\ldots,q)$. Then the generalized hypergeometric function ${}_pF_q$ is defined by

$$ {}_pF_q(z) = {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \quad (p \leq q+1), \tag{1.7} $$

where $(a)_n$ is the Pochhammer symbol, defined in terms of the Gamma function, by

$$ (a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & n = 0, \\ a(a+1)\cdots(a+n-1), & n \in \mathbb{N}. \end{cases} \tag{1.8} $$

In particular, ${}_2F_1$ is called the Gaussian hypergeometric function. We note that the ${}_pF_q$ series in (1.7) converges absolutely for $|z| < \infty$ if $p < q+1$ and for $z \in E$ if $p = q+1$.

We shall also need the following lemma.

Lemma 1.1 (see [5]). *Let $\beta_1 < 1$, $\beta_2 < 1$, and $\eta \in \mathbb{R}$. Then, for p, q analytic in E with $p(0) = q(0) = 1$, the conditions $\Re p(z) > \beta_1$ and $\Re e^{i\eta}(q(z) - \beta_2) > 0$ imply $\Re e^{i\eta}((p * q)(z) - \delta) > 0$, where $1 - \delta = 2(1-\beta_1)(1-\beta_2)$.*

2. Main Results

We use the notations introduced in [4]. Let $\mu \geq 0$ and $\nu \geq 0$ satisfy

$$ \mu + \nu = \alpha - \gamma, \qquad \mu\nu = \gamma. \tag{2.1} $$

When $\gamma = 0$, then μ is chosen to be 0, in which case, $\nu = \alpha \geq 0$. When $\alpha = 1 + 2\gamma$, (2.1) yields $\mu + \nu = 1 + \gamma = 1 + \mu\nu$ or $(\mu - 1)(1 - \nu) = 0$.

(i) For $\gamma > 0$, then choosing $\mu = 1$ gives $\nu = \gamma$.

(ii) For $\gamma = 0$, then $\mu = 0$ and $\nu = \alpha = 1$.

Theorem 2.1. *Let $\mu \geq 0$, $\nu \geq 0$ satisfy* (2.1). *Further, let $\delta < 1$ be given, and define $\beta = \beta(\delta,\mu,\nu)$ by*

$$ 1 - \frac{1-\delta}{2}\left\{1 - \frac{1}{\nu}\int_0^1 \lambda(t)\left(\int_0^1 \frac{ds}{1+ts^{\mu}}\right)dt + \left(\frac{1}{\nu} - 1\right)\int_0^1 \lambda(t)\left(\iint_0^1 \frac{d\eta\, d\zeta}{1+t\eta^{\nu}\zeta^{\mu}}\right)dt\right\}^{-1}, \quad \gamma \neq 0, $$

$$ 1 - \frac{1-\delta}{2}\left\{1 - \frac{1}{\alpha}\int_0^1 \frac{\lambda(t)}{1+t}dt + \left(\frac{1}{\alpha} - 1\right)\int_0^1 \lambda(t)\left(\int_0^1 \frac{d\eta}{1+t\eta^{\alpha}}\right)dt\right\}^{-1}, \quad \gamma = 0 \ (\mu = 0,\ \nu = \alpha > 0). \tag{2.2} $$

If $f \in \mathcal{W}_{\beta}(\alpha,\gamma)$, then $F = V_{\lambda}(f) \in \mathcal{W}_{\delta}(1,0) \subset S$. The value of β is sharp.

Proof. The case $\gamma = 0$ $(\mu = 0,\ \nu = \alpha > 0)$ corresponds to Theorem 1.5 in [2]. So we assume that $\gamma > 0$.

Define

$$(1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma z f''(z)=H(z). \tag{2.3}$$

Writing $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$, it follows that

$$H(z)=1+\sum_{n=1}^{\infty}a_{n+1}(n\nu+1)(n\mu+1)z^n. \tag{2.4}$$

It is a simple exercise to see that

$$f'(z)=H(z)*{}_3F_2\left(2,\frac{1}{\nu},\frac{1}{\mu};\frac{1}{\nu+1},\frac{1}{\mu+1};z\right). \tag{2.5}$$

Let $F(z)=V_\lambda(f)(z)$, where $V_\lambda(f)$ is defined by (1.2). Then for $\gamma\neq 0$, we can write

$$\begin{aligned}F'(z)&=f'(z)*\int_0^1\frac{\lambda(t)}{1-tz}dt\\&=H(z)*{}_3F_2\left(2,\frac{1}{\nu},\frac{1}{\mu};\frac{1}{\nu+1},\frac{1}{\mu+1};z\right)*\int_0^1\frac{\lambda(t)}{1-tz}dt\\&=H(z)*\int_0^1\lambda(t)\,{}_3F_2\left(2,\frac{1}{\nu},\frac{1}{\mu};\frac{1}{\nu+1},\frac{1}{\mu+1};tz\right)dt.\end{aligned} \tag{2.6}$$

Since $f\in\mathcal{W}_\beta(\alpha,\gamma)$, it follows that $\Re\{e^{i\phi}(H(z)-\beta)\}>0$ for some $\phi\in\mathbb{R}$. Now, for each $\gamma>0$, we first claim that

$$\Re\left[\int_0^1\lambda(t)\,{}_3F_2\left(2,\frac{1}{\nu},\frac{1}{\mu};\frac{1}{\nu+1},\frac{1}{\mu+1};tz\right)dt\right]>1-\frac{1-\delta}{2(1-\beta)},\quad z\in E, \tag{2.7}$$

which, by Lemma 1.1, implies that $F\in\mathcal{W}_\delta(1,0)$. Therefore, it suffices to verify the inequality (2.7). Using the identity (which can be checked by comparing the coefficients of z^n on both sides)

$${}_3F_2(2,b,c;d,e;z)=(d-1)\,{}_3F_2(1,b,c;d-1,e;z)-(d-2)\,{}_3F_2(1,b,c;d,e;z), \tag{2.8}$$

it follows that

$${}_3F_2\left(2,\frac{1}{\nu},\frac{1}{\mu};\frac{1}{\nu+1},\frac{1}{\mu+1};z\right)=\frac{1}{\nu}\int_0^1\frac{ds}{1-zs^\mu}+\left(1-\frac{1}{\nu}\right)\iint_0^1\frac{d\eta\,d\zeta}{1-z\eta^\nu\zeta^\mu}. \tag{2.9}$$

Thus,

$$\begin{aligned}&\int_0^1 \lambda(t)\,{}_3F_2\left(2,\frac{1}{\nu},\frac{1}{\mu};\frac{1}{\nu+1},\frac{1}{\mu+1};tz\right)dt\\&\quad=\int_0^1 \lambda(t)\left\{\frac{1}{\nu}\int_0^1\frac{ds}{1-tzs^{\mu}}+\left(1-\frac{1}{\nu}\right)\iint_0^1\frac{d\eta\,d\zeta}{1-tz\eta^{\nu}\zeta^{\mu}}\right\}dt.\end{aligned}\tag{2.10}$$

Therefore, for $\gamma > 0$, we have

$$\begin{aligned}&\Re\left[\int_0^1 \lambda(t)\,{}_3F_2\left(2,\frac{1}{\nu},\frac{1}{\mu};\frac{1}{\nu+1},\frac{1}{\mu+1};tz\right)dt\right]\\&\quad>\frac{1}{\nu}\int_0^1\lambda(t)\left(\int_0^1\frac{ds}{1+ts^{\mu}}\right)dt+\left(\frac{1}{\nu}-1\right)\int_0^1\lambda(t)\left(\iint_0^1\frac{d\eta\,d\zeta}{1+t\eta^{\nu}\zeta^{\mu}}\right)dt\\&\quad=1-\frac{1-\delta}{2(1-\beta)},\end{aligned}\tag{2.11}$$

in the view of (2.2).

To prove the sharpness, let $f \in \mathcal{W}_{\beta}(\alpha,\gamma)$ be the function determined by

$$(1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma z f''(z)=\beta+(1-\beta)\frac{1+z}{1-z}.\tag{2.12}$$

Using a series expansion, we see that we can write

$$f(z)=z+\sum_{n=2}^{\infty}\frac{2(1-\beta)}{(n\nu+1-\nu)(n\mu+1-\mu)}z^n.\tag{2.13}$$

Then,

$$F(z)=V_{\lambda}(f)(z)=z+2(1-\beta)\sum_{n=2}^{\infty}\frac{\varphi_n}{(n\nu+1-\nu)(n\mu+1-\mu)}z^n,\tag{2.14}$$

where $\varphi_n = \int_0^1 \lambda(t)t^{n-1}dt$. Equation (2.2) can be restated as

$$\begin{aligned}
\frac{1}{1-\beta} &= \frac{2}{1-\delta}\left\{1-\frac{1}{\nu}\int_0^1 \lambda(t)\left(\int_0^1 \frac{ds}{1+ts^{\mu}}\right)dt+\left(\frac{1}{\nu}-1\right)\int_0^1 \lambda(t)\left(\iint_0^1 \frac{d\eta\, d\zeta}{1+t\eta^{\nu}\zeta^{\mu}}\right)dt\right\} \\
&= \frac{2}{1-\delta}\left\{1+\int_0^1 \lambda(t)\left(-\frac{1}{\nu}\int_0^1 \frac{ds}{1+ts^{\mu}}+\left(\frac{1}{\nu}-1\right)\iint_0^1 \frac{d\eta\, d\zeta}{1+t\eta^{\nu}\zeta^{\mu}}\right)dt\right\} \\
&= \frac{2}{1-\delta}\int_0^1 \lambda(t)\left\{\sum_{n=2}^{\infty}\frac{(-1)^{n-1}t^{n-1}}{(n\mu+1-\mu)}\left(-\frac{1}{\nu}+\left(\frac{1}{\nu}-1\right)\frac{1}{(n\nu+1-\nu)}\right)\right\}dt \\
&= -\frac{2}{1-\delta}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}n\varphi_n}{(n\nu+1-\nu)(n\mu+1-\mu)}.
\end{aligned} \tag{2.15}$$

Finally,

$$F'(z) = 1+2(1-\beta)\sum_{n=2}^{\infty}\frac{n\varphi_n}{(n\nu+1-\nu)(n\mu+1-\mu)}z^{n-1}, \tag{2.16}$$

which for $z=-1$ takes the value

$$F'(-1) = 1+2(1-\beta)\sum_{n=2}^{\infty}\frac{(-1)^{n-1}n\varphi_n}{(n\nu+1-\nu)(n\mu+1-\mu)} = 1+2(1-\beta)\left\{\frac{-(1-\delta)}{2(1-\beta)}\right\} = \delta. \tag{2.17}$$

This shows that the result is sharp. □

Letting $\gamma=0$ and $\alpha=1$ in Theorem 1.1, we obtain the following result of Ruscheweyh [6].

Corollary 2.2. *Let $\delta<1$, and define $\beta=\beta(\delta,1)<1$ by*

$$\beta(\delta) = 1-\frac{1-\delta}{2}\left\{1-\int_0^1 \frac{\lambda(t)}{1+t}dt\right\}^{-1}. \tag{2.18}$$

If $f\in\mathcal{W}_{\beta}(1,0)\equiv\mathcal{P}_1(\beta)$, then $F=V_{\lambda}(f)\in\mathcal{W}_{\delta}(1,0)\subset S$. The value of β is sharp.

Theorem 2.3. *Let $\delta<1$ and $\alpha,\gamma\geq 0$, and define $\beta=\beta(\delta)<1$ by*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t)\frac{(1-((1+\delta)/(1-\delta))t)}{(1+t)}dt. \tag{2.19}$$

If $f\in\mathcal{W}_{\beta}(\alpha,\gamma)$, then $V_{\lambda}(f)\in\mathcal{W}_{\delta}(\alpha,\gamma)$. The value of β is sharp.

Proof. The idea of the proof is similar to the one used to prove Theorem 2 in [1].

Let $F(z) = V_\lambda(f)(z) = \int_0^1 \lambda(t)(f(tz)/t)dt$. Clearly,

$$F'(z) = \int_0^1 \frac{\lambda(t)}{1-tz}dt * f'(z). \tag{2.20}$$

Since, $f \in \mathcal{W}_\beta(\alpha,\gamma)$, so with

$$g(z) = \frac{(1-\alpha+2\gamma)(f(z)/z) + (\alpha-2\gamma)f'(z) + \gamma z f''(z) - \beta}{1-\beta}, \tag{2.21}$$

we have $\Re[e^{i\phi}g(z)] > 0$, where $\phi \in \mathbb{R}$.

For $\gamma \neq \alpha/2$,

$$f'(z) = \frac{1}{\alpha-2\gamma}(\beta + (1-\beta)g(z)) - \frac{1-\alpha+2\gamma}{\alpha-2\gamma}\frac{f(z)}{z} - \frac{\gamma}{\alpha-2\gamma}zf''(z). \tag{2.22}$$

Putting this value in (2.20),

$$F'(z) = \int_0^1 \frac{\lambda(t)}{1-tz}dt * \left(\frac{1}{\alpha-2\gamma}(\beta + (1-\beta)g(z)) - \frac{1-\alpha+2\gamma}{\alpha-2\gamma}\frac{f(z)}{z} - \frac{\gamma}{\alpha-2\gamma}zf''(z)\right). \tag{2.23}$$

Equivalently,

$$F'(z) = \frac{1}{\alpha-2\gamma}g(z) * \left[\beta + (1-\beta)\int_0^1 \frac{\lambda(t)}{1-tz}dt\right] - \frac{1-\alpha+2\gamma}{\alpha-2\gamma}\frac{F(z)}{z} - \frac{\gamma}{\alpha-2\gamma}zF''(z). \tag{2.24}$$

Thus

$$(1-\alpha+2\gamma)(F(z)/z) + (\alpha-2\gamma)F'(z) + \gamma z F''(z) = g(z) * \left[\beta + (1-\beta)\int_0^1 \frac{\lambda(t)}{1-tz}dt\right]. \tag{2.25}$$

In the case when $\gamma = \alpha/2$,

$$g(z) = \frac{f(z)/z + \gamma z f''(z) - \beta}{1-\beta}. \tag{2.26}$$

Since

$$\frac{f(z)}{z} = \beta + (1-\beta)g(z) - \gamma z f''(z), \tag{2.27}$$

This leads to,

$$\frac{F(z)}{z} + \gamma z F''(z) = g(z) * \left[\beta + (1-\beta)\int_0^1 \frac{\lambda(t)}{1-tz}dt\right], \tag{2.28}$$

which is clearly (2.25) with $\gamma = \alpha/2$.

Further $F \in \mathcal{W}_\delta(\alpha,\gamma)$ if and only if $G(z) := (F(z)-\delta z)/(1-\delta) \in \mathcal{W}_0(\alpha,\gamma)$. Now using (2.25), we obtain

$$(1-\alpha+2\gamma)\frac{G(z)}{z} + (\alpha-\gamma)G'(z) + \gamma z G''(z) = g(z) * \left[\frac{\beta-\delta}{1-\delta} + \frac{1-\beta}{1-\delta}\int_0^1 \frac{\lambda(t)}{1-tz}dt\right]. \tag{2.29}$$

Since $\Re e^{i\phi} g(z) > 0$ for some $\phi \in \mathbb{R}$, it follows by duality principle [8, page 23] that

$$(1-\alpha+2\gamma)\frac{G(z)}{z} + (\alpha-2\gamma)G'(z) + \gamma z G''(z) \neq 0 \tag{2.30}$$

if, and only if,

$$\Re\left[\frac{\beta-\delta}{1-\delta} + \frac{1-\beta}{1-\delta}\int_0^1 \frac{\lambda(t)}{1-tz}dt\right] > \frac{1}{2}. \tag{2.31}$$

Using $\Re(1/(1-tz)) > 1/(1+t)$, we get

$$\Re\left[\frac{\beta-\delta}{1-\delta} + \frac{1-\beta}{1-\delta}\int_0^1 \frac{\lambda(t)}{1-tz}dt\right] > \frac{1-\beta}{1-\delta}\left[\frac{\beta-\delta}{1-\beta} + \int_0^1 \frac{\lambda(t)}{1+t}dt\right]. \tag{2.32}$$

By using (2.19), we have

$$\frac{\beta-(1+\delta)/2}{1-\beta} = -\int_0^1 \frac{\lambda(t)}{(1+t)}dt. \tag{2.33}$$

Thus,

$$\frac{\beta-\delta}{1-\beta} + \int_0^1 \frac{\lambda(t)}{1+t}dt = \frac{1}{2}\frac{1-\delta}{1-\beta}, \tag{2.34}$$

which implies that

$$\Re\left[\frac{\beta-\delta}{1-\delta} + \frac{1-\beta}{1-\delta}\int_0^1 \frac{\lambda(t)}{1-tz}dt\right] > \frac{1-\beta}{1-\delta}\left[\frac{\beta-\delta}{1-\beta} + \int_0^1 \frac{\lambda(t)}{1+t}dt\right] = \frac{1}{2}. \tag{2.35}$$

Thus, we deduce, using duality principle, that $(1-\alpha+2\gamma)(G(z)/z)+(\alpha-\gamma)G'(z)+\gamma zG''(z)$ is contained in a half plane not containing the origin. So, $G \in \mathcal{W}_0(\alpha,\gamma)$ and hence $F \in \mathcal{W}_\delta(\alpha,\gamma)$.

To prove the sharpness, let $f(z) = z + 2(1-\beta)\sum_{n=2}^{\infty}(z^n/(n\mu+1-\mu)(n\nu+1-\nu))$.

$$F(z) = V_\lambda(f)(z) = z + 2(1-\beta)\sum_{n=2}^{\infty}\frac{z^n\omega_n}{(n\mu+1-\mu)(n\nu+1-\nu)}, \quad \text{where } \omega_n = \int_0^1 \lambda(t)t^{n-1}dt. \tag{2.36}$$

Further,

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t)\frac{(1-((1+\delta)/(1-\delta))t)}{(1+t)}dt \tag{2.37}$$

gives

$$\frac{\beta}{1-\beta} = -1 + \int_0^1 \lambda(t)\frac{(1+(1+\delta)/(1-\delta))}{(1+t)}t\,dt, \tag{2.38}$$

or

$$\frac{1}{1-\beta} = \frac{2}{1-\delta}\int_0^1 \frac{t\lambda(t)}{1+t}dt = \frac{2}{1-\delta}\sum_{n=2}^{\infty}(-1)^n\omega_n. \tag{2.39}$$

Further, assume that

$$H(z) = (1-\alpha+2\gamma)\frac{F(z)}{z} + (\alpha-\gamma)F'(z) + \gamma zF''(z). \tag{2.40}$$

Since $F(z) = z + 2(1-\beta)\sum_{n=2}^{\infty}(\omega_n z^n/(n\mu+1-\mu)(n\nu+1-\nu))$,
so,

$$H(z) = 1 + 2(1-\beta)\sum_{n=2}^{\infty}\omega_n z^{n-1}. \tag{2.41}$$

Therefore, for $z = -1$,

$$H(-1) = 1 - 2(1-\beta)\sum_{n=2}^{\infty}\omega_n(-1)^n = 1 - 2(1-\beta)\frac{1-\delta}{2(1-\beta)} = \delta. \tag{2.42}$$

This shows that the result is sharp. □

Letting $\gamma = 0$ in Theorem 2.3 above, we obtain the following result of Kim and Rønning [9].

Corollary 2.4. *Let $\delta < 1$ and $\alpha \geq 0$, and define $\beta = \beta(\delta)$ by*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t) \frac{(1-((1+\delta)/(1-\delta))t)}{(1+t)} dt. \tag{2.43}$$

If $f \in \mathcal{W}_\beta(\alpha, 0) \equiv \mathcal{P}_\alpha(\beta)$, then $V_\lambda(f) \in \mathcal{W}_\delta(\alpha, 0) \equiv \mathcal{P}_\alpha(\delta)$. The value of β is sharp.

Upon setting $\lambda(t) = (1+c)t^c$ with $-1 < c$, we have the following corollary.

Corollary 2.5. *Let $\delta < 1$, $\alpha, \gamma \geq 0$, and $-1 < c \leq 0$ be given, and let $G(z)$ be defined by*

$$G(z) = \frac{(1+c)}{z^c} \int_0^z u^{c-1} f(u) du. \tag{2.44}$$

Suppose that $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then $G \in \mathcal{W}_\delta(\alpha, 0)$, where

$$\beta = \frac{2(1+c)\,{}_2F_1(1, 2+c; 3+c, -1) - (2+c)}{2(1+c)\,{}_2F_1(1, 2+c; 3+c, -1)}. \tag{2.45}$$

The constant β is sharp.

The special case of Corollary 2.5 (with $\gamma = 0$) has been obtained by Aghalary et al. [11].

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