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Research Article

Exact Solutions of ϕ^4 Equation Using Lie Symmetry Approach along with the Simplest Equation and Exp-Function Methods

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This paper obtains the exact solutions of the ϕ^4 equation. The Lie symmetry approach along with the simplest equation method and the Exp-function method are used to obtain these solutions. As a simplest equation we have used the equation of Riccati in the simplest equation method. Exact solutions obtained are travelling wave solutions.

1. Introduction

The research area of nonlinear equations has been very active for the past few decades. There are several kinds of nonlinear equations that appear in various areas of physics and mathematical sciences. Much effort has been made on the construction of exact solutions of nonlinear equations as they play an important role in many scientific areas, such as, in the study of nonlinear physical phenomena [1, 2]. Nonlinear wave phenomena appear in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fiber, biology, oceanology [3], solid state physics, chemical physics, and geometry. In recent years, many powerful and efficient methods to find analytic solutions of nonlinear equation have drawn a lot of interest by a diverse group of scientists. These methods include, the tanh-function method, the extended tanh-function method [2, 4, 5], the sine-cosine method [6], and the (G'/G)-expansion method [7, 8].

In this paper, we study the φ^4 equation, namely,

$$\phi_{tt} - \phi_{xx} - \phi + \phi^3 = 0. \tag{1.1}$$

The purpose of this paper is to use the Lie symmetry method along with the simplest equation method (SEM) and the Exp-function method to obtain exact solutions of the φ^4 equation. The simplest equation method was developed by Kudryashov [9–12] on the basis of a procedure analogous to the first step of the test for the Painlevé property. The Exp-function method is a very powerful method for solving nonlinear equations. This method was introduced by He and Wu [13] and since its appearance in the literature it has been applied by many researchers for solving nonlinear partial differential equations. See for example, [14, 15].

The outline of this paper is as follows. In Section 2 we discuss the methodology of Lie symmetry analysis and obtain the Lie point symmetries of the φ^4 equation. We then use the translation symmetries to reduce this equation to an ordinary differential equation (ODE). In Section 3 we describe the SEM and then we obtain the exact solutions of the reduced ODE using SEM. In Section 4 we explain the basic idea of the Exp-function method and obtain exact solutions of the reduced ODE using the Exp-function method. Concluding remarks are summarized in Section 5.

2. Lie Symmetry Analysis

We recall that a Lie point symmetry of a partial differential equation (PDE) is an invertible transformation of the independent and dependent variables that keep the equation invariant. In general determining all the symmetries of a partial differential equation is a daunting task. However, Sophus Lie (1842–1899) noticed that if we confine ourselves to symmetries that depend continuously on a small parameter and that form a group (continuous one-parameter group of transformations), one can linearize the symmetry condition and end up with an algorithm for calculating continuous symmetries [16–19].

The symmetry group of (1.1) will be generated by the vector field of the form

$$X = \tau(t,x,\phi)\frac{\partial}{\partial t} + \xi(t,x,\phi)\frac{\partial}{\partial x} + \eta(t,x,\phi)\frac{\partial}{\partial \phi}. \tag{2.1}$$

Applying the second prolongation $X^{[2]}$ to (1.1) we obtain

$$X^{[2]}\left(\phi_{tt} - \phi_{xx} - \phi + \phi^3\right)\Big|_{(1.1)} = 0, \tag{2.2}$$

where

$$\begin{gathered}
X^{[2]} = X + \zeta_1\frac{\partial}{\partial \phi_t} + \zeta_2\frac{\partial}{\partial \phi_x} + \zeta_{11}\frac{\partial}{\partial \phi_{tt}} + \zeta_{12}\frac{\partial}{\partial \phi_{tx}} + \zeta_{22}\frac{\partial}{\partial \phi_{xx}}, \\
\zeta_1 = D_t(\eta) - \phi_t D_t(\tau) - \phi_x D_t(\xi), \\
\zeta_2 = D_x(\eta) - \phi_t D_x(\tau) - \phi_x D_x(\xi), \\
\zeta_{11} = D_t(\zeta_1) - \phi_{tt} D_t(\tau) - \phi_{tx} D_t(\xi),
\end{gathered}$$

$$
\begin{aligned}
\zeta_{12} &= D_x(\zeta_1) - \phi_{tt} D_x(\tau) - \phi_{tx} D_x(\xi), \\
\zeta_{22} &= D_x(\zeta_2) - \phi_{tx} D_t(\tau) - \phi_{xx} D_t(\xi), \\
D_t &= \frac{\partial}{\partial t} + \phi_t \frac{\partial}{\partial \phi} + \phi_{tx} \frac{\partial}{\partial \phi_x} + \phi_{tt} \frac{\partial}{\partial \phi_t} + \cdots, \\
D_x &= \frac{\partial}{\partial x} + \phi_x \frac{\partial}{\partial \phi} + \phi_{xx} \frac{\partial}{\partial \phi_x} + \phi_{tx} \frac{\partial}{\partial \phi_t} + \cdots .
\end{aligned}
\tag{2.3}
$$

Expanding the (2.2) we obtain the following overdetermined system of linear partial differential equations:

$$
\begin{gathered}
\eta - \eta_{tt} + \eta_{xx} = 0, \qquad \eta_u - 2\tau_t = 0, \qquad 2\eta_{tu} - \tau_{tt} + \tau_{xx} = 0, \qquad \tau_t - \xi_x = 0, \\
\eta_{uu} - 2\tau_{tu} = 0, \qquad \tau_u = 0, \qquad \xi_{tt} + 2\eta_{xu} - \xi_{xx} = 0, \qquad \tau_{uu} = 0, \qquad \xi_t - \tau_x = 0, \\
\xi_{tu} - \tau_{xu} = 0, \\
\xi_{uu} = 0, \\
\xi_u = 0, \\
\eta_{uu} - 2\xi_{xu} = 0.
\end{gathered}
\tag{2.4}
$$

Solving the above system we obtain the following infinitesimal generators:

$$
X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}. \tag{2.5}
$$

We now use a linear combination of the translation symmetries X_1 and X_2, namely, $X = X_1 + cX_2$ and reduce (1.1) to an ordinary differential equation. The symmetry X yields the following two invariants:

$$
\chi = x - ct, \qquad u = \phi, \tag{2.6}
$$

which gives a group invariant solution $u = u(\chi)$ and consequently using these invariants (1.1) is transformed into the second-order nonlinear ODE

$$
\left(c^2 - 1\right)u'' - u + u^3 = 0. \tag{2.7}
$$

3. Solution of (2.7) Using the Simplest Equation Method

We now use the simplest equation method to solve (2.7). The simplest equation that will be used is the Ricatti equation

$$
G'(\chi) = b\,G(\chi) + d\,G(\chi)^2, \tag{3.1}
$$

where b and d are arbitrary constants. This equation is a well-known nonlinear ordinary differential equation which possess exact solutions given by elementary functions. The solutions can be expressed as

$$G(\chi) = \frac{b\exp[b(\chi + C)]}{1 - d\exp[b(\chi + C)]}, \tag{3.2}$$

for the case when $d < 0$, $b > 0$, and

$$G(\chi) = -\frac{b\exp[b(\chi + C)]}{1 + d\exp[b(\chi + C)]}, \tag{3.3}$$

for $d > 0$, $b < 0$. Here C is a constant of integration.

Let us consider the solution of (2.7) of the form

$$u(\chi) = \sum_{i=0}^{M} A_i (G(\chi))^i, \tag{3.4}$$

where $G(\chi)$ satisfies the Riccati equation (3.1), M is a positive integer that can be determined by balancing procedure, and $A_0, A_1, A_2, \ldots, A_M$ are parameters to be determined.

The balancing procedure yields $M = 1$, so the solution of (2.7) is of form

$$u(\chi) = A_0 + A_1 G(\chi). \tag{3.5}$$

3.1. Solution of (2.7) ***When*** $d < 0$ ***and*** $b > 0$

Substituting (3.5) into (2.7) and making use of the Ricatti equation (3.1) and then equating all coefficients of the functions G^i to zero, we obtain an algebraic system of equations in terms of A_0 and A_1. Solving these algebraic equations, with the aid of Mathematica, we obtain the following values of A_0 and A_1.

Case 1. $A_0 = -1$, $A_1 = -bd + bc^2 d$, $b = \pm\sqrt{2}/\sqrt{1 - c^2}$, $1 - c^2 \neq 0$.

Case 2. $A_0 = 1$, $A_1 = bd - bc^2 d$, $b = \pm\sqrt{2}/\sqrt{1 - c^2}$, $1 - c^2 \neq 0$.

Therefore, when $d < 0$, $b > 0$ the solution of (2.7) and hence the solution of (1.1) for Case 1 is given by

$$\phi_1(x,t) = -1 + \frac{b^2 d(c^2 - 1)\exp[b(x - ct + C)]}{1 - d\exp[b(x - ct + C)]}, \tag{3.6}$$

and the solution of (1.1) for Case 2 is given by

$$\phi_2(x,t) = 1 - \frac{b^2 d(c^2 - 1)\exp[b(x - ct + C)]}{1 - d\exp[b(x - ct + C)]}. \tag{3.7}$$

3.2. Solution of (2.7) *When* $d > 0$ *and* $b < 0$

If $d > 0$, $b < 0$, substituting (3.5) into (2.7) and making use of (3.1) and then proceeding as above, we obtain the following values of A_0 and A_1.

Case 3. $A_0 = -1$, $A_1 = -bd + bc^2 d$, $b = \pm\sqrt{2}/\sqrt{1-c^2}$, $1 - c^2 \neq 0$.

Case 4. $A_0 = 1$, $A_1 = bd - bc^2 d$, $b = \pm\sqrt{2}/\sqrt{1-c^2}$, $1 - c^2 \neq 0$.

Therefore, when $d > 0$, $c < 0$ the solution of (2.7) and hence the solution of (1.1) for Case 3 is given by

$$\phi_3(x,t) = -1 - \frac{b^2 d(c^2 - 1)\exp[b(x - ct + C)]}{1 + d\exp[b(x - ct + C)]}, \tag{3.8}$$

and the solution of (1.1) for Case 4 is given by

$$\phi_4(x,t) = 1 + \frac{b^2 d(c^2 - 1)\exp[b(x - ct + C)]}{1 + d\exp[b(x - ct + C)]}. \tag{3.9}$$

4. Solution of (2.7) Using the Exp-Function Method

In this section we use the Exp-function method for solving (2.7). According to the Exp-function method [13–15], we consider solutions of (2.7) in the form

$$u(\chi) = \frac{\sum_{n=-b}^{d} a_n \exp(n\chi)}{\sum_{m=-p}^{q} b_m \exp(m\chi)}, \tag{4.1}$$

where b, d, p, and q are positive integers which are unknown to be further determined, a_n and b_m are unknown constants. By the balancing procedure of the Exp-function method, we obtain $p = b$ and $q = d$. Furthermore, for simplicity, we set $p = b = 1$ and $q = d = 1$, so (4.1) reduces to

$$u(\chi) = \frac{a_{-1}\exp(-\chi) + a_0 + a_1\exp(\chi)}{b_{-1}\exp(-\chi) + b_0 + b_1\exp(\chi)}. \tag{4.2}$$

Substituting (4.2) into (2.7) and by the help of Mathematica, we obtain

$$\begin{gathered} c = \pm\sqrt{2}, \quad a_{-1} = 0, \quad a_1 = 0, \\ b_{-1} = \frac{a_0^2}{8}, \quad b_0 = 0, \quad b_1 = 1, \end{gathered} \tag{4.3}$$

where a_0 is a free parameter. Substituting these results into (4.2), we obtain the exact solution

$$u(\chi) = \frac{a_0\exp(\chi)}{(a_0^2/8) + \exp(2\chi)}, \tag{4.4}$$

of (2.7). Consequently, if we choose that $a_0 = \sqrt{8}$ then this solution, in terms of the variables x and t becomes

$$\phi(x,t) = \sqrt{2}\,\mathrm{sech}\left(\sqrt{\frac{1}{c^2-1}}(x-ct)\right), \tag{4.5}$$

which is a soliton solution of our φ^4 equation (1.1).

5. Conclusion

In this paper, Lie symmetry analysis in conjunction with the simplest equation method and the Exp-function method have been successfully used to obtain exact solutions of the φ^4 equation. As a simplest equation, we have used the Riccati equation. The solutions obtained were travelling wave solutions. In particular, a soliton solution was also obtained.

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