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*Research Article*

# Bounded Approximate Identities in Ternary Banach Algebras

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Let $A$ be a ternary Banach algebra. We prove that if $A$ has a left-bounded approximating set, then $A$ has a left-bounded approximate identity. Moreover, we show that if $A$ has bounded left and right approximate identities, then $A$ has a bounded approximate identity. Hence, we prove Altman's Theorem and Dixon's Theorem for ternary Banach algebras.

## 1. Introduction

Ternary algebraic operations were considered in the 19th century by several mathematicians such as Cayley [1] who introduced the notion of cubic matrix which in turn was generalized by Kapranov et al. in [2]. The comments on physical applications of ternary structures can be found in [3–7].

A nonempty set $G$ with a ternary operation $[\cdot,\cdot,\cdot] : G \times G \times G \to G$ is called a ternary groupoid and denoted by $(G,[\cdot,\cdot,\cdot])$. The ternary groupoid $(G,[\cdot,\cdot,\cdot])$ is called a ternary semigroup if the operation $[\cdot,\cdot,\cdot]$ is associative, that is, if

$$[[x,y,z],u,v] = [x,[y,z,u],v] = [x,y,[z,u,v]] \tag{1.1}$$

holds for all $x, y, z, u, v \in G$. A ternary semigroup $(G, [\cdot,\cdot,\cdot]))$ is a ternary group if for all $a, b, c \in G$, there are $x, y, z \in G$ such that

$$[x, a, b] = [a, y, b] = [a, b, z] = c, \tag{1.2}$$

where the elements $x, y, z$ are uniquely determined (see [8]).

A ternary Banach algebra is a complex Banach space $A$, equipped with a ternary product $(x, y, z) \mapsto [x, y, z]$ of $A^3$ into $A$, which is associative in the sense that $[[x, y, z], u, v] = [x, y, [z, u, v]] = [x, [y, z, u], v]$ and satisfies $\|[x, y, z]\| \leq \|x\|\|y\|\|z\|$. An element $e \in A$ is an identity of $A$ if $x = [x, e, e] = [e, e, x]$ for all $x \in A$.

Assume that $A$ is a ternary Banach algebra a bounded net $(e_\alpha, f_\alpha)$ is a left-bounded approximate identity for $A$ if $\lim_\alpha [e_\alpha, f_\alpha, a] = a$ for all $a \in A$. Similarly, a bounded net $(e_\alpha, f_\alpha)$ is a right-bounded approximate identity for $A$ if $\lim_\alpha [a, e_\alpha, f_\alpha] = a$ for all $a \in A$. Also, $(e_\alpha, f_\alpha)$ is a middle-bounded approximate identity for $A$ if $\lim_\alpha [e_\alpha, a, f_\alpha] = a$ for all $a \in A$. A net $(e_\alpha, f_\alpha)$ is a bounded approximate identity for $A$ if $(e_\alpha, f_\alpha)$ is a left-, right-,and middle-bounded approximate identity for $A$.

For ternary Banach algebra $A$, a set $U \times V$ is said to be an approximating set for $A$ ($U$ and $V$ are bounded subsets of $A$) if for every $\epsilon > 0$, and every $a \in A$, there exist $u \in U$, $v \in V$ such that $\|[u, v, a] - a\| < \epsilon$, $\|[u, a, v] - a\| < \epsilon$, $\|[a, u, v] - a\| < \epsilon$.

Existence of bounded approximating set for binary Banach algebras guarantees existing of bounded approximate identity (Altman's Theorem [9, Proposition 2, page 58] or [10]) and also this notion generalized for commutative Fréchet algebras [11]. For normed algebra $A$ with left-bounded approximate identity and right-bounded approximate identity, Dixon [12] proved that $A$ has a bounded approximate identity [13, Proposition 2.9.3].

In this paper, we prove Altman's Theorem and Dixon's Theorem for ternary Banach algebras. By "$\circ$", we mean the quasiproduct between elements $x, y$ of binary algebra $A$ which are defined by $x \circ y = x + y - xy$.

## 2. Main Results

We start our work with the following theorem which can be regarded as a version of Altman's Theorem for ternary Banach algebras.

**Theorem 2.1.** *Let $A$ be a ternary Banach algebra and $U, V$ be bounded subsets of $A$ such that for given $a \in A$ and $\epsilon > 0$ there are $u \in U$ and $v \in V$, $\|[u, v, a] - a\| < \epsilon$. Then $A$ possess a left-bounded approximate identity.*

*Proof.* Let $\epsilon > 0$, and set

$$W = UV \circ UV = \{(u_1 v_1) \circ (u_2 v_2) : u_1, u_2 \in U,\ v_1 v_2 \in V\}. \tag{2.1}$$

For proof of theorem, it is sufficient to show that for every finite subset $F \subset A$, there exists $w = uv \circ st \in W$ such that $\|[uv \circ st, a] - a\| < \epsilon$ for every $a \in F$.

*Step 1.* Let $F = \{a\}$ be singleton. Then, there are $u \in U$ and $v \in V$ such that $\|uv\| < M$, and

$$\|[u, v, a] - a\| < \frac{\epsilon}{(M+1)}. \tag{2.2}$$

Letting $w = uv \circ uv$, then

$$\|[uv \circ uv, a] - a\| = \|[u, v, [u, v, a] - a] - ([u, v, a] - a)\| < \epsilon. \tag{2.3}$$

*Step 2.* Let $F = \{a_1, a_2\}$. There is a $(u_1, v_1) \in U \times V$ such that $\|[u_1, v_1, a_1] - a_1\| < \epsilon/(1+M)$, and for $[u_1, v_1, a_2] - a_2 \in A$ there is a $(u_2, v_2) \in U \times V$ such that

$$\|[u_2, v_2, [u_1, v_1, a_2] - a_2] - ([u_1, v_1, a_2] - a_2)\| < \epsilon. \tag{2.4}$$

Put $w_1 = u_1 v_1$ and $w_2 = u_2 v_2$. Then

$$\begin{aligned} \|[w_2 \circ w_1, a_i] - a_i\| &= \|[u_2, v_2, a_i] + [u_1, v_1, a_i] - [u_2, v_2, [u_1, v_1, a_i]] - a_i\| \\ &= \|[u_2, v_2, a_i - [u_1, v_1, a_i]] - (a_i - [u_1, v_1, a_i])\| < \epsilon, \end{aligned} \tag{2.5}$$

for $i = 1, 2$.

*Step 3.* Now, suppose that obtained results in Steps 1 and 2 are true for $i = 1, 2, \ldots, n$. Let $F = \{a_1, a_2, \ldots, a_{n+1}\}$, and let $K = \max\{\|a_i\| : i = 1, \ldots, n\}$. There exist $w_1 \circ w_2 \in W$ such that $\|a_i - [w_2 \circ w_1, a_i]\| < \epsilon/3(M+1)^2$, for $i = 1, 2, \ldots, n$, where $w_1$ and $w_2$ are defined as in Step 2. Also, we can choose $\alpha_1 = \theta_1 \eta_1$ and $\alpha_2 = \theta_2 \eta_2$ such that $\alpha_1 \circ \alpha_2 \in W$,

$$\begin{aligned} &\|[\alpha_2 \circ \alpha_1, w_2 \circ w_1] - w_2 \circ w_1\| \\ &\quad = \left\| [\theta_2, \eta_2, w_2 \circ w_1 - [\theta_1, \eta_1, w_2 \circ w_1]] - (w_2 \circ w_1 - [\theta_1, \eta_1, w_2 \circ w_1]) \right\| \\ &\quad < \frac{\epsilon}{3K}, \end{aligned} \tag{2.6}$$

and $\|[\alpha_2 \circ \alpha_1, a_{n+1}] - a_{n+1}\| < \epsilon$. Then for every $j = 1, 2, \ldots, n$ we have

$$\begin{aligned} \left\| [\alpha_2 \circ \alpha_1, a_j] - a_j \right\| &\le \left\| a_j - [w_2 \circ w_1, a_j] \right\| + \left\| [\alpha_2 \circ \alpha_1, a_j] - [\alpha_2 \circ \alpha_1, [w_2 \circ w_1, a_j]] \right\| \\ &\quad + \left\| [\alpha_2 \circ \alpha_1, [w_2 \circ w_1, a_j]] - [w_2 \circ w_1, a_j] \right\| \\ &\le \left\| a_{j-[w_2 \circ w_1, a_j]} \right\| + \|\alpha_2 \circ \alpha_1\| \left\| a_j - [w_2 \circ w_1, a_j] \right\| \\ &\quad + \|[\alpha_2 \circ \alpha_1, w_2 \circ w_1] - w_2 \circ w_1\| \left\| a_j \right\| \\ &< \epsilon. \end{aligned} \tag{2.7}$$

Let $F(A)$ be the collection of all finite subsets of $A$ and $\Lambda = \mathbb{N} \times \mathbb{N} \times F(A)$. Then $\Lambda$ is a direct set with the following partial order:

$$(n_1, m_1, F_1) \leq (n_2, m_2, F_2) \quad \text{iff } F_1 \subseteq F_2,\ n_1 \leq n_2,\ m_1 \leq m_2. \tag{2.8}$$

Now, we can choose a bounded approximate identity $(e_\lambda, f_\lambda)_{\lambda \in \Lambda}$ for $A$. □

Now, we prove Dixon's Theorem for ternary Banach algebras. Hence, we prove that if a ternary Banach algebra has both left- and right-bounded approximate identities, then it has a bounded approximate identity.

**Theorem 2.2.** *Let $A$ be a ternary Banach algebra and $(e_\alpha, f_\alpha)$ and $(e_\beta, f_\beta)$ be bounded left and right approximate identities of $A$, respectively. Then $A$ has a bounded approximate identity.*

*Proof.* Consider $(c_{\alpha,\beta}, d_{\alpha,\beta}) = (e_\alpha f_\alpha \circ e_\beta f_\beta, e_\beta f_\beta \circ e_\alpha f_\alpha)$. We claim that $(c_{\alpha,\beta}, d_{\alpha,\beta})$ is a bounded approximate identity for $A$. Boundedness of mentioned net is clear. Therefore, we have to show that $(c_{\alpha,\beta}, d_{\alpha,\beta})$ is a right, left, and middle approximate identity for $A$.

*Step 1.* $(c_{\alpha,\beta}, d_{\alpha,\beta})$ is a left approximate identity. Because

$$\begin{aligned}
&\left\| [e_\alpha f_\alpha \circ e_\beta f_\beta, e_\beta f_\beta \circ e_\alpha f_\alpha, a] - a \right\| \\
&\quad = \left\| [e_\alpha f_\alpha, e_\beta f_\beta \circ e_\alpha f_\alpha, a] + [e_\beta f_\beta, e_\beta f_\beta \circ e_\alpha f_\alpha, a] - [e_\alpha f_\alpha e_\beta f_\beta, e_\beta f_\beta \circ e_\alpha f_\alpha, a] - a \right\| \\
&\quad \leq \left\| [e_\alpha f_\alpha, e_\alpha f_\alpha, a] - a \right\| + \left\| [e_\alpha f_\alpha, e_\beta f_\beta, a] - [e_\alpha f_\alpha e_\beta f_\beta, e_\alpha f_\alpha, a] \right\| \\
&\qquad + \left\| [e_\alpha f_\alpha e_\beta f_\beta, e_\alpha f_\alpha, a] - [e_\beta f_\beta, e_\alpha f_\alpha, a] \right\| + \left\| [e_\alpha f_\alpha e_\beta f_\beta, e_\beta f_\beta, a] - [e_\beta f_\beta, e_\beta f_\beta, a] \right\| \\
&\qquad + \left\| [e_\alpha f_\alpha e_\beta f_\beta, e_\beta f_\beta e_\alpha f_\alpha, a] - [e_\beta f_\beta e_\beta f_\beta, e_\alpha f_\alpha, a] \right\| \\
&\quad \leq \left\| [e_\alpha, f_\alpha, [e_\alpha, f_\alpha, a]] - [e_\alpha, f_\alpha, a] \right\| + \left\| [e_\alpha, f_\alpha, a] - a \right\| + \| e_\alpha f_\alpha \| \| e_\beta f_\beta \| \left\| a - [e_\alpha, f_\alpha, a] \right\| \\
&\qquad + \left\| [e_\alpha, f_\alpha, [e_\beta f_\beta, e_\alpha f_\alpha, a]] - [e_\beta f_\beta, e_\alpha f_\alpha, a] \right\| \\
&\qquad + \left\| [e_\alpha, f_\alpha, [e_\beta f_\beta, e_\beta f_\beta, a]] - [e_\beta f_\beta, e_\beta f_\beta, a] \right\| \\
&\qquad + \left\| [e_\alpha, f_\alpha, [e_\beta f_\beta, e_\beta f_\beta e_\alpha f_\alpha, a]] - [e_\beta f_\beta e_\beta f_\beta, e_\alpha f_\alpha, a] \right\| \\
&\quad \leq \frac{5\epsilon}{MN+1} + MN \frac{\epsilon}{MN+1} < \epsilon,
\end{aligned} \tag{2.9}$$

where $\|e_\alpha f_\alpha\| \leq \|e_\alpha\| \|f_\alpha\| \leq M$, and $\|e_\beta f_\beta\| \leq \|e_\beta\| \|f_\beta\| \leq N$.

*Step 2.* $(c_{\alpha,\beta}, d_{\alpha,\beta})$ is a right approximate identity because

$$\begin{aligned}
&\| [a, e_\alpha f_\alpha \circ e_\beta f_\beta, e_\beta f_\beta \circ e_\alpha f_\alpha] - a\| \\
&\quad = \| [a, e_\alpha f_\alpha, e_\beta f_\beta \circ e_\alpha f_\alpha] + [a, e_\beta f_\beta, e_\beta f_\beta \circ e_\alpha f_\alpha] - [a, e_\alpha f_\alpha e_\beta f_\beta, e_\beta f_\beta \circ e_\alpha f_\alpha] - a\| \\
&\quad \leq \| [a, e_\beta f_\beta, e_\beta f_\beta] - a\| + \| [a, e_\alpha f_\alpha, e_\beta f_\beta] - [a, e_\alpha f_\alpha e_\beta f_\beta, e_\beta f_\beta] \| \\
&\qquad + \| [a, e_\alpha f_\alpha e_\beta f_\beta, e_\alpha f_\alpha] - [a, e_\alpha f_\alpha, e_\alpha f_\alpha] \| + \| [a, e_\beta f_\beta, e_\beta f_\beta e_\alpha f_\alpha] - [a, e_\beta f_\beta, e_\alpha f_\alpha] \| \\
&\qquad + \| [a, e_\alpha f_\alpha e_\beta f_\beta, e_\beta f_\beta e_\alpha f_\alpha] - [a, e_\alpha f_\alpha, e_\beta f_\beta e_\alpha f_\alpha] \| \\
&\quad \leq \| [[a, e_\beta, f_\beta], e_\beta, f_\beta] - [a, e_\beta, f_\beta] \| + \| [a, e_\beta, f_\beta] - a\| \\
&\qquad + \| [a, e_\alpha, f_\alpha] - [[a, e_\alpha, f_\alpha], e_\beta, f_\beta] \| \|e_\beta f_\beta\| + \| [[a, e_\alpha, f_\alpha], e_\beta, f_\beta] - [a, e_\alpha, f_\alpha] \| \|e_\alpha f_\alpha\| \\
&\qquad + \| [[a, e_\beta, f_\beta], e_\beta, f_\beta] - [a, e_\beta, f_\beta] \| \|e_\alpha f_\alpha\| \\
&\qquad + \| [[a, e_\alpha f_\alpha, e_\beta f_\beta], e_\beta f_\beta] - [a, e_\alpha f_\alpha, e_\beta f_\beta] \| \|e_\alpha f_\alpha\| \\
&\quad \leq \frac{2\epsilon}{MN+1} + \frac{3M\epsilon}{MN+1} + \frac{N\epsilon}{MN+1} < \epsilon.
\end{aligned} \tag{2.10}$$

*Step 3.* By the similar method, we show that the net $(c_{\alpha,\beta}, d_{\alpha,\beta})$ is a middle approximate identity:

$$\begin{aligned}
&\| [e_\alpha f_\alpha \circ e_\beta f_\beta, a, e_\beta f_\beta \circ e_\alpha f_\alpha] - a\| \\
&\quad = \| [e_\alpha f_\alpha, a, e_\beta f_\beta \circ e_\alpha f_\alpha] + [e_\beta f_\beta, a, e_\beta f_\beta \circ e_\alpha f_\alpha] - [e_\alpha f_\alpha e_\beta f_\beta, a, e_\beta f_\beta \circ e_\alpha f_\alpha] - a\| \\
&\quad \leq \| [e_\alpha f_\alpha, a, e_\beta f_\beta] - a\| + \| [e_\alpha f_\alpha, a, e_\alpha f_\alpha] - [e_\alpha f_\alpha, a, e_\beta f_\beta e_\alpha f_\alpha] \| \\
&\qquad + \| [e_\alpha f_\alpha e_\beta f_\beta, a, e_\beta f_\beta] - [e_\beta f_\beta, a, e_\beta f_\beta] \| + \| [e_\alpha f_\alpha e_\beta f_\beta, a, e_\alpha f_\alpha] - [e_\beta f_\beta, a, e_\alpha f_\alpha] \| \\
&\qquad + \| [e_\alpha f_\alpha e_\beta f_\beta, a, e_\beta f_\beta e_\alpha f_\alpha] - [e_\beta f_\beta, a, e_\beta f_\beta e_\alpha f_\alpha] \| \\
&\quad \leq \| [e_\alpha, f_\alpha, [a, e_\beta, f_\beta]] - [a, e_\beta, f_\beta] \| + \| [a, e_\beta, f_\beta] - a\| \\
&\qquad + \| [e_\alpha, f_\alpha, a] - [[e_\alpha, f_\alpha, a], e_\beta, f_\beta] \| \|e_\alpha f_\alpha\| \\
&\qquad + \| [e_\alpha, f_\alpha, [e_\beta f_\beta, a, e_\beta f_\beta]] - [e_\beta f_\beta, a, e_\beta f_\beta] \| \\
&\qquad + \| [e_\alpha, f_\alpha, [e_\beta f_\beta, a, e_\alpha f_\alpha]] - [e_\beta f_\beta, a, e_\alpha f_\alpha] \| \\
&\qquad + \| [e_\alpha, f_\alpha, [e_\beta f_\beta, a, e_\beta f_\beta e_\alpha f_\alpha]] - [e_\beta f_\beta, a, e_\beta f_\beta e_\alpha f_\alpha] \| \\
&\quad \leq \frac{5\epsilon}{MN+1} + \frac{M\epsilon}{MN+1} < \epsilon.
\end{aligned} \tag{2.11}$$

This completes the proof of theorem. □

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