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Research Article

Uniqueness of Traveling Waves for a Two-Dimensional Bistable Periodic Lattice Dynamical System

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We study traveling waves for a two-dimensional lattice dynamical system with bistable nonlinearity in periodic media. The existence and the monotonicity in time of traveling waves can be derived in the same way as the one-dimensional lattice case. In this paper, we derive the uniqueness of nonzero speed traveling waves by using the comparison principle and the sliding method.

1. Introduction

In this paper, we study the following two-dimensional (2D) lattice dynamical system:

$$\dot{u}_{i,j}(t) := \frac{d}{dt}u_{i,j}(t) = D_2\left[u_{i,j}\right](t) + f\left(u_{i,j}(t)\right), \quad i, j \in \mathbb{Z},\ t > 0, \tag{1.1}$$

where f is a C^2 function in $\mathbb{R}$ and

$$\begin{gathered} D_2\left[u_{i,j}\right] := p_{i+1,j}u_{i+1,j} + p_{i,j}u_{i-1,j} + q_{i,j+1}u_{i,j+1} + q_{i,j}u_{i,j-1} - d_{i,j}u_{i,j}, \\ d_{i,j} := p_{i+1,j} + p_{i,j} + q_{i,j+1} + q_{i,j}. \end{gathered} \tag{1.2}$$

We assume that the coefficients $p_{i,j}, q_{i,j}$ are positive and bounded such that

$$p_{i+N,j} = p_{i,j} = p_{i,j+N}, \qquad q_{i+N,j} = q_{i,j} = q_{i,j+N}, \quad \forall i, j \in \mathbb{Z}, \tag{1.3}$$

for some positive integer N. Furthermore, we consider the case of bistable nonlinearity, namely,

$$f(0) = f(1) = f(a) = 0, \qquad f'(0) < 0 < f'(a), \qquad f'(1) < 0, \tag{1.4}$$

for some constant $a \in (0,1)$. For simplicity, we only consider the case when $a \in (0, 1/2]$.

We are interested in (planar) traveling wave solutions of (1.1) such that

$$u_{i,j}\left(t + \frac{kNr + mNs}{c}\right) = u_{i-kN,j-mN}(t) \quad \forall k, m \in \mathbb{Z},\ t \in \mathbb{R}, \tag{1.5}$$

for some (speed) $c \neq 0$ and

$$\lim_{ri+sj \to -\infty} u_{i,j}(t) = 0, \qquad \lim_{ri+sj \to +\infty} u_{i,j}(t) = 1, \tag{1.6}$$

for any $t \in \mathbb{R}$ in the direction $(r, s) := (\cos\theta, \sin\theta)$ for some $\theta \in (0, \pi/2)$.

The study of lattice dynamical systems has attracted a lot of attention for past years. In particular, traveling wave solutions are important due to the wide applications of these special solutions. For example, the invading of one species to another can be described by traveling wave solutions (see, e.g., [1, 2]). The lattice dynamical system arises, for example, when the habitat is divided into discrete niches in certain biology models. We refer the reader to, for example, [3–9] for monostable nonlinearity and [10–16] for bistable nonlinearity in a one-dimensional lattice. In particular, in [16] the authors studied a very general model with bistable nonlinearity in a 1D lattice. Our purpose of this paper is to extend the result of [16] to the case of multidimensional lattice. For the study of multidimensional lattice, we refer to [17–19]. For the simplicity of presentation, we will only consider the 2D lattice dynamical system (1.1). Our results can be easily extended to the more general case with a convection term or spatially dependent nonlinearity as in [16].

In a similar manner as that in [16] for 1D lattice case, we can prove the existence of traveling wave solutions of (1.1)–(1.6) with profile $\{U_{i,j}\}_{i,j\in\mathbb{Z}}$ and speed $c \in \mathbb{R}$, by transforming the problem (1.1)–(1.6) into an integral formulation. Moreover, if the speed $c > 0$, then we can obtain $\dot{U}_{i,j}(\mathrm{t}) < 0$ for all $i, j \in \mathbb{Z}$ and $t \in \mathbb{R}$. We will not repeat the proof here and focus on the study of the uniqueness of nonzero speed traveling waves. The uniqueness is in the sense that if there exist two traveling waves with nonzero speeds, then these two speeds are the same, and two wave profiles are the same except a translation. Due to that the nonlinearity is independent of spatial variable, our proof of the uniqueness is simpler and more transparent than that in [16]. In fact, motivated by the work of Fife and McLeod [20], Lemma 3.1 (below) provides some estimations in terms of a given traveling wave solution for the solution to the initial value problem for (1.1) with certain initial condition. Moreover, with Lemma 3.1, we employ the idea of moving coordinate and a sliding method to complete the proof of uniqueness (see Theorem 3.3).

This paper is organized as follows. In Section 2, we give some preliminaries including a comparison principle. Then we prove uniqueness of traveling wave with nonzero speed in Section 3.

2. Preliminaries

The following lemma can be easily deduced from (1.5) and (1.6).

Lemma 2.1. *Let $\{u_{i,j}\}_{i,j\in\mathbb{Z}}$ be a solution of (1.1)–(1.6). If $c > 0(< 0)$, then $u_{i,j}(t) \to 0(\to 1)$ as $t \to \infty$ and $u_{i,j}(t) \to 1(\to 0)$ as $t \to -\infty$ for each i, j.*

We can determine the sign of the speed c (when $c \neq 0$) as follows.

Lemma 2.2. *Suppose that $c \neq 0$, then c has the same sign as $[-\int_0^1 f(s)ds]$.*

Proof. For $K \geq \max\{1, N/|c|\}$, an integration by parts gives

$$\begin{aligned}
\int_{-K}^{K}\left[\dot{u}_{i,j}(t)\right]^2 dt &= \int_{-K}^{K}\dot{u}_{i,j}(t)\left\{D_2\left[u_{i,j}\right](t)+f\left(u_{i,j}(t)\right)\right\}dt \\
&= \int_{-K}^{K}p_{i+1,j}\dot{u}_{i,j}u_{i+1,j}dt - \int_{-K}^{K}p_{i,j}u_{i,j}\dot{u}_{i-1,j}dt + p_{i,j}u_{i-1,j}u_{i,j}\big|_{-K}^{K} \\
&\quad + \int_{-K}^{K}q_{i,j+1}\dot{u}_{i,j}u_{i,j+1}dt - \int_{-K}^{K}q_{i,j}u_{i,j}\dot{u}_{i,j-1}dt + q_{i,j}u_{i,j-1}u_{i,j}\big|_{-K}^{K} \\
&\quad - \frac{1}{2}d_{i,j}u_{i,j}^2\big|_{-K}^{K} + \int_{-K}^{K}\dot{u}_{i,j}f\left(u_{i,j}\right)dt \quad \text{for } 1 \leq i, j \leq N.
\end{aligned} \tag{2.1}$$

Then $1 \leq \forall j \leq N$,

$$\begin{aligned}
\sum_{i=1}^{N}\int_{-K}^{K}\left[\dot{u}_{i,j}(t)\right]^2 dt &= \int_{-K}^{K}p_{N+1,j}\dot{u}_{N,j}u_{N+1,j}dt - \int_{-K}^{K}p_{1,j}\dot{u}_{0,j}u_{1,j}dt \\
&\quad + \sum_{i=1}^{N}p_{i,j}u_{i-1,j}u_{i,j}\big|_{-K}^{K} + \sum_{i=1}^{N}\int_{-K}^{K}q_{i,j+1}\dot{u}_{i,j}u_{i,j+1}dt + \sum_{i=1}^{N}q_{i,j}u_{i,j-1}u_{i,j}\big|_{-K}^{K} \\
&\quad - \sum_{i=1}^{N}\int_{-K}^{K}q_{i,j}u_{i,j}\dot{u}_{i,j-1}dt - \sum_{i=1}^{N}\frac{1}{2}d_{i,j}u_{i,j}^2\big|_{-K}^{K} + \sum_{i=1}^{N}\int_{-K}^{K}\dot{u}_{i,j}f\left(u_{i,j}\right)dt.
\end{aligned} \tag{2.2}$$

Therefore,

$$\begin{aligned}
\sum_{j=1}^{N}\sum_{i=1}^{N}\int_{-K}^{K}\left[\dot{u}_{i,j}(t)\right]^2dt &= \sum_{j=1}^{N}\mathrm{p}_{1,j}\left\{\int_{-K}^{K}\dot{u}_{N,j}u_{N+1,j}dt-\int_{-K}^{K}\dot{u}_{0,j}u_{1,j}dt\right\}\\
&\quad+\sum_{i=1}^{N}q_{i,1}\left\{\int_{-K}^{K}\dot{u}_{i,N}u_{i,N+1}dt-\int_{-K}^{K}\dot{u}_{i,0}u_{i,1}dt\right\}\\
&\quad+\sum_{j=1}^{N}\sum_{i=1}^{N}\left\{p_{i,j}u_{i-1,j}u_{i,j}|_{-K}^{K}+q_{i,j}u_{i,j-1}u_{i,j}|_{-K}^{K}-\frac{1}{2}d_{i,j}u_{i,j}^{2}|_{-K}^{K}\right\}\\
&\quad+\sum_{j=1}^{N}\sum_{i=1}^{N}\int_{-K}^{K}\dot{u}_{i,j}f(u_{i,j})dt.
\end{aligned} \tag{2.3}$$

Sending K to ∞, using (1.5) and Lemma 2.1, it follows that

$$\sum_{j=1}^{N}\sum_{i=1}^{N}\int_{-\infty}^{\infty}\left[\dot{u}_{i,j}(t)\right]^2dt=-\operatorname{sgn}(c)N^2\int_0^1 f(s)ds. \tag{2.4}$$

Hence, the lemma follows. □

As a simple consequence of Lemma 2.2, we have $c = 0$ if f is of balanced type, that is, $\int_0^1 f(s)ds = 0$. Notice that we cannot guarantee the speed c is zero or not by using the method developed in [16]. In fact, for the 1D lattice case, the classical work of Keener [21] indicates that the propagation failure (i.e., $c = 0$) occurs when the diffusion coefficient is sufficiently small, even when f is of unbalanced type. A similar result for 1D periodic case can be found in [22]. For our model, the problem for the propagation failure is still open.

Set $\mathbf{u} := \{u_{i,j}\}_{i,j\in\mathbb{Z}}$. Define $\mathcal{N}_{i,j}\mathbf{u}(t) := \dot{u}_{i,j}(t) - D_2[u_{i,j}](t) - f(u_{i,j}(t))$. Then we have the following comparison principle.

Lemma 2.3. *Assume that $t_0 \in \mathbb{R}$, $i_0, j_0 \in \mathbb{Z}\cup\{\infty\}$ and $c \in \mathbb{R}$. Suppose that $\mathbf{u}:= \{u_{i,j}(t)\}_{i,j\in\mathbb{Z}}$ and $\mathbf{v} := \{v_{i,j}(t)\}_{i,j\in\mathbb{Z}}$ are bounded and continuous on*

$$\left\{(i,j,t)\in\mathbb{Z}^2\times\mathbb{R}\mid t\geq t_0,\ ri+sj+ct\leq ri_0+sj_0+1\right\}, \tag{2.5}$$

such that

$$\begin{gathered}
\mathcal{N}_{i,j}\mathbf{u}\geq\mathcal{N}_{i,j}\mathbf{v}\quad\forall t>t_0,\ ri+sj+ct\leq ri_0+sj_0,\\
u_{i,j}(t_0)\geq v_{i,j}(t_0)\quad\forall ri+sj+ct_0<ri_0+sj_0,\\
u_{i,j}(t)\geq v_{i,j}(t)\quad\forall t\geq t_0,\ ri_0+sj_0\leq ri+sj+ct\leq ri_0+sj_0+1,
\end{gathered} \tag{2.6}$$

then $u_{i,j}(t) \geq v_{i,j}(t)$ *for all* $t > t_0$, $ri + sj + ct \leq ri_0 + sj_0$. *Moreover, if there exists some* (i_1, j_1) *with* $ri_1 + sj_1 + ct_0 < ri_0 + sj_0$ *such that* $u_{i_1,j_1}(t_0) > v_{i_1,j_1}(t_0)$, *then* $u_{i,j}(t) > v_{i,j}(t)$ *for all* $t > t_0$, $ri + sj + ct < ri_0 + sj_0$.

Since the proof is quite similar to the one given in [16, Lemma 1], we safely omit it here.

3. Uniqueness

In this section, we will study the uniqueness of traveling waves of (1.1)–(1.6). Firstly, applying a method of Fife-McLeod [20], we can derive the following result.

Lemma 3.1. *Suppose that* $\mathbf{u} := \{u_{i,j}(t)\}_{i,j\in\mathbb{Z}}$ *is a solution of* (1.1) *for* $t > 0$, *such that*

$$\limsup_{ri+sj\to-\infty} u_{i,j}(0) < a < \liminf_{ri+sj\to\infty} u_{i,j}(0) \tag{3.1}$$

and $0 < u_{i,j}(t) < 1$ *for all* $t \geq 0$. *If* $U^c_{i,j}(t)$ *is a traveling wave solution with speed* $c \neq 0$, *then there exist a sufficiently large positive integer* i_0 *and* j_0, *depending on the initial value, and positive numbers* β_0 *(depending on the value of* $f'(s)$ *near* $s = 0$ *and* $s = 1$*),* $\delta_0 = \delta_0(\beta_0, l, f, U^c) > 0$, *such that for all* $i, j \in \mathbb{Z}$, $t > 0$,

$$U^c_{i-i_0N,\, j-j_0N}\left(t - \delta e^{-\beta t}\right) - le^{-\beta t} \leq u_{i,j}(t) \leq U^c_{i+i_0N,\, j+j_0N}\left(t + \delta e^{-\beta t}\right) + le^{-\beta t} \tag{3.2}$$

if $\beta \in (0, \beta_0]$, $\delta \geq \delta_0$,

$$l \in \left[\max\left\{\limsup_{ri+sj\to-\infty} u_{i,j}(0), 1 - \liminf_{ri+sj\to\infty} u_{i,j}(0)\right\}, \min\{a - \epsilon, 1 - a - \epsilon\}\right], \tag{3.3}$$

for some small $\epsilon > 0$.

Proof. We will only consider the case when $c > 0$. In this case, we have $\dot{U}^c_{i,j}(t) < 0$ for all $t \in \mathbb{R}$.

First, we let

$$\Phi(0, p) = \begin{cases} [f(0) - f(p)]/p, & p > 0, \\ -f'(0), & p = 0. \end{cases} \tag{3.4}$$

Clearly, $\Phi(0, p)$ is continuous. Fixing ϵ with

$$0 < \epsilon < \min\left\{a - \limsup_{ri+sj\to-\infty} u_{i,j}(0), \liminf_{ri+sj\to\infty} u_{i,j}(0) - a\right\}, \tag{3.5}$$

there exists $\mu_1 > 0$, such that $[f(0)-f(p)]/p \geq 2\mu_1$ for all $0 < p \leq a-\epsilon$. Since $f(s)$ is continuous, we would find $\Delta_1 > 0$, such that $(f(u) - f(u + p))/p \geq \mu_1$ for all $0 \leq u \leq \Delta_1$, $0 < p \leq a - \epsilon$. By the same reasoning, there exist $\mu_2 > 0$ and $\Delta_2 > 0$, such that $(f(u) - f(u+p))/p \geq \mu_2$ for all $p \leq a - \epsilon$, $1 - \Delta_2 \leq u \leq 1$.

Define $\tilde{\mathbf{u}} := \{\tilde{u}_{i,j}\}_{i,j\in\mathbb{Z}}$, where

$$\tilde{u}_{i,j}(t) := \min\left\{1, U^c_{i+\tilde{i}_0 N, j+\tilde{j}_0 N}\left(t + \tilde{\delta}e^{-\tilde{\beta}t}\right) + \tilde{l}e^{-\tilde{\beta}t}\right\}. \tag{3.6}$$

Choose $\tilde{l}$ satisfying $\limsup_{ri+sj\to-\infty} u_{i,j}(0) < \tilde{l} < a - \epsilon < a$, and let i_0, j_0 be determined later. We claim that

$$u_{i,j}(0) \leq U^c_{i+\tilde{i}_0 N, j+\tilde{j}_0 N}\left(\tilde{\delta}\right) + \tilde{l}, \quad \forall i, j \in \mathbb{Z}, \tag{3.7}$$

for some $\tilde{i}_0$ and $\tilde{j}_0$. Since $\limsup_{ri+sj\to-\infty} u_{i,j}(0) < \tilde{l}$, there exists m, such that

$$u_{i,j}(0) < \tilde{l}, \quad \forall ri + sj \leq m. \tag{3.8}$$

Moreover, since $\lim_{ri+sj\to\infty} U_{i,j}(0) = 1$, there exist $\tilde{i}_0$ and $\tilde{j}_0$, such that

$$u_{i,j}(0) \leq U^c_{i+\tilde{i}_0 N, j+\tilde{j}_0 N}\left(\tilde{\delta}\right) + \tilde{l}, \quad \forall ri + sj \geq m. \tag{3.9}$$

Combining (3.8) and (3.9), we have proved the claim (3.7).

Now, we prove $\mathcal{N}_{i,j}\tilde{\mathbf{u}} \geq 0$. If $\tilde{u}_{i,j}(t) = U^c_{i+\tilde{i}_0 N, j+\tilde{j}_0 N}(t + \tilde{\delta}e^{-\tilde{\beta}t}) + \tilde{l}e^{-\tilde{\beta}t}$, then

$$\begin{aligned}
\mathcal{N}_{i,j}\tilde{\mathbf{u}} &= \left(1 - \tilde{\delta}\tilde{\beta}e^{-\tilde{\beta}t}\right)\left(U^c_{i+\tilde{i}_0 N, j+\tilde{j}_0 N}\right)'\left(t + \tilde{\delta}e^{-\tilde{\beta}t}\right) - \tilde{l}\tilde{\beta}e^{-\tilde{\beta}t} \\
&\quad - D_2\left[U^c_{i+\tilde{i}_0 N, j+\tilde{j}_0 N}\left(t + \tilde{\delta}e^{-\tilde{\beta}t}\right) + \tilde{l}e^{-\tilde{\beta}t}\right] - f\left(U^c_{i+\tilde{i}_0 N, j+\tilde{j}_0 N}\left(t + \tilde{\delta}e^{-\tilde{\beta}t}\right) + \tilde{l}e^{-\tilde{\beta}t}\right) \\
&= -\tilde{\delta}\tilde{\beta}e^{-\tilde{\beta}t}\left(U^c_{i+\tilde{i}_0 N, j+\tilde{j}_0 N}\right)'\left(t + \tilde{\delta}e^{-\tilde{\beta}t}\right) - \tilde{l}\tilde{\beta}e^{-\tilde{\beta}t} \\
&\quad + f\left(U^c_{i+\tilde{i}_0 N, j+\tilde{j}_0 N}\left(t + \tilde{\delta}e^{-\tilde{\beta}t}\right)\right) - f\left(U^c_{i+\tilde{i}_0 N, j+\tilde{j}_0 N}\left(t + \tilde{\delta}e^{-\tilde{\beta}t}\right) + \tilde{l}e^{-\tilde{\beta}t}\right).
\end{aligned} \tag{3.10}$$

Divide into three cases.

Case 1. $0 \leq U^c_{i,j} \leq \Delta := \min\{\Delta_1, \Delta_2\}$. From the above discussion, since $\tilde{l}e^{-\tilde{\beta}t} \leq \tilde{l} < a - \epsilon$, we have

$$f\left(U^c_{i,j}\right) - f\left(U^c_{i,j} + \tilde{l}e^{-\tilde{\beta}t}\right) \geq \mu_1\tilde{l}e^{-\tilde{\beta}t}. \tag{3.11}$$

Since $\dot{U}^c_{i,j} < 0$, $\mathcal{N}_{i,j}\tilde{\mathbf{u}} \geq -\tilde{l}\tilde{\beta}e^{-\tilde{\beta}t} + \mu_1\tilde{l}e^{-\tilde{\beta}t} = (\mu_1 - \tilde{\beta})\tilde{l}e^{-\tilde{\beta}t}$. Choosing $\tilde{\beta} \leq \mu_1$, we have $\mathcal{N}_{i,j}\tilde{\mathbf{u}} \geq 0$ in this case.

Case 2. $1-\Delta \leq U_{i,j}^{c} \leq 1$. As in Case 1, we have $\mu_2 > 0$ such that $\mathcal{N}_{i,j}\tilde{\mathbf{u}} \geq -\tilde{l}\tilde{\beta}e^{-\tilde{\beta}t} + \mu_2\tilde{l}e^{-\tilde{\beta}t} = (\mu_2 - \tilde{\beta})\tilde{l}e^{-\tilde{\beta}t} \geq 0$ if $\tilde{\beta} \leq \mu_2$.

Case 3. $\Delta \leq U_{i,j}^{c} \leq 1-\Delta$. If $\sigma := \min_{\Delta \leq U_{i,j}^{c} \leq 1-\Delta}|\dot{U}_{i,j}^{c}| > 0$ and $k := \max_{s\in[0,1]}|f'(s)| > 0$, then $\mathcal{N}_{i,j}\tilde{\mathbf{u}} \geq \tilde{\delta}\tilde{\beta}\sigma e^{-\tilde{\beta}t} - \tilde{l}\tilde{\beta}e^{-\tilde{\beta}t} - k\tilde{l}e^{-\tilde{\beta}t} = (\tilde{\delta}\tilde{\beta}\sigma - \tilde{l}\tilde{\beta} - k\tilde{l})e^{-\tilde{\beta}t}$. Choosing $\tilde{\delta} \geq \tilde{l}(\tilde{\beta}+k)/\tilde{\beta}\sigma$, we have $\mathcal{N}_{i,j}\tilde{\mathbf{u}} \geq 0$ in this case.

Then $\mathcal{N}_{i,j}\tilde{\mathbf{u}} \geq 0$ for all cases. Hence, the second inequality of (3.2) follows from a comparison principle. By the same way, we have the first inequality of (3.2). This proves the lemma. □

Note that we have the following different type of super- and sub-solutions which can be verified by a similar way as that of Lemma 3.1.

Lemma 3.2. *Suppose that* $\mathbf{w} := \{w_{i,j}(t)\}_{i,j\in\mathbb{Z}}$ *is a solution of* (1.1)–(1.6). *For any* $t_0 \in R$, $\tilde{l} \in (0, l_0)$, $l_0 := \min\{a-\epsilon, 1-a-\epsilon\}$, ϵ *is a small number, and* $\sigma_1 \geq \sigma_0 = \sigma_0(\beta, f, w_{i,j}^{c}) > 0$, *and let* $\mathbf{w}^{\pm} = \{w_{i,j}^{\pm}(t)\}_{i,j\in\mathbb{Z}}$, *where*

$$w_{i,j}^{\pm}(t) := w_{i,j}\left(t + t_0 \mp \sigma_1\tilde{l}\left(1-e^{-\beta t}\right)\right) \pm \tilde{l}e^{-\beta t}, \tag{3.12}$$

then $\pm\mathcal{N}_{i,j}\mathbf{w}^{\pm} \geq 0$.

We now prove the following uniqueness result.

Theorem 3.3. *Suppose that* $\{U_{i,j}^{c}\}$, $\{\overline{U}_{i,j}^{\overline{c}}\}$ *are two traveling wave solutions of* (1.1)–(1.6) *with* c, $\overline{c} \neq 0$, *then one has* $c = \overline{c}$ *and* $U_{i,j}^{c}(\xi) = \overline{U}_{i,j}^{\overline{c}}(\xi + \xi^*)$ *for some* $\xi^* \in \mathbb{R}$.

Proof. As before, we only consider the case when both c and $\overline{c}$ are positive. Since

$$\lim_{ri+sj\to-\infty} U_{i,j}(0) = 0 < a < \lim_{ri+sj\to\infty} U_{i,j}(0) = 1, \tag{3.13}$$

by Lemma 3.1, we have

$$U_{i-i_0N,\, j-j_0N}^{c}\left(t - \delta e^{-\beta t}\right) - le^{-\beta t} \leq \overline{U}_{i,j}^{\overline{c}}(t) \leq U_{i+i_0N,\, j+j_0N}^{c}\left(t + \delta e^{-\beta t}\right) + le^{-\beta t}, \tag{3.14}$$

for all $t > 0$, $i, j \in \mathbb{Z}$ with the constants i_0, j_0, l, β, δ defined in Lemma 3.1.

Let $I_k = I + kN$, $J_m = J + mN$, and $I, J \in \{1, \ldots, N\}$, $k, m \in \mathbb{Z}$. Take $(i, j) = (I_k, J_m)$. We get for all $t > 0$, $k, m \in \mathbb{Z}$,

$$\begin{aligned} &U_{I+kN-i_0N,\, J+mN-j_0N}^{c}\left(t - \delta e^{-\beta t}\right) - le^{-\beta t} \leq \overline{U}_{I+kN,\, J+mN}^{\overline{c}}(t) \\ &\quad \leq U_{I+kN+i_0N,\, J+mN+j_0N}^{c}\left(t + \delta e^{-\beta t}\right) + le^{-\beta t}, \quad \forall I, J. \end{aligned} \tag{3.15}$$

By the property (1.5) for all $t > 0$, $k, m \in \mathbb{Z}$,

$$\begin{aligned} U^{c}_{I-i_0N,\,J-j_0N}\left(t - \frac{(rk+sm)N}{c} - \delta e^{-\beta t}\right) - le^{-\beta t} &\le \overline{U}^{\overline{c}}_{I,J}\left(t - \frac{(rk+sm)N}{\overline{c}}\right) \\ &\le U^{c}_{I+i_0N,\,J+j_0N}\left(t - \frac{(rk+sm)N}{c} + \delta e^{-\beta t}\right) + le^{-\beta t}, \quad \forall I, J. \end{aligned} \tag{3.16}$$

Setting the moving coordinate

$$\xi := t - \frac{(rk+sm)N}{\overline{c}}, \tag{3.17}$$

we have for any $\xi \in \mathbb{R}$, for all I, J,

$$\begin{aligned} &U^{c}_{I-i_0N,J-j_0N}\left(\xi + (rk+sm)\left(\frac{N}{\overline{c}} - \frac{N}{c}\right) - \delta e^{-\beta t}\right) - le^{-\beta t} \\ &\quad\le \overline{U}^{\overline{c}}_{I,J}(\xi) \\ &\quad\le U^{c}_{I+i_0N,J+j_0N}\left(\xi + (rk+sm)\left(\frac{N}{\overline{c}} - \frac{N}{c}\right) + \delta e^{-\beta t}\right) + le^{-\beta t}. \end{aligned} \tag{3.18}$$

Suppose that $c \neq \overline{c}$. We may assume that $c < \overline{c}$. Fixing ξ and sending $t \to \infty$, this leads that either $\overline{U}^{\overline{c}}_{I,J}(\xi) \equiv 0$ or $\overline{U}^{\overline{c}}_{I,J}(\xi) \equiv 1$, which is a contradiction. Hence, $c = \overline{c}$.

We now suppress the dependence of c, and we obtain

$$U_{I-i_0N,\,J-j_0N}(\xi) \le \overline{U}_{I,J}(\xi) \le U_{I+i_0N,\,J+j_0N}(\xi), \quad \forall \xi \in \mathbb{R},\ \forall I, J. \tag{3.19}$$

For $\xi_0 := (ri_0 + sj_0)N/c$, we have

$$U_{I,J}(\xi + \xi_0) \le \overline{U}_{I,J}(\xi) \le U_{I,J}(\xi - \xi_0), \quad \forall \xi \in \mathbb{R},\ \forall I, J. \tag{3.20}$$

Define

$$\begin{aligned} \xi_* &:= \inf\left\{\tilde{\xi} \mid U_{I,J}\left(\xi + \tilde{\xi}\right) \le \overline{U}_{I,J}(\xi),\ \forall \xi \in \mathbb{R},\ \forall I, J\right\}, \\ \xi^* &:= \sup\left\{\tilde{\xi} \mid \overline{U}_{I,J}(\xi) \le U_{I,J}\left(\xi + \tilde{\xi}\right),\ \forall \xi \in \mathbb{R},\ \forall I, J\right\}. \end{aligned} \tag{3.21}$$

Since $\dot{U}_{i,j}(t) < 0$ due to $c > 0$, we have $\xi^* \le \xi_*$. Assume that $\xi^* < \xi_*$. By the strong comparison principle, we know that

$$\overline{U}_{I,J}(\xi) < U_{I,J}(\xi + \xi^*), \quad \forall \xi \in \mathbb{R},\ \forall I, J. \tag{3.22}$$

Since $\lim_{|\xi|\to\infty}\dot{U}_{I,J}(\xi)=0$, there exists $M>0$ such that

$$2\sigma_1\left|\dot{U}_{I,J}(\xi)\right|\le 1,\quad \forall|\xi\,|\ge M,\ \forall I,J, \tag{3.23}$$

where σ_1 is the number mentioned in Lemma 3.2. If $|\xi+\xi^*|\le M+1$, by the continuity of $U_{I,J}$, we would find $h\in(0,\eta)$, $\eta:=\min\{1/(2\sigma_1),l_0\}$, such that

$$\overline{U}_{I,J}(\xi)<U_{I,J}(\xi+\xi^*+2\sigma_1 h),\quad \forall I,J. \tag{3.24}$$

If $|\xi+\xi^*|\ge M+1$, then

$$\begin{aligned} U_{I,J}(\xi+\xi^*+2\sigma_1 h)-\overline{U}_{I,J}(\xi) &> U_{I,J}(\xi+\xi^*+2\sigma_1 h)-U_{I,J}(\xi+\xi^*)\\ &=\dot{U}_{I,J}(\xi+\xi^*+\theta 2\sigma_1 h)(2\sigma_1 h),\quad \theta\in(0,1)\\ &\ge -h,\quad \forall I,J. \end{aligned} \tag{3.25}$$

Combining (3.24) and (3.25), we get

$$\overline{U}_{I,J}(\xi)\le U_{I,J}(\xi+\xi^*+2\sigma_1 h)+h,\quad \forall\xi\in\mathbb{R},\forall I,J. \tag{3.26}$$

Hence,

$$\overline{U}_{I,J}(0)\le U_{I,J}(\xi^*+2\sigma_1 h)+h,\quad \forall I,J. \tag{3.27}$$

By Lemma 3.2 and the comparison principle, we have

$$\overline{U}_{I,J}(\xi)\le U_{I,J}\left(\xi+\xi^*+2\sigma_1 h-\sigma_1 h\left(1-e^{-\beta t}\right)\right)+he^{-\beta t},\quad \forall\xi\in\mathbb{R},\ \forall I,J. \tag{3.28}$$

Fixing ξ and sending $t\to\infty$,

$$\overline{U}_{I,J}(\xi)\le U_{I,J}(\xi+\xi^*+\sigma_1 h). \tag{3.29}$$

This contradicts with the definition of ξ^*. Hence, $U_{i,j}(\xi)=\overline{U}_{i,j}(\xi+\xi^*)$. □

Hence, we obtain the uniqueness (up to translations) of the traveling wave solution with nonzero speed.

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