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Research Article

On a Differential Equation Involving Hilfer-Hadamard Fractional Derivative

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This paper studies a fractional differential inequality involving a new fractional derivative (Hilfer-Hadamard type) with a polynomial source term. We obtain an exponent for which there does not exist any global solution for the problem. We also provide an example to show the existence of solutions in a wider space for some exponents.

1. Introduction

Fractional derivatives have proved to be very efficient and adequate to describe many phenomena with memory and hereditary processes. These phenomena are abundant in science, engineering (viscoelasticity, control, porous media, mechanics, electrical engineering, electromagnetism, etc.) as well as in geology, rheology, finance, and biology. Unlike the classical derivatives, fractional derivatives have the ability to characterize adequately processes involving a past history. We are witnessing a huge development of fractional calculus and methods in the theory of differential equations. Indeed, after the appearance of the papers by Bagley and Torvik [1–3], researchers started to deal directly with differential equations containing fractional derivatives instead of ignoring them as it used to be the case. For analytical treatments, we may refer the reader to [4–36], and for some applications, one can consult [1–3, 8, 25, 26, 26, 27, 27–31, 33, 34, 37–49] to cite but a few.

We will consider the problem:

$$\begin{gathered}\left(\mathfrak{D}_{a^+}^{\alpha,\beta}u\right)(t)=f[t,u(t)],\quad 0<\alpha<1,\ 0\le\beta\le1,\ t>a>0,\\ \left(\mathfrak{D}_{a^+}^{(\beta-1)(1-\alpha)}u\right)(a)=u_0\ge0,\end{gathered}\tag{1.1}$$

where $\mathfrak{D}_{a^+}^{\alpha,\beta}u$ is a new type of fractional derivative we will define below and u_0 is a given constant. This new fractional derivative interpolates the Hadamard fractional derivative and its Caputo counterpart [26, 34], in the same way the Hilfer fractional derivative interpolates the Riemann-Liouville fractional derivative and the Caputo fractional derivative. That is why we are naming it after Hilfer and Hadamard.

A nonexistence result for global solutions of the problem (1.1) will be proved when $f[t,u(t)] \geq (\log(t/a))^{\mu}|u(t)|^m$ for some $m > 1$ and $\mu \in \mathbf{R}$. That is we consider the Cauchy problem:

$$\begin{gathered}\left(\mathfrak{D}_{a^+}^{\alpha,\beta}u\right)(t) \geq \left(\log\frac{t}{a}\right)^{\mu}|u(t)|^m, \quad t > a > 0,\ m > 1,\ \mu \in \mathbf{R},\\ \left(\mathfrak{D}_{a^+}^{\gamma-1}u\right)(a) = u_0 \geq 0,\end{gathered} \tag{1.2}$$

where $\gamma = \alpha+\beta-\alpha\beta$ and show that no solutions can exist for all time for certain values of μ and m. Clearly, sufficient conditions for nonexistence provide necessary conditions for existence of solutions. In addition, we construct an example for which there exist solutions for some powers m and in some appropriate space.

The existence and uniqueness of solutions for problem (1.1) has been discussed in [50] in the space $C_{\delta;1-\gamma,\mu}^{\alpha,\beta}[a,b]$ defined by

$$C_{\delta;1-\gamma,\mu}^{\alpha,\beta}[a,b] = \left\{y \in C_{1-\gamma,\log}[a,b], \mathfrak{D}_{a^+}^{\alpha,\beta}y \in C_{\mu,\log}[a,b]\right\}, \tag{1.3}$$

where

$$C_{\gamma,\log}[a,b] = \left\{g : (a,b] \longrightarrow \mathbf{R} : \left(\log\frac{x}{a}\right)^{\gamma}g(x) \in C[a,b]\right\} \tag{1.4}$$

for $0 \leq \mu < 1$ and $C_{0,\log}[a,b] = C[a,b]$.

We also point out here that the case where $\mathfrak{D}_{a^+}^{\alpha,\beta}$ is the usual Riemann-Liouville fractional derivative has been studied in [26] (see also references therein). There are very few papers [26, 29] dealing with the pure Hadamard case, that is, when $\beta = 0$.

The rest of the paper is divided into three sections. In Section 2, we present some definitions, notations, and lemmas which will be needed later in our proof. Section 3 is devoted to the nonexistence result and Section 4 contains an example of existence of solutions.

2. Preliminaries

In this section, we present some background material for the forthcoming analysis. For more details, see [25, 26, 33, 42, 51, 52].

Definition 2.1. The space $X_c^p(a,b)$ $(c \in \mathbf{R}, 1 \le p \le \infty)$ consists of those real-valued Lebesgue measurable functions g on (a,b) for which $\|g\|_{X_c^p} < \infty$, where

$$\|g\|_{X_c^p} = \left(\int_a^b \left|t^c g(t)\right|^p \frac{dt}{t}\right)^{1/p}, \quad 1 \le p < \infty,\ c \in \mathbf{R},$$
$$\|g\|_{X_c^\infty} = \operatorname*{ess\,sup}_{a \le x \le b} \left|x^c g(x)\right|, \quad c \in \mathbf{R}. \tag{2.1}$$

In particular, when $c = 1/p$, we see that $X_{1/p}^p(a,b) = L_p(a,b)$.

Definition 2.2. Let $\Omega = [a,b]$ $(0 < a < b < \infty)$ be a finite interval and $0 \le \gamma < 1$, we introduce the weighted space $C_{\gamma,log}[a,b]$ of continuous functions g on $(a,b]$:

$$C_{\gamma,\log}[a,b] = \left\{g \in C(a,b] : \left(\log\frac{x}{a}\right)^\gamma g(x) \in C[a,b]\right\}. \tag{2.2}$$

In the space $C_{\gamma,\log}[a,b]$, we define the norm:

$$\|g\|_{C_{\gamma,\log}} = \left\|\left(\log\frac{x}{a}\right)^\gamma g(x)\right\|_C, \qquad \|g\|_{C_{0,\log}} = \|g\|_\infty. \tag{2.3}$$

Definition 2.3. Let $\delta = x(d/dx)$ be the δ-derivative, for $n \in \mathbf{N}$, we denote by $C_{\delta,\gamma}^n[a,b]$ $(0 \le \gamma < 1)$ the Banach space of functions g which have continuous δ-derivatives on $[a,b]$ up to order $n-1$ and have the derivative $\delta^n g$ of order n on $(a,b]$ such that $\delta^n g \in C_{\gamma,\log}[a,b]$:

$$C_{\delta,\gamma}^n[a,b] = \left\{g : (a,b] \longrightarrow \mathbf{R} : \delta^k g \in C[a,b], k = 0,\ldots,n-1, \delta^n g \in C_{\gamma,\log}[a,b]\right\} \tag{2.4}$$

with the norm:

$$\|g\|_{C_{\delta,\gamma}^n} = \sum_{k=0}^{n-1}\left\|\delta^k g\right\|_C + \|\delta^n g\|_{C_{\gamma,\log}}. \tag{2.5}$$

When $n = 0$, we set

$$C_{\delta,\gamma}^0[a,b] = C_{\gamma,\log}[a,b]. \tag{2.6}$$

Definition 2.4. Let (a,b) $(0 \le a < b \le \infty)$ be a finite or infinite interval of the half-axis $\mathbf{R}^+$ and let $\alpha > 0$. The Hadamard left-sided fractional integral $\mathcal{J}_{a^+}^\alpha f$ of order $\alpha > 0$ is defined by

$$\left(\mathcal{J}_{a^+}^\alpha f\right)(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \left(\log\frac{x}{t}\right)^{\alpha-1}\frac{f(t)dt}{t}, \quad a < x < b \tag{2.7}$$

provided that the integral exists. When $\alpha = 0$, we set

$$\mathcal{J}_{a^+}^{0} f = f. \tag{2.8}$$

Definition 2.5. Let (a,b) $(0 \le a < b \le \infty)$ be a finite or infinite interval of the half-axis $\mathbf{R}^+$ and let $\alpha > 0$. The Hadamard right-sided fractional integral $\mathcal{J}_{b^-}^{\alpha} f$ of order $\alpha > 0$ is defined by

$$\left(\mathcal{J}_{b^-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left(\log \frac{t}{x}\right)^{\alpha-1} \frac{f(t)dt}{t}, \quad a < x < b, \tag{2.9}$$

provided that the integral exists. When $\alpha = 0$, we set

$$\mathcal{J}_{b^-}^{0} f = f. \tag{2.10}$$

Definition 2.6. The left-sided Hadamard fractional derivative of order $0 \le \alpha < 1$ on (a,b) is defined by

$$\left(\mathfrak{D}_{a^+}^{\alpha} f\right)(x) := \delta\left(\mathcal{J}_{a^+}^{1-\alpha} f\right)(x), \tag{2.11}$$

that is,

$$\left(\mathfrak{D}_{a^+}^{\alpha} f\right)(x) = \left(x \frac{d}{dx}\right) \frac{1}{\Gamma(1-\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{-\alpha} \frac{f(t)dt}{t}, \quad a < x < b. \tag{2.12}$$

When $\alpha = 0$, we set

$$\mathfrak{D}_{a^+}^{0} f = f. \tag{2.13}$$

Definition 2.7. The right-sided Hadamard fractional derivative of order α $(0 \le \alpha < 1)$ on (a,b) is defined by

$$\left(\mathfrak{D}_{b^-}^{\alpha} f\right)(x) := -\delta\left(\mathcal{J}_{b^-}^{1-\alpha} f\right)(x), \tag{2.14}$$

that is,

$$\left(\mathfrak{D}_{b^-}^{\alpha} f\right)(x) = -\left(x \frac{d}{dx}\right) \frac{1}{\Gamma(1-\alpha)} \int_x^b \left(\log \frac{t}{x}\right)^{-\alpha} \frac{f(t)dt}{t}, \quad a < x < b. \tag{2.15}$$

When $\alpha = 0$, we set

$$\mathfrak{D}_{b^-}^{0} f = f. \tag{2.16}$$

Lemma 2.8. *If $\alpha > 0$, $\beta > 0$ and $0 < a < b < \infty$, then*

$$\left(\mathcal{J}_{a^+}^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(\log\frac{x}{a}\right)^{\beta+\alpha-1},$$
$$\left(\mathfrak{D}_{a^+}^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{x}{a}\right)^{\beta-\alpha-1}. \tag{2.17}$$

In particular, if $\beta = 1$, then the Hadamard fractional derivative of a constant is not equal to zero:

$$\left(\mathfrak{D}_{a^+}^{\alpha}1\right)(x) = \frac{1}{\Gamma(1-\alpha)}\left(\log\frac{x}{a}\right)^{-\alpha}, \tag{2.18}$$

when $0 < \alpha < 1$.

Lemma 2.9. *Let $0 < a < b < \infty$, $\alpha > 0$, and $0 \le \mu < 1$.*

(a) *If $\mu > \alpha > 0$, then $\mathcal{J}_{a^+}^{\alpha}$ is bounded from $C_{\mu,\log}[a,b]$ into $C_{\mu-\alpha,\log}[a,b]$. In particular, $\mathcal{J}_{a^+}^{\alpha}$ is bounded in $C_{\mu,\log}[a,b]$.*

(b) *If $\mu \le \alpha$, then $\mathcal{J}_{a^+}^{\alpha}$ is bounded from $C_{\mu,\log}[a,b]$ into $C[a,b]$. In particular, $\mathcal{J}_{a^+}^{\alpha}$ is bounded in $C_{\mu,\log}[a,b]$.*

This lemma justifies the following one

Lemma 2.10 (the semigroup property of the fractional integration operator $\mathcal{J}_{\mathrm{a}^+}^{\alpha}$). *Let $\alpha > 0$, $\beta > 0$, and $0 \le \mu < 1$. If $0 < a < b < \infty$, then, for $f \in C_{\mu,\log}[a,b]$,*

$$\mathcal{J}_{a^+}^{\alpha}\mathcal{J}_{a^+}^{\beta}f = \mathcal{J}_{a^+}^{\alpha+\beta}f \tag{2.19}$$

holds at any point $x \in (a,b]$. When $f \in C[a,b]$, this relation is valid at any point $x \in [a,b]$.

Lemma 2.11. *Let $0 \le \alpha < 1$ and $0 \le \gamma < 1$. If $f \in C_{\gamma,\log}^1[a,b]$, then the fractional derivatives $\mathfrak{D}_{a^+}^{\alpha}$ and $\mathfrak{D}_{b^-}^{\alpha}$ exist on $(a,b]$ and $[a,b)$, respectively, $(a > 0)$ and can be represented in the forms:*

$$\left(\mathfrak{D}_{a^+}^{\alpha}f\right)(x) = \frac{f(a)}{\Gamma(1-\alpha)}\left(\log\frac{x}{a}\right)^{-\alpha} + \frac{1}{\Gamma(1-\alpha)}\int_a^x\left(\log\frac{x}{t}\right)^{-\alpha}f'(t)dt,$$
$$\left(\mathfrak{D}_{b^-}^{\alpha}f\right)(x) = \frac{f(b)}{\Gamma(1-\alpha)}\left(\log\frac{b}{x}\right)^{-\alpha} - \frac{1}{\Gamma(1-\alpha)}\int_x^b\left(\log\frac{t}{x}\right)^{-\alpha}f'(t)dt, \tag{2.20}$$

respectively.

Lemma 2.12 (fractional integration by Parts). *Let $\alpha > 0$ and $1 \leq p \leq \infty$. If $\varphi \in L_p(\mathbf{R}^+)$ and $\psi \in X^q_{-1/p}$, then*

$$\int_0^\infty \varphi(x)\left(\mathcal{J}^\alpha_+\psi\right)(x)\frac{dx}{x} = \int_0^\infty \psi(x)\left(\mathcal{J}^\alpha_-\varphi\right)(x)\frac{dx}{x}, \tag{2.21}$$

where $1/p + 1/q = 1$.

Definition 2.13. The fractional derivative ${}^c\mathfrak{D}^\alpha_{a^+}f$ of order α $(0 < \alpha < 1)$ on (a,b) defined by

$${}^c\mathfrak{D}^\alpha_{a^+}f = \mathcal{J}^{1-\alpha}_{a^+}\delta f, \tag{2.22}$$

where $\delta = x(d/dx)$, is called the Hadamard-Caputo fractional derivative of order α.

Now, motivated by the Hilfer fractional derivative introduced in [41, 42], we introduce the new fractional derivative which we call Hilfer-Hadamard fractional derivative of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$:

$$\left(\mathfrak{D}^{\alpha,\beta}_{a^+}u\right)(t) = \mathcal{J}^{\beta(1-\alpha)}_{a^+}\left(t\frac{d}{dt}\right)\left(\mathcal{J}^{(1-\beta)(1-\alpha)}_{a^+}u\right)(t). \tag{2.23}$$

The Hilfer fractional derivative interpolates the Riemann-Liouville fractional derivative and the Caputo fractional derivative. This new one interpolates the Hadamard fractional derivative and its caputo counterpart. Indeed, for $\beta = 0$, we find the Hadamard fractional derivative as defined in Definition 2.6 and, for $\beta = 1$, we find its Caputo type counterpart (Definition 2.13).

Theorem 2.14 (Young's inequality). *If a and b are nonnegative real numbers and p and q are positive real numbers such that $1/p + 1/q = 1$, then one has*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{2.24}$$

Equality holds if and only if $a^p = b^q$.

3. Nonexistence Result

Before we state and prove our main result, we will start with the following lemma.

Lemma 3.1. *If $\alpha > 0$ and $f \in C[a,b]$, then*

$$\begin{aligned}\left(\mathcal{J}^\alpha_{a+}f\right)(a) &= \lim_{t\to a}\left(\mathcal{J}^\alpha_{a+}f\right)(t) = 0,\\ \left(\mathcal{J}^\alpha_{b-}f\right)(b) &= \lim_{t\to b}\left(\mathcal{J}^\alpha_{b-}f\right)(t) = 0.\end{aligned} \tag{3.1}$$

Proof. Since $f \in C[a,b]$, then on $[a,b]$ we have $|f(t)| < M$ for some positive constant M. Therefore,

$$\begin{aligned} \left|\left(\mathcal{J}_{a+}^{\alpha} f\right)(t)\right| &\leq \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s)| \frac{ds}{s} \leq \frac{M}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\ &\leq \frac{M}{\Gamma(\alpha+1)} \left(\log \frac{t}{a}\right)^{\alpha}. \end{aligned} \tag{3.2}$$

As $\alpha > 0$, we see that

$$\left(\mathcal{J}_{a+}^{\alpha} f\right)(a) = \lim_{t \to a} \left(\mathcal{J}_{a+}^{\alpha} f\right)(t) = 0. \tag{3.3}$$

In a similar manner, we prove the second part of the lemma. □

The proof of the next result is based on the test function method developed by Mitidieri and Pokhozhaev in [52].

Theorem 3.2. *Assume that $\mu \in \mathbf{R}$ and $m < (1+\mu)/(1-\gamma)$. Then, Problem* (1.2) *does not admit global nontrivial solutions in $C_{1-\gamma,\log}^{\gamma}[a,b]$, where*

$$C_{1-\gamma,\log}^{\gamma}[a,b] = \left\{ y \in C_{1-\gamma,\log}[a,b] : \mathfrak{D}_{a^+}^{\gamma} y \in C_{1-\gamma,\log}[a,b] \right\} \tag{3.4}$$

when $u_0 \geq 0$.

Proof. Assume that a nontrivial solution exists for all time $t > a$. Let $\varphi(t) \in C^1([a,\infty))$ be a test function satisfying $\varphi(t) \geq 0$, $\varphi(t)$ is non-increasing and such that

$$\varphi(t) := \begin{cases} 1, & a \leq t \leq \theta T, \\ 0, & t \geq T, \end{cases} \tag{3.5}$$

for some $T > a$ and some θ $(\theta < 1)$ such that $a < \theta T < T$. Multiplying the inequality in (1.2) by $\varphi(t)/t$ and integrating over $[a,T]$, we get

$$\int_a^T \varphi(t) \left(\mathfrak{D}_{a^+}^{\alpha,\beta} u\right)(t) \frac{dt}{t} \geq \int_a^T \left(\log \frac{t}{a}\right)^{\mu} |u(t)|^m \varphi(t) \frac{dt}{t}. \tag{3.6}$$

Observe that the integral in left-hand side exists and the one in the right-hand side exists for $m < (1+\mu)/(1-\gamma)$ when $u \in C_{1-\gamma,\log}^{\gamma}[a,b]$. Moreover, from the definition of $(\mathfrak{D}_{a^+}^{\alpha,\beta} u)(t)$, we can rewrite (3.6) as

$$\int_a^T \varphi(t) \left(\mathcal{J}_{a^+}^{\beta(1-\alpha)} t \frac{d}{dt} \mathcal{J}_{a^+}^{1-\gamma} u\right)(t) \frac{dt}{t} \geq \int_a^T \left(\log \frac{t}{a}\right)^{\mu} |u(t)|^m \varphi(t) \frac{dt}{t}. \tag{3.7}$$

By virtue of Lemma 2.12 (after extending by zero outside $[a,T]$), we may deduce from (3.7) that

$$\int_a^T \frac{d}{dt}\left(\mathcal{J}_{a^+}^{1-\gamma}u\right)(t)\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi(t)\right)(t)dt \geq \int_a^T \left(\log\frac{t}{a}\right)^{\mu}|u(t)|^m\varphi(t)\frac{dt}{t}. \tag{3.8}$$

Notice that Lemma 2.12 is valid in our case since $((\log(t/a))^{(1-\gamma)}(\mathfrak{D}_{a^+}^{\gamma}u) \in C[a,T]$ implies that $|(\log(t/a))^{(1-\gamma)}(\mathfrak{D}_{a^+}^{\gamma}u)(t)| \leq M$ on $[a,T]$ for some positive constant M)

$$\begin{aligned}\int_a^T \left|t^{-1/p}\left(\mathfrak{D}_{a^+}^{\gamma}u\right)(t)\right|^{p'}\frac{dt}{t} &\leq M\int_a^T t^{1-p'}\left(\log\frac{t}{a}\right)^{-p'(1-\gamma)}\frac{dt}{t}\\ &\leq M\int_a^{\infty} t^{1-p'}\left(\log\frac{t}{a}\right)^{-p'(1-\gamma)}\frac{dt}{t}.\end{aligned} \tag{3.9}$$

Let $s = (p'-1)(\log(t/a))$, then by the definition of the Gamma function,

$$\begin{aligned}\int_a^T \left|t^{-1/p}\left(\mathfrak{D}_{a^+}^{\gamma}u\right)(t)\right|^{p'}\frac{dt}{t} &\leq \frac{Ma^{1-p'}}{(p'-1)^{1-p'(1-\gamma)}}\int_0^{\infty} s^{-p'(1-\gamma)}e^{-s}ds\\ &\leq \frac{Ma^{1-p'}}{(p'-1)^{1-p'(1-\gamma)}}\Gamma\left(1-p'(1-\gamma)\right) < \infty.\end{aligned} \tag{3.10}$$

Hence, $t(d/dt)(\mathcal{J}_{a^+}^{1-\gamma}u)t = (\mathfrak{D}_{a^+}^{\gamma}u)(t) \in X_{-1/p}^{p'}$ (and $\varphi \in L_p$) for some $p > 1/\gamma$.

An integration by parts in (3.8) yields

$$\begin{aligned}&\left[\left(\mathcal{J}_{a^+}^{1-\gamma}u\right)(t)\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi\right)(t)\right]_{t=a}^{T} - \int_a^T \left(\mathcal{J}_{a^+}^{1-\gamma}u\right)(t)\frac{d}{dt}\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi\right)(t)dt\\ &\quad\geq \int_a^T \left(\log\frac{t}{a}\right)^{\mu}|u(t)|^m\varphi(t)\frac{dt}{t},\end{aligned} \tag{3.11}$$

or

$$\begin{aligned}&-u_0\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi\right)(a^+) - \int_a^T \left(\mathcal{J}_{a^+}^{1-\gamma}u\right)(t)\frac{d}{dt}\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi\right)(t)dt\\ &\quad\geq \int_a^T \left(\log\frac{t}{a}\right)^{\mu}|u(t)|^m\varphi(t)\frac{dt}{t}\end{aligned} \tag{3.12}$$

because $(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi)(T) = 0$ (see Lemma 3.1) and

$$\left(\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+) = \left(\mathfrak{D}_{a^+}^{\gamma-1}u\right)(a^+) = u_0. \tag{3.13}$$

Multiplying by t/t inside the integral in the left hand side of (3.12), we see that

$$\begin{aligned} L &:= \int_a^T \left(\mathcal{J}_{a^+}^{1-\gamma} u\right)(t) \left(-t\frac{d}{dt}\right)\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)} \varphi\right)(t)\frac{dt}{t} \\ &\geq \int_a^T \left(\log\frac{t}{a}\right)^{\mu} |u(t)|^m \varphi(t)\frac{dt}{t}. \end{aligned} \tag{3.14}$$

It appears from Definition 2.7 that

$$L = \int_a^T \left(\mathcal{J}_{a^+}^{1-\gamma} u\right)(t)\left(\mathfrak{D}_{T^-}^{1-\beta(1-\alpha)} \varphi\right)(t)\frac{dt}{t}, \tag{3.15}$$

and from Lemma 2.11, we see that

$$L = \int_a^T \left(\mathcal{J}_{a^+}^{1-\gamma} u\right)(t)\left[\frac{\varphi(T)}{\Gamma(\beta(1-\alpha))}\left(\log\frac{T}{t}\right)^{\beta(1-\alpha)-1} - \frac{1}{\Gamma(\beta(1-\alpha))}\int_t^T \left(\log\frac{s}{t}\right)^{\beta(1-\alpha)-1}\varphi'(s)ds\right]\frac{dt}{t}. \tag{3.16}$$

Since $\varphi(T) = 0$ and

$$\frac{1}{\Gamma(\beta(1-\alpha))}\int_t^T \left(\log\frac{s}{t}\right)^{\beta(1-\alpha)-1}\varphi'(s)ds = \left(\mathcal{J}_{T^-}^{\beta(1-\alpha)} \delta\varphi\right)(t), \tag{3.17}$$

the last equality becomes

$$\begin{aligned} L &= -\int_a^T \left(\mathcal{J}_{a^+}^{1-\gamma} u\right)(t)\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)} \delta\varphi\right)(t)\frac{dt}{t} \\ &\geq \int_a^T \left(\log\frac{t}{a}\right)^{\mu} |u(t)|^m \varphi(t)\frac{dt}{t}. \end{aligned} \tag{3.18}$$

Note that $\delta\varphi \in L_p$ and by the same argument as the one used at the beginning of the proof we may show that $\mathcal{J}_{a^+}^{1-\gamma} u \in X_{-1/p}^{p'}$ since $\mathcal{J}_{a^+}^{1-\gamma} u \in C_{1-\gamma,\log}[a,T]$.

Therefore, Lemma 2.12 again allows us to write

$$L = -\int_a^T \delta\varphi(t)\left(\mathcal{J}_{a^+}^{\beta(1-\alpha)}\mathcal{J}_{a^+}^{1-\gamma} u\right)(t)\frac{dt}{t}, \tag{3.19}$$

and by the semigroup property Lemma 2.10

$$L = -\int_a^T \delta\varphi(t)\left(\mathcal{J}_{a^+}^{1-\alpha} u\right)(t)\frac{dt}{t}. \tag{3.20}$$

On the other hand,

$$\int_a^T \delta\varphi(t)\left(\mathcal{J}_{a^+}^{1-\alpha}u\right)(t)\frac{dt}{t} = \frac{1}{\Gamma(1-\alpha)}\int_a^T \delta\varphi(t)\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}\frac{u(s)}{s}ds\frac{dt}{t}$$
$$\leq \frac{1}{\Gamma(1-\alpha)}\int_a^T \left|\delta\varphi(t)\right|\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}\frac{|u(s)|}{s}ds\frac{dt}{t}. \tag{3.21}$$

As φ is nonincreasing, we have $\varphi(s) \geq \varphi(t)$ for all $t \geq s$ and $1/\varphi^{1/m}(s) \leq 1/\varphi^{1/m}(t)$, $m > 1$. Also, it is clear that

$$\varphi'(t) = 0, \quad t \in [a, \theta T]. \tag{3.22}$$

Therefore,

$$L \leq \frac{1}{\Gamma(1-\alpha)}\int_a^T \left|\delta\varphi(t)\right|\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}\frac{|u(s)|\varphi^{1/m}(s)}{s\varphi^{1/m}(s)}ds\frac{dt}{t}$$
$$\leq \frac{1}{\Gamma(1-\alpha)}\int_{\theta T}^T \frac{\left|\delta\varphi(t)\right|}{\varphi^{1/m}(t)}\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}\frac{|u(s)|\varphi^{1/m}(s)}{s}ds\frac{dt}{t}. \tag{3.23}$$

Definition 2.4 allows us to write

$$L \leq \int_{\theta T}^T \frac{\left|\delta\varphi(t)\right|}{\varphi^{1/m}(t)}\left(\mathcal{J}_{a^+}^{1-\alpha}|u|\varphi^{1/m}\right)(t)\frac{dt}{t}. \tag{3.24}$$

By the same argument as the one used at the beginning of the proof, we may show that $|u(t)|\varphi^{1/m}(t) \in X_{-1/p}^{p'}$ ($|u(t)|\varphi^{1/m}(t) \leq |u(t)|$). Moreover, it is easy to see that $\delta\varphi(t)/\varphi^{1/m}(t) \in L_p$ (for, otherwise, we consider $\varphi^{\lambda}(t)$ with some sufficiently large λ). Thus, we can apply Lemma 2.12 to get

$$L \leq \int_{\theta T}^T |u(t)|\varphi^{1/m}(t)\left(\mathcal{J}_{T^-}^{1-\alpha}\frac{\left|\delta\varphi\right|}{\varphi^{1/m}}\right)(t)\frac{dt}{t}. \tag{3.25}$$

Next, we multiply by $(\log(t/a))^{\mu/m}.(\log(t/a))^{-\mu/m}$ inside the integral in the right-hand side of (3.25):

$$L \leq \int_{\theta T}^T \left(\mathcal{J}_{T^-}^{1-\alpha}\frac{\left|\delta\varphi\right|}{\varphi^{1/m}}\right)(t)|u(t)|\varphi^{1/m}(t)\frac{(\log(t/a))^{\mu/m}}{(\log(t/a))^{\mu/m}}\frac{dt}{t}. \tag{3.26}$$

For $\mu \geq 0$, we have $(\log(t/a))^{-\mu/m} \leq (\log(\theta T/a))^{-\mu/m}$ (because $-\mu/m < 0$ and $t > \theta T$). It follows that

$$L \leq \left(\log\frac{\theta T}{a}\right)^{-\mu/m}\int_{\theta T}^T \left(\mathcal{J}_{T^-}^{1-\alpha}\frac{\left|\delta\varphi\right|}{\varphi^{1/m}}\right)(t)\left(\log\frac{t}{a}\right)^{\mu/m}|u(t)|\varphi^{1/m}(t)\frac{dt}{t}. \tag{3.27}$$

By using the Young inequality (see Theorem 2.14), with m and m' such that $1/m + 1/m' = 1$, in the right-hand side of (3.27), we find

$$\begin{aligned} L &\leq \frac{1}{m}\int_{\theta T}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t} + \frac{(\log(\theta T/a))^{-\mu m'/m}}{m'}\int_{\theta T}^{T}\left(\mathcal{D}_{T^-}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)^{m'}(t)\frac{dt}{t} \\ &\leq \frac{1}{m}\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t} + \frac{(\log(\theta T/a))^{-\mu m'/m}}{m'}\int_{\theta T}^{T}\left(\mathcal{D}_{T^-}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)^{m'}(t)\frac{dt}{t}. \end{aligned} \tag{3.28}$$

Clearly, from (3.14) and (3.28), we see that

$$\begin{aligned} &\frac{(\log(\theta T/a))^{-\mu m'/m}}{m'}\int_{\theta T}^{T}\left(\mathcal{D}_{T^-}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)^{m'}(t)\frac{dt}{t} \\ &\quad\geq \left(1-\frac{1}{m}\right)\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t}, \end{aligned} \tag{3.29}$$

or

$$\frac{1}{m'}\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t} \leq \frac{(\log(\theta T/a))^{-\mu m'/m}}{m'}\int_{\theta T}^{T}\left(\mathcal{D}_{T^-}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)^{m'}(t)\frac{dt}{t}. \tag{3.30}$$

Therefore, by Definition 2.5, we have

$$\begin{aligned} &\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t} \\ &\quad\leq \frac{(\log(\theta T/a))^{-\mu m'/m}}{\Gamma^{m'}(1-\alpha)}\int_{\theta T}^{T}\left(\int_{t}^{T}\left(\log\frac{s}{t}\right)^{-\alpha}\frac{|\delta\varphi(s)|}{\varphi^{1/m}(s)}\frac{ds}{s}\right)^{m'}\frac{dt}{t}. \end{aligned} \tag{3.31}$$

The change of variable $\sigma T = t$ yields

$$\begin{aligned} &\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t} \\ &\quad\leq \frac{(\log(\theta T/a))^{-\mu m'/m}}{\Gamma^{m'}(1-\alpha)}\int_{\theta}^{1}\left(\int_{\sigma T}^{T}\left(\log\frac{s}{\sigma T}\right)^{-\alpha}\frac{|\varphi'(s)|}{\varphi(s)^{1/m}}ds\right)^{m'}\frac{d\sigma}{\sigma}. \end{aligned} \tag{3.32}$$

Another change of variable $r = s/T$ gives

$$\begin{aligned} &\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t} \\ &\quad\leq \frac{(\log(\theta T/a))^{-\mu m'/m}}{\Gamma^{m'}(1-\alpha)}\int_{\theta}^{1}\left(\int_{\sigma}^{1}\left(\log\frac{r}{\sigma}\right)^{-\alpha}\frac{|\varphi'(r)|}{\varphi(r)^{1/m}}dr\right)^{m'}\frac{d\sigma}{\sigma}. \end{aligned} \tag{3.33}$$

We may assume that the integral term in the right-hand side of (3.33) is convergent, that is,

$$\frac{1}{\Gamma^{m'}(1-\alpha)}\int_{\theta}^{1}\left(\int_{\sigma}^{1}\left(\ln\frac{r}{\sigma}\right)^{-\alpha}\frac{|\varphi'(r)|}{\varphi(r)^{1/m}}dr\right)^{m'}d\sigma \le C, \tag{3.34}$$

for some positive constant C, for otherwise we consider $\varphi^{\lambda}(r)$ with some sufficiently large λ. Therefore

$$\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t} \le C\left(\log\frac{\theta T}{a}\right)^{-\mu m'/m}. \tag{3.35}$$

If $\mu > 0$, then

$$\left(\log\frac{\theta T}{a}\right)^{-\mu m'/m} \longrightarrow 0, \tag{3.36}$$

as $T \longrightarrow \infty$. Finally, from (3.35), we obtain

$$\lim_{T\to\infty}\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t} = 0. \tag{3.37}$$

We reach a contradiction since the solution is not supposed to be trivial.

In the case $\mu = 0$ we have $-\mu m'/m = 0$ and the relation (3.35) ensures that

$$\lim_{T\to\infty}\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t} \le C. \tag{3.38}$$

Moreover, it is clear that

$$\begin{aligned}
&\left(\log\frac{\theta T}{a}\right)^{-\mu/m}\int_{\theta T}^{T}\left(\mathcal{D}_{T^-}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)(t)\left(\log\frac{t}{a}\right)^{\mu/m}|u(t)|\varphi^{1/m}(t)\frac{dt}{t}\\
&\quad\le\left(\log\frac{\theta T}{a}\right)^{-\mu/m}\left[\int_{\theta T}^{T}\left(\mathcal{D}_{T^-}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)^{m'}(t)\frac{dt}{t}\right]^{1/m'}\left[\int_{\theta T}^{T}\left(\log\frac{t}{a}\right)^{\mu}|u(t)|^{m}\varphi(t)\frac{dt}{t}\right]^{1/m}.
\end{aligned} \tag{3.39}$$

This relation, together with (3.27) (relations (3.28) and (3.31) also are used without θ), implies that

$$\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t} \le K\left[\int_{\theta T}^{t}\left(\log\frac{t}{a}\right)^{\mu}|u(t)|^{m}\varphi(t)\frac{dt}{t}\right]^{1/m} \tag{3.40}$$

for some positive constant K, with

$$\lim_{T\to\infty}\int_{\theta T}^{T}\left(\log\frac{t}{a}\right)^{\mu}|u(t)|^{m}\varphi(t)\frac{dt}{t}=0 \tag{3.41}$$

due to the convergence of the integral in (3.38). This is again a contradiction.

If $\mu < 0$, we have $(\log(t/a))^{-\mu/m} \leq (\log(T/a))^{-\mu/m}$ (because $-\mu/m > 0$ and $t < T$). Then, the change of variables $t = (T/a)^{\sigma}$ and $s = (T/a)^{r}$ in (3.27) yields

$$\begin{aligned}&\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t}\\&\quad\leq\frac{(\log(T/a))^{1-\mu m'/m}}{\Gamma^{m'}(1-\alpha)}\int_{\ln\theta T/\ln(T/a)}^{\ln T/\ln(T/a)}\left(\int_{\sigma}^{\ln T/\ln(T/a)}\left(\ln\frac{(T/a)^{r}}{(T/a)^{\sigma}}\right)^{-\alpha}\frac{|\varphi'(r)|}{\varphi^{1/m}(r)}dr\right)^{m'}d\sigma,\end{aligned} \tag{3.42}$$

or

$$\begin{aligned}&\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t}\\&\quad\leq\frac{(\log(T/a))^{1-\alpha m'-\mu m'/m}}{\Gamma^{m'}(1-\alpha)}\int_{\ln\theta T/\ln(T/a)}^{\ln T/\ln(T/a)}\left(\int_{\sigma}^{\ln T/\ln(T/a)}(r-\sigma)^{-\alpha}\frac{|\varphi'(r)|}{\varphi^{1/m}(r)}dr\right)^{m'}d\sigma.\end{aligned} \tag{3.43}$$

The expression $|\varphi'(r)|/\varphi^{1/m}(r)$ may be assumed bounded (or else we use $\varphi^{\lambda}(r)$ with a large value of λ). Hence,

$$\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t}\leq C\left(\log\frac{T}{a}\right)^{-m'-\mu m'/m} \tag{3.44}$$

for some positive constant C. □

Although we are concerend here about nonexistence of solutions, using standard techniques, one may show the existence of local solutions of Problem (1.1) with $1 < m < (1+\mu)/(1-\gamma)$. However, according to Theorem 3.2, such a solution cannot be continued for all time in $C^{\gamma}_{1-\gamma,\log}[a,b]$. This is a phenomenon which occurs often in parabolic and hyperbolic problems with sources of polynomial type. In the absence of strong dissipations, these sources are the cause of blowup in finite time (of local solutions). For this reason, they are called blowup terms.

4. Example

For our example, we need the following lemma.

Lemma 4.1. *The following result holds for the fractional derivative operator* $\mathfrak{D}_{a^+}^{\alpha,\beta}$*:*

$$\left(\mathfrak{D}_{a^+}^{\alpha,\beta}\left[\left(\log\frac{s}{a}\right)^{\gamma-1}\right]\right)(t)=\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}\left(\log\frac{t}{a}\right)^{\gamma-\alpha-1},\qquad t>a;\ \gamma>0, \tag{4.1}$$

where $0<\alpha<1$ *and* $0\leq\beta\leq 1$.

Proof. We observe from Lemma 2.8 that

$$\left(\mathcal{J}_{a^+}^{(1-\alpha)(1-\beta)}\left(\log\frac{s}{a}\right)^{\gamma-1}\right)(t)=\frac{\Gamma(\gamma)}{\Gamma((1-\alpha)(1-\beta)+\gamma)}\left(\log\frac{t}{a}\right)^{\gamma+(1-\alpha)(1-\beta)-1}. \tag{4.2}$$

Therefore,

$$\begin{aligned}&\left(t\frac{d}{dt}\right)\left(\mathcal{J}_{a^+}^{(1-\alpha)(1-\beta)}\left(\log\frac{s}{a}\right)^{\gamma-1}\right)(t)\\&\quad=\frac{[\gamma+(1-\alpha)(1-\beta)-1]\Gamma(\gamma)}{\Gamma((1-\alpha)(1-\beta)+\gamma)}\left(\log\frac{t}{a}\right)^{\gamma+(1-\alpha)(1-\beta)-2},\end{aligned} \tag{4.3}$$

which, in light of the definition of $\mathfrak{D}_{a^+}^{\alpha,\beta}$, yields

$$\begin{aligned}&\left(\mathfrak{D}_{a^+}^{\alpha,\beta}\left[\left(\log\frac{s}{a}\right)^{\gamma-1}\right]\right)(t)\\&\quad=\frac{\Gamma(\gamma)}{\Gamma((1-\alpha)(1-\beta)+\gamma-1)}\left(\mathcal{J}_{a^+}^{\beta(1-\alpha)}\left(\log\frac{s}{a}\right)^{\gamma+(1-\alpha)(1-\beta)-2}\right)(t).\end{aligned} \tag{4.4}$$

From Lemma 2.8 again, we have

$$\begin{aligned}&\left(\mathfrak{D}_{a^+}^{\alpha,\beta}\left[\left(\log\frac{s}{a}\right)^{\gamma-1}\right]\right)(t)\\&\quad=\frac{\Gamma(\gamma)}{\Gamma(\beta(1-\alpha)+\gamma+(1-\alpha)(1-\beta)-1)}\left(\log\frac{t}{a}\right)^{\beta(1-\alpha)+\gamma+(1-\alpha)(1-\beta)-2}\\&\quad=\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}\left(\log\frac{t}{a}\right)^{\gamma-\alpha-1}.\end{aligned} \tag{4.5}$$

The proof is complete. □

Example 4.2. Consider the following differential equation of Hilfer-Hadamard-type fractional derivative of order $0<\alpha<1$ and type $0\leq\beta\leq 1$:

$$\left(\mathfrak{D}_{a^+}^{\alpha,\beta}y\right)(t)=\lambda\left(\log\frac{t}{a}\right)^{\mu}[y(t)]^m\quad(t>a>0;m>1) \tag{4.6}$$

with real $\lambda, \mu \in \mathbf{R}^+$ ($\lambda \neq 0$). Suppose that the solution has the following form:

$$y(t) = c\left(\log \frac{t}{a}\right)^{\nu}. \tag{4.7}$$

Our aim next is to find the values of c and ν. By using Lemma 4.1 we have

$$\left(\mathfrak{D}_{a^+}^{\alpha,\beta}\left[c\left(\log \frac{s}{a}\right)^{\nu}\right]\right)(t) = \frac{c\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)}\left(\log \frac{t}{a}\right)^{\nu-\alpha}. \tag{4.8}$$

Therefore,

$$\frac{c\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)}\left(\log \frac{t}{a}\right)^{\nu-\alpha} = \lambda\left(\log \frac{t}{a}\right)^{\mu}\left[c\left(\log \frac{t}{a}\right)^{\nu}\right]^{m}. \tag{4.9}$$

It can be directly shown that $\nu = (\alpha+\mu)/(1-m)$ and $c = [\Gamma((\alpha+\mu)/(1-m)+1)/\lambda\Gamma((m\alpha+\mu)/(1-m)+1)]^{1/(m-1)}$. If $(m\alpha+\mu)/(1-m) > -1$, that is, $m > (1+\mu)/(1-\alpha)$, then (4.6) has the exact solution:

$$y(t) = \left[\frac{\Gamma((\alpha+\mu)/(1-m)+1)}{\lambda\Gamma((m\alpha+\mu)/(1-m)+1)}\right]^{1/(m-1)}\left(\log \frac{t}{a}\right)^{(\alpha+\mu)/(1-m)}. \tag{4.10}$$

This solution satisfies the initial condition when $(\alpha+\mu)/(1-m) \geq \gamma - 1 > -1$. Note that there is an overlap of the interval of existence in this example and the interval of nonexistence in the previous theorem. This may be explained by the fact that this solution is in $C_{1-\gamma,\log}[a,b]$ but not in $C^{\gamma}_{1-\gamma,\log}[a,b]$.

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