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Research Article

A Godunov-Mixed Finite Element Method on Changing Meshes for the Nonlinear Sobolev Equations

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A Godunov-mixed finite element method on changing meshes is presented to simulate the nonlinear Sobolev equations. The convection term of the nonlinear Sobolev equations is approximated by a Godunov-type procedure and the diffusion term by an expanded mixed finite element method. The method can simultaneously approximate the scalar unknown and the vector flux effectively, reducing the continuity of the finite element space. Almost optimal error estimates in L^2-norm under very general changes in the mesh can be obtained. Finally, a numerical experiment is given to illustrate the efficiency of the method.

1. Introduction

We consider the following nonlinear Sobolev equations:

$$\begin{gathered} u_t - \nabla \cdot (a(x,t,u)\nabla u_t + b(x,t,u)\nabla u) + \mathbf{c}(x,t,u)\cdot \nabla u = f(x,t,u), \quad x \in \Omega,\ t \in (0,T], \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t \in [0,T], \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded subset of R^n $(n \le 3)$ with smooth boundary $\partial\Omega$, $u_0(x)$ and $f(x,t,u)$ are known functions, and the coefficients $a(x,t,u), b(x,t,u), \mathbf{c}(x,t,u) = (c_1(x,t,u), c_2(x,t,u))^{\mathrm{T}}$ satisfy the following condition:

$$0 < a_0 \le a(x,t,u) \le a_1, \qquad \left|\frac{\partial a(x,t,u)}{\partial t}\right| \le K_1,$$
$$0 < b_0 \le b(x,t,u) \le b_1,$$

$$|\mathbf{c}(x,t,u)| = \sqrt{c_1^2(x,t,u) + c_2^2(x,t,u)} \le c_0, \qquad \left|\frac{\partial \mathbf{c}(x,t,u)}{\partial u}\right| \le K_2,$$
$$a(x,t,u), b(x,t,u), \mathbf{c}(x,t,u), f(x,t,u), \frac{\partial \mathbf{c}(x,t,u)}{\partial u} \text{ are Lipschitz continuous about } u, \tag{1.2}$$

where a_0, a_1, b_0, b_1, c_0, K_1, and K_2 are positive constants. We assume that $u(x,t)$ satisfy the smooth condition in the following analysis.

For time-changing localized phenomena, such as sharp fronts and layers, the finite element method on changing meshes [1–3] is advantageous over fixed finite element method. The reason is that the former treats the problem with the finite element method on space domain by using different meshes and different basic functions at different time levels so that it has the capability of self-adaptive local grid modification (refinement or unrefinement) to efficiently capture propagating fronts or moving layers. The work [4] had combined this method with mixed finite element method to study parabolic problems. In [5], an upwind-mixed method on changing meshes was considered for two-phase miscible flow in porous media.

Sobolev equations have important applications in many mathematical and physical problems, such as the percolation theory when the fluid flows through the cracks [6], the transfer problem of the moisture in the soil [7], and the heat conduction problem in different materials [8]. So there exists great and actual significance to research Sobolev equations. Many works had researched on numerical treatments for Sobolev equations. More attentions were paid for treating a damping term $\nabla \cdot [a\nabla u_t]$, which is a distinct character of Sobolev equations different from parabolic equation. For example, time stepping Galerkin method was presented for nonlinear Sobolev equations in [9, 10]. In [11, 12], nonlinear Sobolev equations with convection term were researched by using finite difference streamline-diffusion method and discontinuous Galerkin method, respectively. Two new least-squares mixed finite element procedures were formulated for solving convection-dominated Sobolev equations in [13].

Methods which combined Godunov-type schemes for advection with mixed finite elements for diffusion were introduced in [14] and had been applied to flow problems in reservoir engineering, contaminant transport, and computational fluid dynamics. Applications of these types of methods to single and two-phase flow in oil reservoirs were discussed in [15, 16]; application to the Navier-Stokes equations was given in [17]. These methods had proven useful for advective flow problems because they combined element-by-element conservation of mass with numerical stability and minimal numerical diffusion. Dawson [18] researched advective flow problems in one space dimension by high-order Godunov-mixed method. In 1993, he expanded this method to multidimensions [19] and presented three variations. In these methods, advection was approximated by a Godunov-type procedure, and diffusion was approximated by a low-order-mixed finite element method.

The object of this paper is to present a Godunov-mixed finite element method on changing meshes for the nonlinear Sobolev equations. The convection term $\mathbf{c} \cdot \nabla u$ of the nonlinear Sobolev equations is approximated by a Godunov-type procedure and the diffusion term by an expanded mixed finite element method. This method can simultaneously approximate the scalar unknown and the vector flux effectively, reducing the continuity of the finite element space. We describe this method in Section 2. In Section 3, we introduce three projections and a lemma. We derive almost optimal error

estimates in L^2-norm under very general changes in the mesh in Section 4. In Section 5, we present results of numerical experiment, which confirm our theoretical results.

Throughout the analysis, the symbol K will denote a generic constant, which is independent of mesh parameters Δt and h and not necessarily the same at different occurrences.

2. The Godunov-Mixed Method on Changing Meshes

At first we give some notation and basic assumptions. The usual Sobolev spaces and norms are adopted on Ω. The inner product on $L^2(\Omega)$ is denoted by $(f,g) = \int_\Omega fg\,dx$. Define the following spaces and norms:

$$\begin{gathered} H^m(\mathrm{div},\Omega) = \left\{\mathbf{f} = \left(f_x, f_y\right); f_x, f_y, \nabla\cdot\mathbf{f} \in H^m(\Omega)\right\}, \quad m \geq 0, \\ \|\mathbf{f}\|^2_{H^m(\mathrm{div})} = \left\|f_x\right\|^2_m + \left\|f_y\right\|^2_m + \|\nabla\cdot\mathbf{f}\|^2_m, \\ H(\mathrm{div},\Omega) = H^0(\mathrm{div},\Omega), \qquad W = \frac{L^2(\Omega)}{\{\varphi \equiv \text{ constant on } \Omega\}}, \\ V = \{\mathbf{v} \in H(\mathrm{div};\Omega) \mid \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega, \mathbf{n} \text{ is the unit outward norm to } \partial\Omega\}. \end{gathered} \tag{2.1}$$

Let $\Delta t^n > 0$ $(n = 1,2,\ldots,N^*)$ denote different time steps, $t^n = \sum_{k=1}^n \Delta t^k$, $T = \sum_{n=1}^{N^*} \Delta t^n$, $\Delta t = \max_n \Delta t^n$. Assume that the time steps Δt^n do not change too rapidly; that is, we assume there exist positive constants t_* and t^* which are independent of n and Δt such that

$$t_* \leq \frac{\Delta t^n}{\Delta t^{n-1}} \leq t^*. \tag{2.2}$$

For a given function $g(x,t)$, let $g^n = g(x,t^n)$.

Assume $\Omega = (0,1)\times(0,1)$. At each time level t^n, we construct a quasiuniform rectangular partition $K_h^n = \{e_i^n\}$ of Ω:

$$\begin{gathered} \delta_x^n : 0 = x_{1/2}^n < x_{3/2}^n < \cdots < x_{i+1/2}^n < \cdots < x_{I^n+1/2}^n = 1, \\ \delta_y^n : 0 = y_{1/2}^n < y_{3/2}^n < \cdots < y_{j+1/2}^n < \cdots < y_{J^n+1/2}^n = 1. \end{gathered} \tag{2.3}$$

Let $h_{i,x}^n = x_{i+1/2}^n - x_{i-1/2}^n$, $h_{i,y}^n = y_{j+1/2}^n - y_{j-1/2}^n$, $h^n = \max_{i,j}\{h_{i,x}^n, h_{i,y}^n\}$, and $h = \max_n h^n$. And $x_i^n = (x_{i+1/2}^n + x_{i-1/2}^n)/2$ is the midpoint of $[x_{i-1/2}^n, x_{i+1/2}^n]$, $h_{i+1/2,x}^n = x_{i+1}^n - x_i^n$, $\Delta t^n = O(h^n)$. Let $y_j^n, h_{j+1/2,y}^n$ be defined analogously.

Let

$$M_{-1}^k(\delta_x^n) = \left\{f : f|_{B_{i,x}^n} \in P^k\left(B_{i,x}^n\right), i = 1,\ldots,I^n\right\}, \qquad M_0^k(\delta_x^n) = M_{-1}^k(\delta_x^n) \cap C^0([0,1]), \tag{2.4}$$

where $B_{i,x}^n = [x_{i-1/2}^n, x_{i+1/2}^n]$ and $P^k(B_{i,x}^n)$ is the set of all polynomials of degree less than or equal to k defined on $B_{i,x}^n$. Similar definitions are given to $M_{-1}^0(\delta_y^n)$ and $M_0^1(\delta_y^n)$.

Then the lowest order Raviart-Thomas spaces W_h^n and V_h^n are given by

$$\begin{aligned} W_h^n &= M_{-1}^0(\delta_x^n) \otimes M_{-1}^0\left(\delta_y^n\right), \qquad V_h^n = V_{h,x}^n \times V_{h,y}^n, \\ V_{h,x}^n &= M_0^1(\delta_x^n) \otimes M_{-1}^0\left(\delta_y^n\right), \qquad V_{h,y}^n = M_{-1}^0(\delta_x^n) \otimes M_0^1\left(\delta_y^n\right). \end{aligned} \tag{2.5}$$

That is, the space W_h^n is the space of functions which are constant on each element $e_i^n \in K_h^n$, and V_h^n is the space of vector valued functions whose components are continuous and linear on each element $e_i^n \in K_h^n$. The degrees of freedom of a function $\mathbf{v}^n \in V_h^n$ correspond to the values of $\mathbf{v}^n \cdot \boldsymbol{\gamma}$ at the midpoints of the sides of e_i^n, where $\boldsymbol{\gamma}$ is the unit outward normal to ∂e_i^n. It is easy to see that $W_h^n \times V_h^n \subset W \times V$ and $\operatorname{div} V_h^n = W_h^n$.

By introducing variables $\overline{\mathbf{z}} = -\nabla u$, $\mathbf{z} = b\overline{\mathbf{z}} + a\overline{\mathbf{z}}_t$, $\mathbf{g} = \mathbf{c}u = (c_1 u, c_2 u)^{\mathrm{T}} = (g_1, g_2)^{\mathrm{T}}$, and $\overline{\mathbf{c}}(x,t,u) = (\partial c_1/\partial u, \partial c_2/\partial u)^{\mathrm{T}}$, we modify the first equation in (1.1) as

$$u_t + \nabla \cdot \mathbf{z} + (\nabla \cdot \mathbf{g} + u\overline{\mathbf{c}}(u) \cdot \overline{\mathbf{z}}) = f(u). \tag{2.6}$$

Here we are using the so-called "expanded" mixed finite element method, proposed by Arbogast et al. [20], which gives a gradient approximation $\overline{\mathbf{z}}$ as well as an approximation to the diffusion term $\mathbf{z}$.

The weak form of (2.6) is

$$(u_t, w) + (\nabla \cdot \mathbf{z}, w) + (\nabla \cdot \mathbf{g}, w) + (u\overline{\mathbf{c}}(u) \cdot \overline{\mathbf{z}}, w) = \big(f(u), w\big), \tag{2.7}$$

so that

$$\begin{aligned} (u_t, w) + (\nabla \cdot \mathbf{z}, w) + (\nabla \cdot \mathbf{g}, w) + (u\overline{\mathbf{c}}(u) \cdot \overline{\mathbf{z}}, w) &= \big(f(u), w\big), \quad \forall w \in W, \\ (\overline{\mathbf{z}}, \mathbf{v}) &= (u, \nabla \cdot \mathbf{v}), \quad \forall \mathbf{v} \in V, \\ (\mathbf{z}, \mathbf{v}) &= (b(u)\overline{\mathbf{z}} + a(u)\overline{\mathbf{z}}_t, \mathbf{v}), \quad \forall \mathbf{v} \in V. \end{aligned} \tag{2.8}$$

The Godunov-mixed method on changing meshes is presented as follows: at each time level n, find $U^n \in W_h^n$, $\mathbf{Z}^n \in V_h^n$ such that

$$\begin{aligned} &\left(\frac{U^n - R^n U^{n-1}}{\Delta t^n}, w^n\right) + (\nabla \cdot \mathbf{Z}^n, w^n) + \left(U^n \overline{\mathbf{c}}(U^n) \cdot \overline{Z}^n, w^n\right) \\ &\quad = \big(f(U^n), w^n\big) - \left(\nabla \cdot \mathbf{G}^{n-1}, w^n\right), \quad \forall w^n \in W_h^n, \end{aligned}$$

$$\left(R^n U^{n-1} - U^{n-1}, w^n\right) = 0, \quad \forall w^n \in W_h^n,$$
$$\left(\overline{\mathbf{Z}}^n, \mathbf{v^n}\right) = (U^n, \nabla \cdot \mathbf{v^n}), \quad \forall \mathbf{v^n} \in V_h^n,$$
$$(\mathbf{Z}^n, \mathbf{v^n}) = \left(b(U^n)\overline{\mathbf{Z}}^n + a(U^n)\frac{\overline{\mathbf{Z}}^n - R^n\overline{\mathbf{Z}}^{n-1}}{\Delta t^n}, \mathbf{v^n}\right), \quad \forall \mathbf{v^n} \in V_h^n,$$
$$\left(a(U^n)\left(R^n\tilde{\mathbf{Z}}^{n-1} - \tilde{\mathbf{Z}}^{n-1}\right), \mathbf{v^n}\right) = 0, \quad \forall \mathbf{v^n} \in V_h^n. \tag{2.9}$$

When different finite element spaces are used at time levels t^n and t^{n-1}, the second and fifth equations of (2.9) give the L^2-projection $\{R^n U^{n-1}, R^n\tilde{\mathbf{Z}}^{n-1}\}$ of the previous approximate solution $\{U^{n-1}, \tilde{\mathbf{Z}}^{n-1}\}$ into the current finite element space $W_h^n \times V_h^n$. This projection is used in the first and third equations of (2.9) as initial value to calculate $\{U^n, \tilde{\mathbf{Z}}^n\}$. If the finite element spaces are the same at time levels t^n and t^{n-1}, we know that $R^n U^{n-1} = U^{n-1}$, $R^n\tilde{\mathbf{Z}}^{n-1} = \tilde{\mathbf{Z}}^{n-1}$.

Let $\mathbf{G}^n$ in (2.9) be constructed by the Godunov method. We adopt the Godunov flux [19] $H_w(w^L, w^R)$ for a given flux function $w(s)$:

$$H_w\left(w^L, w^R\right) = \begin{cases} \min\limits_{w^L \le s \le w^R} w(s), & \text{if } w^L \le w^R, \\ \max\limits_{w^R \le s \le w^L} w(s), & \text{otherwise}, \end{cases} \tag{2.10}$$

where w^L, w^R are the left and right states, respectively. It is easily seen that H_w is Lipschitz in its arguments if w is Lipschitz in s and H_w is consistent, that is, $H_w(s, s) = w(s)$.

Let $\mathbf{G} = (\overline{g}_1, \overline{g}_2)$, $g_i = c_i u$, $\mathbf{G}_{i+1/2,j+1/2}^{n-1} = (\overline{g}_{1,i+1/2,j}^{n-1}, \overline{g}_{2,i,j+1/2}^{n-1})$. Now we give the calculation of $\mathbf{G}$ by H_w by the following three steps.

First Step

We construct a piecewise linear function RU^{n-1} on each element e_i^{n-1}:

$$RU^{n-1}|_{e_i^{n-1}} = U_{i,j}^{n-1} + (x - x_i)\delta_x U_{i,j}^{n-1} + (y - y_j)\delta_y U_{i,j}^{n-1}, \tag{2.11}$$

where $\delta_x U_{i,j}^{n-1}$ and $\delta_y U_{i,j}^{n-1}$ are x, y slopes. The x slope calculation can be performed as

$$\delta_x U_{i,j}^{n-1} = \begin{cases} \triangle_+^x U_{i,j}^{n-1}, & \text{if } \left|\triangle_+^x U_{i,j}^{n-1}\right| \le \left|\triangle_-^x U_{i,j}^{n-1}\right|, \\ \triangle_-^x U_{i,j}^{n-1}, & \text{otherwise}, \end{cases} \tag{2.12}$$

where $\triangle_+^x$ and $\triangle_-^x$ are forward and backward difference operators in the x direction, respectively:

$$\begin{aligned} \triangle_+^x U_{i,j}^{n-1} &= \frac{U_{i+1,j}^{n-1} - U_{i,j}^{n-1}}{h_{i+1/2,x}^{n-1}}, \\ \triangle_-^x U_{i,j}^{n-1} &= \frac{U_{i,j}^{n-1} - U_{i-1,j}^{n-1}}{h_{i-1/2,x}^{n-1}}, \end{aligned} \tag{2.13}$$

and δ_x satisfy

$$\begin{aligned} \frac{h_{i,x}^{n-1}}{2}\left|\delta_x U_{i,j}^{n-1}\right| &\le C \sum_{j-L\le j\le j+L} \sum_{i-R\le i\le i+R} \left|U_{i,j}^{n-1}\right|, \\ \delta_x u_{i,j}^{n-1} &= u_x\left(x_i, y_j, t^{n-1}\right) + O(h), \end{aligned} \tag{2.14}$$

where L and R are positive integers independent of h. The y slope is defined analogously.

Second Step

By the Taylor expansion, for u with smooth second derivatives, we have

$$u\left(x_{i+1/2}, y_j, t^{n-1/2}\right) = u + \frac{h_{i,x}^n}{2} u_x + \frac{\Delta t^n}{2} u_t + O\left(h^2 + \Delta t^2\right). \tag{2.15}$$

By (2.6), we see that

$$u_t = f - u\bar{\mathbf{c}}(u)\cdot\bar{z} - g_{1u}u_x - g_{2y} - \nabla\cdot z, \tag{2.16}$$

$$u_{i+1/2,y}^{n-1/2} = u + \left(\frac{h_{i,x}^n}{2} - \frac{\Delta t^n}{2} g_{1u}\right) u_x - \frac{\Delta t^n}{2} g_{2y} + \sigma, \tag{2.17}$$

where

$$\sigma = \frac{\Delta t^n}{2}\left(f - u\bar{\mathbf{c}}(u)\cdot\bar{z} - \nabla\cdot z\right) + O\left(h^2 + \Delta t^2\right) = O\left(h^2 + \Delta t\right). \tag{2.18}$$

Based on (2.17) and a similar expansion about (x_{i+1}, y_j, t^{n-1}), we define U^L, U^R:

$$\begin{aligned} U_{i+1/2,j}^{L,n-1} &\triangleq U_{i,j}^{n-1} + \left(\frac{h_{i,x}^{n-1}}{2} - \frac{\Delta t^{n-1}}{2} g_{1u}\left(U_{i,j}^{n-1}\right)\right)\delta_x U_{i,j}^{n-1} - \frac{\Delta t^{n-1}}{2h_{j,y}^{n-1}}\left[\Gamma_{i,j+1/2}^{n-1} - \Gamma_{i,j-1/2}^{n-1}\right], \\ U_{i+1/2,j}^{R,n-1} &\triangleq U_{i+1,j}^{n-1} + \left(\frac{h_{i,x}^{n-1}}{2} - \frac{\Delta t^{n-1}}{2} g_{1u}\left(U_{i+1,j}^{n-1}\right)\right)\delta_x U_{i+1,j}^{n-1} - \frac{\Delta t^{n-1}}{2h_{j,y}^{n-1}}\left[\Gamma_{i+1,j+1/2}^{n-1} - \Gamma_{i+1,j-1/2}^{n-1}\right], \end{aligned} \tag{2.19}$$

where

$$\Gamma_{i,j+1/2}^{n-1} = H_g\left(U_{i,j+1/2}^{L,n-1}, U_{i,j+1/2}^{R,n-1}\right),$$
$$U_{i,j+1/2}^{L,n-1} = U_{i,j}^{n-1} + \frac{h_{j,y}^{n-1}}{2}\delta_y U_{i,j}^{n-1}, \tag{2.20}$$
$$U_{i,j+1/2}^{R,n-1} = U_{i,j+1}^{n-1} + \frac{h_{j+1,y}^{n-1}}{2}\delta_y U_{i,j+1}^{n-1}.$$

Third Step

With the above definitions, $(\overline{g}_{1,i+1/2,j}^{n-1}, \overline{g}_{2,i,j+1/2}^{n-1})$ is calculated as follows.

If $c_1(U_{i+1/2,j}^{n-1}) > 0$, define

$$\overline{g}_{1,i+1/2,j}^{n-1} \triangleq H_{g_1}\left(U_{i+1/2,j}^{L,n-1}, U_{i+1/2,j}^{L,n-1}\right) \triangleq c_1\left(U_{i+1/2,j}^{L,n-1}\right)U_{i+1/2,j}^{L,n-1}. \tag{2.21}$$

If $c_1(U_{i+1/2,j}^{n-1}) \leq 0$, define

$$\overline{g}_{1,i+1/2,j}^{n-1} \triangleq H_{g_1}\left(U_{i+1/2,j}^{R,n-1}, U_{i+1/2,j}^{R,n-1}\right) \triangleq c_1\left(U_{i+1/2,j}^{R,n-1}\right)U_{i+1/2,j}^{R,n-1}. \tag{2.22}$$

For $\overline{g}_{2,i,j+1/2}^{n-1}$, the definition is similar.

The equations (2.19)–(2.22) hold for elements at least one element away from the boundary. At the left and right boundaries, we can set

$$U_{1/2,j}^{L,n-1} = U_{I^{n-1}+1/2,j}^{R,n-1} = 0, \quad j = 1, 2, \ldots, J^{n-1},\ n = 1, 2, \ldots, N, \tag{2.23}$$

and in the slope calculation procedure, we define $\delta_x U_{1,j}^{n-1}$ and $\delta_x U_{I^{n-1},j}^{n-1}$ using (2.12) with

$$\triangle_-^x U_{1,j}^{n-1} = \frac{2U_{1,j}^{n-1}}{h_{1,x}^{n-1}}, \qquad \triangle_+^x U_{I^{n-1},j}^{n-1} = \frac{-2U_{I^{n-1},j}^{n-1}}{h_{I^{n-1},x}^{n-1}}. \tag{2.24}$$

On the bottom and top boundaries, we set

$$\Gamma_{i,1/2}^{n-1} = \Gamma_{i,J^{n-1}+1/2}^{n-1} = 0, \quad i = 1, 2, \ldots, I^{n-1}, \tag{2.25}$$

in (2.19).

3. Projections and Lemma

We introduce three projections and a lemma to obtain error estimates. Let $\{\tilde{u}^n, \tilde{\mathbf{z}}^n\} : J \to W_h^n \times V_h^n$ be the projection of $\{u, \bar{\mathbf{z}}\}$ in mixed finite element space such that

$$(\tilde{\mathbf{z}}^n, \mathbf{v}^n) - (\tilde{u}^n, \nabla \cdot \mathbf{v}^n) = 0, \quad \forall \mathbf{v}^n \in V_h^n, \tag{3.1}$$

and define $\pi \mathbf{z}^n$ as the π-projection of $\mathbf{z}^n$ such that

$$(\nabla \cdot (\mathbf{z}^n - \pi \mathbf{z}^n), w^n) = 0, \quad \forall w^n \in W_h^n. \tag{3.2}$$

These projections satisfy ([21, 22]):

$$\begin{gathered} \|u^n - \tilde{u}^n\| + \left\|\bar{z}^n - \tilde{\mathbf{z}}^n\right\|_{H(\mathrm{div})} \le K h^n, \\ \left\|\frac{\partial}{\partial t}(u^n - \tilde{u}^n)\right\| + \left\|\frac{\partial}{\partial t}(\bar{\mathbf{z}}^n - \tilde{\mathbf{z}}^n)\right\|_{H(\mathrm{div})} \le K h^n, \end{gathered} \tag{3.3}$$

and it is easy to see that $\|\tilde{u}^n\|_L^\infty \le K$.

Let $\tilde{\mathbf{g}} \in V_h^n$ satisfies that the value of $\tilde{\mathbf{g}} \cdot \boldsymbol{\gamma}$ at the midpoint of the boundaries is equal to the average value of $\mathbf{g} \cdot \boldsymbol{\gamma}$ on boundaries, that is,

$$(\nabla \cdot (\tilde{\mathbf{g}}^n - \mathbf{g}^n), w^n) = 0, \quad \forall w^n \in W_h^n. \tag{3.4}$$

Let $\tilde{\mathbf{g}} = (\tilde{g}_1, \tilde{g}_2)$. From (3.4), we can define

$$\begin{aligned} \tilde{g}_1(x_{i+1/2}, y_j, t^n) &\triangleq \frac{1}{h_{j,y}^n} \int_{y_{j-1/2}}^{y_{j+1/2}} c_1(x_{i+1/2}, y_j, t^n, u) u(x_{i+1/2}, y, t^n) dy \\ &\triangleq \frac{1}{h_{j,y}^n} \int_{y_{j-1/2}}^{y_{j+1/2}} g_1(u(x_{i+1/2}, y, t^n)) dy. \end{aligned} \tag{3.5}$$

Then, $\tilde{g}_2(x_i, y_{j+1/2}, t^n)$ can be defined similarly.

Lemma 3.1. *Let $\xi = U - \tilde{u}$ and assume that u is sufficiently smooth. Then*

$$\left\|\mathbf{G}^{n-1}\left(\cdot, U^{n-1}\right) - \tilde{\mathbf{g}}^{n-1}\right\| \le K\left(\left\|\xi^{n-1}\right\| + h + \Delta t\right). \tag{3.6}$$

Proof. Firstly, construct $\tilde{u}_{i+1/2,j}^{L,n-1}, \tilde{u}_{i+1/2,j}^{R,n-1}$ from $\tilde{u}$ given in (3.1) and (3.2) by

$$\begin{aligned} \tilde{u}_{i+1/2,j}^{L,n-1} = \tilde{u}_{i,j}^{n-1} &+ \left(\frac{h_{i,x}^{n-1}}{2} - \frac{\Delta t^n}{2} g_{1u}\left(\tilde{u}_{i,j}^{n-1}\right)\right) \delta_x \tilde{u}_{i,j}^{n-1} \\ &- \frac{\Delta t^n}{2 h_{j,y}^{n-1}} \left[\Gamma_{i,j+1/2}^{n-1}(\tilde{u}) - \Gamma_{i,j-1/2}^{n-1}(\tilde{u})\right], \end{aligned} \tag{3.7}$$

at interior edges, where

$$\begin{aligned}
\delta_x \tilde{u}_{i,j}^{n-1} &= \begin{cases} \triangle_+^x \tilde{u}_{i,j}^{n-1}, & \text{if } \delta_x U_{i,j}^{n-1} = \triangle_+^x U_{i,j}^{n-1}, \\ \triangle_-^x \tilde{u}_{i,j}^{n-1}, & \text{otherwise,} \end{cases} \\
\Gamma_{i,j+1/2}^{n-1}(\tilde{u}) &= H_g\left(\tilde{u}_{i,j+1/2}^{L,n-1}, \tilde{u}_{i,j+1/2}^{R,n-1}\right), \\
\tilde{u}_{i,j+1/2}^{L,n-1} &= \tilde{u}_{i,j}^{n-1} + \frac{h_{j,y}^{n-1}}{2}\delta_y \tilde{u}_{i,j}^{n-1}, \\
\tilde{u}_{i,j+1/2}^{R,n-1} &= \tilde{u}_{i,j+1}^{n-1} + \frac{h_{j+1,y}^{n-1}}{2}\delta_y \tilde{u}_{i,j+1}^{n-1}.
\end{aligned} \tag{3.8}$$

And define $\tilde{u}_{i+1/2,j}^{R,n-1}$ similarly.

Then, assuming g_1 is twice differentiable and Lipschitz continuous, and using the Lipschitz continuity of the Godunov flux and (2.14), it can be shown that

$$\left|U_{i+1/2,j}^{L,n-1} - \tilde{u}_{i+1/2,j}^{L,n-1}\right| \le K \sum_{l=i-1}^{i+2} \sum_{m=j-1}^{j+2} \left|U_{l,m}^{n-1} - \tilde{u}_{l,m}^{n-1}\right|. \tag{3.9}$$

A similar bound holds for $|U_{i+1/2,j}^{R,n-1} - \tilde{u}_{i+1/2,j}^{R,n-1}|$.

At the boundaries, we follow (2.23) and (2.25) and define

$$\begin{aligned}
\tilde{u}_{1/2,j}^{L,n-1} &= \tilde{u}_{I^{n-1}+1/2,j}^{R,n-1} = 0, \quad j = 1, 2, \ldots, J^{n-1}, \\
\Gamma_{i,1/2}^{n-1}(\tilde{u}) &= \Gamma_{i,J^{n-1}+1/2}^{n-1}(\tilde{u}) = 0, \quad i = 1, 2, \ldots, I^{n-1}.
\end{aligned} \tag{3.10}$$

Define $u_{i+1/2,j}^{L,n-1}, u_{i+1/2,j}^{R,n-1}$ analogous to $\tilde{u}_{i+1/2,j}^{L,n-1}, \tilde{u}_{i+1/2,j}^{R,n-1}$.

If $c_1(U_{i+1/2,j}^{n-1}) > 0$, we consider

$$\begin{aligned}
&\overline{g}_{1,i+1/2,j}^{n-1} - \tilde{g}_1\left(x_{i+1/2}, y_j, t^{n-1}\right) \\
&\quad = H_{g_1}\left(U_{i+1/2,j}^{L,n-1}, U_{i+1/2,j}^{L,n-1}\right) - \frac{1}{h_{j,y}^{n-1}} \int_{y_{j-1/2}}^{y_{j+1/2}} g_1\left(u\left(x_{i+1/2}, y, t^{n-1}\right)\right) dy
\end{aligned}$$

$$
\begin{aligned}
&= \left[H_{g_1}\left(U_{i+1/2,j}^{L,n-1}, U_{i+1/2,j}^{L,n-1}\right) - H_{g_1}\left(\tilde{u}_{i+1/2,j}^{L,n-1}, \tilde{u}_{i+1/2,j}^{L,n-1}\right)\right] \\
&\quad + \left[H_{g_1}\left(\tilde{u}_{i+1/2,j}^{L,n-1}, \tilde{u}_{i+1/2,j}^{L,n-1}\right) - H_{g_1}\left(u_{i+1/2,j}^{L,n-1}, u_{i+1/2,j}^{L,n-1}\right)\right] \\
&\quad + \left[H_{g_1}\left(u_{i+1/2,j}^{L,n-1}, u_{i+1/2,j}^{L,n-1}\right) - g_1\left(u_{i+1/2,j}^{n-1/2}\right)\right] \\
&\quad + \left[g_1\left(u_{i+1/2,j}^{n-1/2}\right) - \frac{1}{h_{j,y}^{n-1}} \int_{y_{j-1/2}}^{y_{j+1/2}} g_1\left(u\left(x_{i+1/2}, y, t^{n-1}\right)\right) dy\right] \\
&= R_1 + R_2 + R_3 + R_4.
\end{aligned}
\tag{3.11}
$$

By the Lipschitz continuity of H_{g_1} and (3.9), we have

$$
\begin{aligned}
|R_1| &\le K\left\{\left|U_{i+1/2,j}^{L,n-1} - \tilde{u}_{i+1/2,j}^{L,n-1}\right| + \left|U_{i+1/2,j}^{R,n-1} - \tilde{u}_{i+1/2,j}^{R,n-1}\right|\right\} \\
&\le K \sum_{l=i-1}^{i+2} \sum_{m=j-1}^{j+2} \left|U_{l,m}^{n-1} - \tilde{u}_{l,m}^{n-1}\right|.
\end{aligned}
\tag{3.12}
$$

The similar arguments are applied to R_2 to get

$$
|R_2| \le K \sum_{l=i-1}^{i+2} \sum_{m=j-1}^{j+2} \left|\tilde{u}_{l,m}^{n-1} - u_{i+1/2,j}^{n-1}\right|. \tag{3.13}
$$

By the consistency of H_{g_1}, we see

$$
|R_3| \le K\left|u_{i+1/2,j}^{L,n-1} - u_{i+1/2,j}^{n-1/2}\right| = O\left(\Delta t + h^2\right). \tag{3.14}
$$

For R_4, it is easy to have

$$
|R_4| \le K\left(\Delta t + h^2\right). \tag{3.15}
$$

Using (3.3), (3.12)–(3.15), and equivalence of norms, we have

$$
\begin{aligned}
\left\|\overline{g}_1^{n-1} - \tilde{g}_1^{n-1}\right\| &\le K \sum_j \sum_i \left|\overline{g}_{1,i+1/2,j}^{n-1} - \tilde{g}_{1,i+1/2,j}^{n-1}\right|^2 h_{i+1/2,x}^{n-1} h_{j,y}^{n-1} \\
&\le K\left(\left\|\xi^{n-1}\right\| + \Delta t + h\right).
\end{aligned}
\tag{3.16}
$$

The similar approach can be used when $c_1(U_{i+1/2,j}^{n-1}) \le 0$. Defining $\overline{g}_2$ and $\widetilde{g}_2$ similarly and following analogous arguments yields

$$\left\|\overline{g}_2^{n-1} - \widetilde{g}_2^{n-1}\right\| \le C\left(\left\|\xi^{n-1}\right\| + \Delta t + h\right). \tag{3.17}$$

Thus, Lemma 3.1 holds. □

4. Error Estimates

At time level t^n, for all $w^n \in W_h^n$, $\mathbf{v^n} \in V_h^n$, the exact solutions satisfy

$$\begin{gathered}
\left(\frac{u^n - u^{n-1}}{\Delta t^n}, w^n\right) + (\nabla \cdot \mathbf{z}^n, w^n) + \left(u^n \overline{\mathbf{c}}(u^n) \cdot \overline{\mathbf{z}}^n, w^n\right) = \left(f(u^n), w^n\right) - (\nabla \cdot \mathbf{g}^n, w^n) - \left(\rho^n, w^n\right), \\
(\overline{\mathbf{z}}, \mathbf{v^n}) = (u^n, \nabla \cdot \mathbf{v^n}), \\
(\mathbf{z}^n, \mathbf{v^n}) = \left(b(u^n)\overline{\mathbf{z}}^n + a(u^n)\frac{\overline{\mathbf{z}}^n - \overline{\mathbf{z}}^{n-1}}{\Delta t^n}, \mathbf{v^n}\right) + (\mathbf{r}^n, \mathbf{v^n}),
\end{gathered} \tag{4.1}$$

where $\rho^n = u_t^n - (u^n - u^{n-1})/\Delta t^n$, $\mathbf{r}^n = a(u^n)(\overline{\mathbf{z}}_t^n - (\overline{\mathbf{z}}^n - \overline{\mathbf{z}}^{n-1})/\Delta t^n)$.

Let

$$\begin{aligned}
\xi_u &= U - \widetilde{u}, & \beta_u &= u - \widetilde{u}, \\
\overline{\boldsymbol{\xi}}_{\mathbf{z}} &= \overline{\mathbf{Z}} - \widetilde{\mathbf{z}}, & \overline{\boldsymbol{\beta}}_{\mathbf{z}} &= \overline{\mathbf{z}} - \widetilde{\mathbf{z}}, \\
\boldsymbol{\xi}_{\mathbf{z}} &= \mathbf{Z} - \pi\mathbf{z}, & \boldsymbol{\beta}_{\mathbf{z}} &= \mathbf{z} - \pi\mathbf{z}.
\end{aligned} \tag{4.2}$$

Using projections (3.1) and (3.2), we subtract (2.9) from (4.1) to get

$$\begin{gathered}
\left(\frac{\xi_u^n - \xi_u^{n-1}}{\Delta t^n}, w^n\right) + \left(\nabla \cdot \boldsymbol{\xi}_{\mathbf{z}}^n, w^n\right) + \left(U^n \overline{\mathbf{c}}(U^n) \cdot \overline{\mathbf{Z}}^n - u^n \overline{\mathbf{c}}(u^n) \cdot \overline{\mathbf{z}}^n, w^n\right) \\
= \left(f(U^n) - f(u^n), w^n\right) + \left(\rho^n, w^n\right) + \left(\nabla \cdot \left(\widetilde{\mathbf{g}}^n - \mathbf{G}^{n-1}\right), w^n\right) + \left(\frac{\beta_u^n - \beta_u^{n-1}}{\Delta t^n}, w^n\right), \\
\left(\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n, \mathbf{v^n}\right) = \left(\xi_u^n, \nabla \cdot \mathbf{v^n}\right), \\
\left(\boldsymbol{\xi}_{\mathbf{z}}^n, \mathbf{v^n}\right) = \left(\left[b(U^n)\overline{\mathbf{Z}}^{\mathbf{n}} - b(u^n)\overline{\mathbf{z}}^{\mathbf{n}}\right], \mathbf{v^n}\right) - \left(\mathbf{r}^n, \mathbf{v^n}\right) - \left(\boldsymbol{\beta}_{\mathbf{z}}^n, \mathbf{v^n}\right) \\
+ \left(a(U^n)\frac{\overline{\mathbf{Z}}^{\mathbf{n}} - \overline{\mathbf{Z}}^{\mathbf{n-1}}}{\Delta t^n} - a(u^n)\frac{\overline{\mathbf{z}}^{\mathbf{n}} - \overline{\mathbf{z}}^{\mathbf{n-1}}}{\Delta t^n}, \mathbf{v^n}\right),
\end{gathered} \tag{4.3}$$

where the last term of the first equation in (4.3) is related to the changing meshes. If the meshes do not change, this term will be zero.

Taking $w = \xi_u^n$, $\mathbf{v} = \boldsymbol{\xi}_{\mathbf{z}}^n$ in the second equation and $\mathbf{v} = \overline{\boldsymbol{\xi}}_{\mathbf{z}}^n$ in the third, then adding them together, we have

$$\begin{aligned}
&\left(\frac{\xi_u^n - \xi_u^{n-1}}{\Delta t^n}, \xi_u^n\right) + \left(b(U^n)\overline{\mathbf{Z}}^n - b(u^n)\overline{\mathbf{z}}^n, \overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right) + \left(a(U^n)\frac{\overline{\mathbf{Z}}^n - \overline{\mathbf{Z}}^{n-1}}{\Delta t^n}, \overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right) - \left(a(u^n)\frac{\overline{\mathbf{z}}^n - \overline{\mathbf{z}}^{n-1}}{\Delta t^n}, \overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right) \\
&\quad = \left(f(U^n) - f(u^n), \xi_u^n\right) + \left(\rho^n, \xi_u^n\right) + \left(\nabla \cdot \left(\widetilde{\mathbf{g}}^n - \mathbf{G}^{n-1}\right), \xi_u^n\right) \\
&\qquad - \left(U^n \overline{\mathbf{c}}(U^n) \cdot \overline{\mathbf{Z}}^n - u^n \overline{\mathbf{c}}(u^n) \cdot \overline{\mathbf{z}}^n, \xi_u^n\right) + \left(\mathbf{r}^n, \overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right) - \left(\boldsymbol{\beta}_{\mathbf{z}}^n, \overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right) + \left(\frac{\beta_u^n - \beta_u^{n-1}}{\Delta t^n}, \xi_u^n\right).
\end{aligned} \tag{4.4}$$

For (4.4), we see

(I)

$$\begin{aligned}
\left(b(U^n)\overline{\mathbf{Z}}^n - b(u^n)\overline{\mathbf{z}}^n, \overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right) &= \left(b(U^n)\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n, \overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right) - \left(b(U^n)\overline{\boldsymbol{\beta}}_{\mathbf{z}}^n, \overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right) \\
&\quad + \left([b(U^n) - b(u^n)]\overline{\mathbf{z}}^n, \overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right),
\end{aligned} \tag{4.5}$$

(II)

$$\begin{aligned}
&\left(a(U^n)\frac{\overline{\mathbf{Z}}^n - \overline{\mathbf{Z}}^{n-1}}{\Delta t^n}, \overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right) - \left(a(u^n)\frac{\overline{\mathbf{z}}^n - \overline{\mathbf{z}}^{n-1}}{\Delta t^n}, \overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right) \\
&\quad = \left(a(U^n)\frac{\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n - \overline{\boldsymbol{\xi}}_{\mathbf{z}}^{n-1}}{\Delta t^n}, \overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right) - \left(a(U^n)\frac{\overline{\boldsymbol{\beta}}_{\mathbf{z}}^n - \overline{\boldsymbol{\beta}}_{\mathbf{z}}^{n-1}}{\Delta t^n}, \overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right) + \left([a(U^n) - a(u^n)]\frac{\overline{\mathbf{z}}^n - \overline{\mathbf{z}}^{n-1}}{\Delta t^n}, \overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right),
\end{aligned} \tag{4.6}$$

(III)

$$\begin{aligned}
&\left(U^n \overline{\mathbf{c}}(U^n) \cdot \overline{\mathbf{Z}}^n - u^n \overline{\mathbf{c}}(u^n) \cdot \overline{\mathbf{z}}^n, \xi_u^n\right) \\
&\quad = \left([\xi_u^n - \beta_u^n]\overline{\mathbf{c}}(U^n) \cdot \overline{\mathbf{Z}}^n, \xi_u^n\right) + \left(u^n \overline{\mathbf{c}}(U^n) \cdot \left[\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n - \overline{\boldsymbol{\beta}}_{\mathbf{z}}^n\right], \xi_u^n\right) + \left(u^n[\overline{\mathbf{c}}(U^n) - \overline{\mathbf{c}}(u^n)] \cdot \overline{\mathbf{z}}^n, \xi_u^n\right).
\end{aligned} \tag{4.7}$$

The last second term in (4.6) is related to the changing meshes. If the meshes do not change, this term will be zero.

Substitute (4.5)–(4.7) into (4.4) to get

$$\begin{aligned}
&\left(\frac{\xi_u^n-\xi_u^{n-1}}{\Delta t^n},\xi_u^n\right)+\left(b(U^n)\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n,\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right)+\left(a(U^n)\frac{\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n-\overline{\boldsymbol{\xi}}_{\mathbf{z}}^{n-1}}{\Delta t^n},\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right)\\
&\quad=\left(f(U^n)-f(u^n),\xi_u^n\right)+\left(\rho^n,\xi_u^n\right)+\left(\nabla\cdot\left(\tilde{\mathbf{g}}^n-\mathbf{G}^{n-1}\right),\xi_u^n\right)+\left(\mathbf{r}^n,\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right)\\
&\qquad-\left(\boldsymbol{\beta}_{\mathbf{z}}^n,\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right)-\left(\left[\xi_u^n-\beta_u^n\right]\overline{\mathbf{c}}(U^n)\cdot\overline{\mathbf{Z}}^n,\xi_u^n\right)-\left(u^n\overline{\mathbf{c}}(U^n)\cdot\left[\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n-\overline{\boldsymbol{\beta}}_{\mathbf{z}}^n\right],\xi_u^n\right)\\
&\qquad-\left(u^n[\overline{\mathbf{c}}(U^n)-\overline{\mathbf{c}}(u^n)]\cdot\overline{\mathbf{z}}^n,\xi_u^n\right)+\left(\frac{\beta_u^n-\beta_u^{n-1}}{\Delta t^n},\xi_u^n\right)+\left(b(U^n)\overline{\boldsymbol{\beta}}_{\mathbf{z}}^n,\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right)\\
&\qquad+\left([b(u^n)-b(U^n)]\overline{\mathbf{z}}^n,\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right)+\left([a(u^n)-a(U^n)]\frac{\overline{\mathbf{z}}^n-\overline{\mathbf{z}}^{n-1}}{\Delta t^n},\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right)\\
&\qquad+\left(a(U^n)\frac{\overline{\boldsymbol{\beta}}_z^n-\overline{\boldsymbol{\beta}}_z^{n-1}}{\Delta t^n},\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right)=T_1+T_2+\cdots+T_{13}.
\end{aligned}\tag{4.8}$$

Furthermore, for the first and third terms on the left-hand side of (4.8), we have

$$\begin{aligned}
&\left(\frac{\xi_u^n-\xi_u^{n-1}}{\Delta t^n},\xi_u^n\right)=\frac{1}{2\Delta t^n}\left[\left(\xi_u^n,\xi_u^n\right)-\left(\xi_u^{n-1},\xi_u^{n-1}\right)\right]+\frac{1}{2\Delta t^n}\left\|\xi_u^n-\xi_u^{n-1}\right\|^2,\\
&\left(a(U^n)\frac{\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n-\overline{\boldsymbol{\xi}}_{\mathbf{z}}^{n-1}}{\Delta t^n},\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right)\geq\frac{1}{2\Delta t^n}\left[\left(a(U^n)\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n,\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right)-\left(a\left(U^{n-1}\right)\overline{\boldsymbol{\xi}}_{\mathbf{z}}^{n-1},\overline{\boldsymbol{\xi}}_{\mathbf{z}}^{n-1}\right)\right]\\
&\qquad-\frac{a_2}{2}\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^{n-1}\right\|^2+\frac{a_0}{2\Delta t^n}\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n-\overline{\boldsymbol{\xi}}_{\mathbf{z}}^{n-1}\right\|^2.
\end{aligned}\tag{4.9}$$

By (4.9), we modify (4.8) as

$$\begin{aligned}
&\frac{1}{2\Delta t^n}\left[\left(\xi_u^n,\xi_u^n\right)-\left(\xi_u^{n-1},\xi_u^{n-1}\right)\right]+\frac{1}{2\Delta t^n}\left\|\xi_u^n-\xi_u^{n-1}\right\|^2+\left(b(U^n)\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n,\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right)\\
&\quad+\frac{1}{2\Delta t^n}\left[\left(a(U^n)\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n,\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right)-\left(a\left(U^{n-1}\right)\overline{\boldsymbol{\xi}}_{\mathbf{z}}^{n-1},\overline{\boldsymbol{\xi}}_{\mathbf{z}}^{n-1}\right)\right]+\frac{a_0}{2\Delta t^n}\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n-\overline{\boldsymbol{\xi}}_{\mathbf{z}}^{n-1}\right\|^2\\
&\quad\leq\frac{a_2}{2}\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^{n-1}\right\|^2+T_1+T_2+\cdots+T_{13}.
\end{aligned}\tag{4.10}$$

By the Lipschitz continuity of $f(u)$, $a(u)$, $b(u)$, $\partial \mathbf{c}/\partial u$, and (3.3), the following terms on the right-hand side of (4.10) can be obtained:

$$
\begin{aligned}
T_1 &\le K\|U^n - u^n\|^2 + K\|\xi_u^n\|^2 \le K\|\xi_u^n\|^2 + Kh^2,\\
T_2 &\le K\Delta t^n \left\| \frac{\partial^2 u}{\partial t^2} \right\|^2_{L^2(t^{n-1},t^n;H^1)} + K\|\xi_u^n\|^2,\\
T_4 &\le K\Delta t^n \left\| \frac{\partial^2 u}{\partial t^2} \right\|^2_{L^2(t^{n-1},t^n;H^1)} + K\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right\|^2,\\
T_5 &\le K\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right\|^2 + Kh^2,\\
T_6 &\le K\|\xi_u^n\|^2 + Kh^2,\\
T_7 &\le K\|\xi_u^n\|^2 + K\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right\|^2 + Kh^2,\\
T_8 &\le K\|U^n - u^n\|^2 + K\|\xi_u^n\|^2 \le K\|\xi_u^n\|^2 + Kh^2,\\
T_{10} &\le K(b_1)h^2 + K\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right\|^2,\\
T_{11} &\le K\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right\|^2 + K\|\xi_u^n\|^2 + Kh^2,\\
T_{12} &\le K\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right\|^2 + K\|U^n - u^n\|^2 + K\left\| \frac{\overline{\mathbf{z}}^n - \overline{\mathbf{z}}^{n-1}}{\Delta t^n} \right\|^2\\
&\le K\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right\|^2 + K\|\xi_u^n\|^2 + K(\Delta t^n)^{-1}\left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(t^{n-1},t^n;H^1)} + Kh^2.
\end{aligned}
\tag{4.11}
$$

We turn to consider T_3. Letting $\mathbf{v} = \widetilde{\mathbf{g}}^n - \mathbf{G}^{n-1}$ in the second equation of (4.10), then we have

$$
\left(\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n, \widetilde{\mathbf{g}}^n - \mathbf{G}^{n-1}\right) = \left(\xi_u^n, \nabla \cdot \left(\widetilde{\mathbf{g}}^n - \mathbf{G}^{n-1}\right)\right). \tag{4.12}
$$

By Lemma 3.1, we have

$$
\begin{aligned}
T_3 &\le \frac{1}{2}\left(b(U^n)\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n, \overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right) + K\left(b_0^{-1}\right)\left\|\widetilde{\mathbf{g}}^n - \mathbf{G}^{n-1}\right\|^2\\
&\le \frac{1}{2}\left(b(U^n)\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n, \overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right) + K\left(b_0^{-1}\right)\left\|\widetilde{\mathbf{g}}^n - \widetilde{\mathbf{g}}^{n-1}\right\|^2 + \left\|\widetilde{\mathbf{g}}^{n-1} - \mathbf{G}^{n-1}\right\|^2\\
&\le \frac{1}{2}\left(b(U^n)\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n, \overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right) + K\left(b_0^{-1}\right)\left(\left\|\xi_u^{n-1}\right\|^2 + h^2 + \Delta t^2\right).
\end{aligned}
\tag{4.13}
$$

Substituting (4.11) and (4.13) into (4.10), we have

$$\begin{aligned}
&\frac{1}{2\Delta t^n}\left[(\xi_u^n,\xi_u^n)-\left(\xi_u^{n-1},\xi_u^{n-1}\right)\right]+\frac{1}{2\Delta t^n}\left\|\xi_u^n-\xi_u^{n-1}\right\|^2+\frac{1}{2}\left(b(U^n)\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n,\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right)\\
&\quad+\frac{a_0}{2\Delta t^n}\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n-\overline{\boldsymbol{\xi}}_{\mathbf{z}}^{n-1}\right\|^2+\frac{1}{2\Delta t^n}\left[\left(a^n\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n,\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right)-\left(a^{n-1}\overline{\boldsymbol{\xi}}_{\mathbf{z}}^{n-1},\overline{\boldsymbol{\xi}}_{\mathbf{z}}^{n-1}\right)\right]\\
&\quad\le K\left(\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right\|^2+\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^{n-1}\right\|^2\right)+K\left(\|\xi_u^n\|^2+\left\|\xi_u^{n-1}\right\|^2+\Delta t^2+h^2\right)\\
&\quad+K(\Delta t^n)\left(\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(t^{n-1},t^n;H^1)}+\left\|\frac{\partial^2 u}{\partial t^2}\right\|^2_{L^2(t^{n-1},t^n;H^1)}\right)+T_9+T_{13}.
\end{aligned}\tag{4.14}$$

Let N be the time step at which $\|\xi_u^n\|$ is maximum, that is,

$$\left\|\xi_u^N\right\|^2=\max_{1\le n\le N^*}\|\xi_u^n\|^2.\tag{4.15}$$

Multiplying (4.14) by $2\Delta t^n$ and summing on n from 1 to N, we obtain

$$\begin{aligned}
&\left\|\xi_u^N\right\|^2+\sum_{n=1}^N\left\|\xi_u^n-\xi_u^{n-1}\right\|^2+\sum_{n=1}^N\left(b(U^n)\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n,\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right)\Delta t^n+a_0\sum_{n=1}^N\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n-\overline{\boldsymbol{\xi}}_{\mathbf{z}}^{n-1}\right\|^2+a_0\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^N\right\|^2\\
&\quad\le K\left(\Delta t^2+h^2\right)+K\sum_{n=1}^N\|\xi_u^n\|^2\Delta t^n+\left\|\xi_u^0\right\|^2+a_1\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^0\right\|^2+K\sum_{n=1}^N\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right\|^2\Delta t^n\\
&\quad+\sum_{n=1}^N\left(\frac{\beta_u^n-\beta_u^{n-1}}{\Delta t^n},\xi_u^n\right)\Delta t^n+\sum_{n=1}^N\left(a(U^n)\frac{\overline{\boldsymbol{\beta}}_{\mathbf{z}}^n-\overline{\boldsymbol{\beta}}_{\mathbf{z}}^{n-1}}{\Delta t^n},\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right)\Delta t^n.
\end{aligned}\tag{4.16}$$

Assuming that the mesh is modified at most M times, and $M\le M^*$, where M^* is independent of h and Δt, we get [3]:

$$\begin{aligned}
&\sum_{n=1}^N\left(\frac{\beta_u^n-\beta_u^{n-1}}{\Delta t^n},\xi_u^n\right)\Delta t^n=\sum_{n=1}^N\left[(\beta_u^n,\xi_u^n)-\left(\beta_u^{n-1},\xi_u^n\right)\right]\le K(M^*h)^2+\frac{1}{4}\left\|\xi_u^N\right\|^2,\\
&\sum_{n=1}^N\left(a(U^n)\frac{\overline{\boldsymbol{\beta}}_{\mathbf{z}}^n-\overline{\boldsymbol{\beta}}_{\mathbf{z}}^{n-1}}{\Delta t^n},\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right)\Delta t^n\le Kh^2+\varepsilon\sum_{n=1}^N\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right\|^2\Delta t^n,
\end{aligned}\tag{4.17}$$

where ε is a sufficiently small positive constant.

Substituting (4.17) into (4.16), we find

$$\begin{aligned}
&\left\|\xi_u^N\right\|^2 + \sum_{n=1}^{N}\left\|\xi_u^n - \xi_u^{n-1}\right\|^2 + \sum_{n=1}^{N}\left(b(U^n)\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n, \overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right)\Delta t^n + a_0\sum_{n=1}^{N}\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n - \overline{\boldsymbol{\xi}}_{\mathbf{z}}^{n-1}\right\|^2 + a_0\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^N\right\|^2 \\
&\quad \le K\left(\Delta t^2 + h^2 + (M^*h)^2\right) + \left\|\xi_u^0\right\|^2 + a_1\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^0\right\|^2 + K\sum_{n=1}^{N}\|\xi_u^n\|^2\Delta t^n \\
&\quad + K\sum_{n=1}^{N}\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right\|^2\Delta t^n + \varepsilon\sum_{n=1}^{N}\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right\|^2\Delta t^n + \frac{1}{4}\left\|\xi_u^N\right\|^2.
\end{aligned} \tag{4.18}$$

Using the discrete Gronwall's lemma, we have

$$\max_n\|\xi_u^n\|^2 \le K\left(h^2 + \Delta t^2 + (M^*h)^2\right). \tag{4.19}$$

Similarly as (4.14), we can derive

$$\max_n\left\|\overline{\boldsymbol{\xi}}_{\mathbf{z}}^n\right\|^2 \le K\left(h^2 + \Delta t^2 + (M^*h)^2\right). \tag{4.20}$$

By the triangle inequality and (3.3), we can obtain the following result.

Theorem 4.1. *Assuming that the coefficients satisfy condition* (1.2), *the mesh is modified at most M times, $M \le M^*$, and $u \in L^2(H^1)$, $u_t \in L^2(H^1)$, $u_{tt} \in L^2(H^1)$, then we have*

$$\begin{aligned}
\max_n\|u^n - U^n\| &\le K(h + \Delta t) + K(M^*)h, \\
\max_n\left\|\overline{\mathbf{z}}^n - \overline{\mathbf{Z}}^n\right\| &\le K(h + \Delta t) + K(M^*)h,
\end{aligned} \tag{4.21}$$

where K and M^* are positive constants independent of h and Δt.

5. Numerical Example

In this section, we give numerical experiment of the following nonlinear equations to illustrate the efficiency of the Godunov-mixed finite element method on changing meshes:

$$\begin{aligned}
u_t = (a(x,t)u_{xt} + b(x,t,u)u_x)_x - c(x,t)u_x + f(x,t,u), \quad (x,t) \in (0,1)\times(0,1], \\
u(0,t) = u(1,t) = 0, \quad t \in [0,1], \\
u(\cdot,0) = \sin\pi x, \quad x \in [0,1],
\end{aligned} \tag{5.1}$$

where $(x,t) \in [0,1]\times[0,1]$, $a(x,t) = t^2(x+0.25) + 0.25$, $b(x,t,u) = 0.005ut^2(x+0.05)$, $c(x,t) = 0.05t(x+1) + 1$, and $f(x,t,u)$ is chosen properly so that the exact solution is $u = e^{-t}\sin\pi x$.

Table 1: L^2-norm estimates of $U^n - u^n$.

Time	Case 1	Case 2	Case 3	Case 4
$t = 0.4$	0.0454	0.0232	0.0117	0.0057
$t = 0.6$	0.0253	0.0135	0.0064	0.0032
$t = 1.0$	0.0288	0.0160	0.0080	0.0039

Table 2: L^2-norm estimates of $\overline{\mathbf{Z}}^n - \overline{\mathbf{z}}^n$.

Time	Case 1	Case 2	Case 3	Case 4
$t = 0.4$	0.0537	0.0307	0.0161	0.0069
$t = 0.6$	0.0646	0.0353	0.0164	0.0080
$t = 1.0$	0.0643	0.0395	0.0198	0.0099

Table 3: Convergence rate of L^2-norm.

Time	Convergence rates of $\|u^n - U^n\|$			Convergence rates of $\|\overline{\mathbf{z}}^n - \overline{\mathbf{Z}}^n\|$		
	Case 1/Case 2	Case 2/Case 3	Case 3/Case 4	Case 1/Case 2	Case 2/Case 3	Case 3/Case 4
$t = 0.4$	0.9670	0.9852	1.0392	0.8052	0.9271	1.2167
$t = 0.6$	0.9023	1.0724	0.9974	0.8692	1.1121	1.0290
$t = 1.0$	0.8464	1.0075	1.0242	0.7009	0.9963	1.0007

In the numerical example, h and Δt change as the following four cases.

Case 1. $t \in [0, 0.4]$, set $h = \Delta t = 0.1$; $t \in (0.4, 0.6]$, set $h = \Delta t = 0.05$; $t \in (0.6, 1.0]$, set $h = \Delta t = 0.1$ and calculate 40 steps at every time interval.

Case 2. $t \in [0, 0.4]$, set $h = \Delta t = 0.05$; $t \in (0.4, 0.6]$, set $h = \Delta t = 0.025$; $t \in (0.6, 1.0]$, set $h = \Delta t = 0.05$ and calculate 80 steps at every time interval.

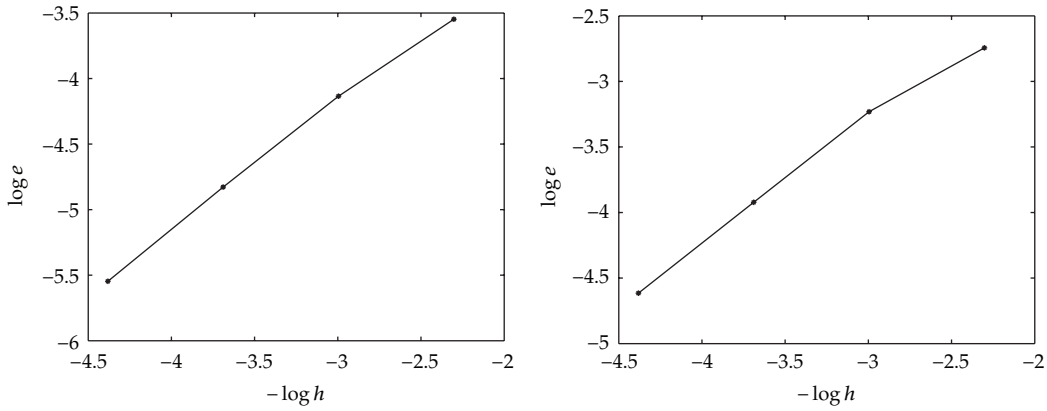
Case 3. $t \in [0, 0.4]$, set $h = \Delta t = 0.025$; $t \in (0.4, 0.6]$, set $h = \Delta t = 0.0125$; $t \in (0.6, 1.0]$, set $h = \Delta t = 0.025$ and calculate 160 steps at every time interval.

Case 4. $t \in [0, 0.4]$, set $h = \Delta t = 0.0125$; $t \in (0.4, 0.6]$, set $h = \Delta t = 0.00625$; $t \in (0.6, 1.0]$, set $h = \Delta t = 0.0125$ and calculate 320 steps at every time interval.

The numerical solutions $U^n, \overline{\mathbf{Z}}^n$ are calculated and the L^2 error estimates of $U^n - u^n, \overline{\mathbf{Z}}^n - \overline{\mathbf{z}}^n$ are obtained; see Tables 1 and 2.

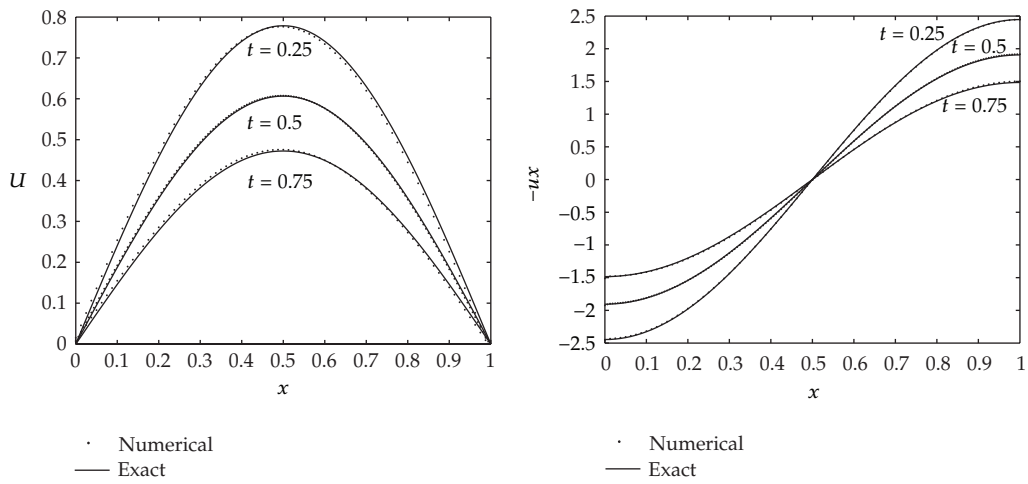
In Table 3, convergence rates of $U^n - u^n, \overline{\mathbf{Z}}^n - \overline{\mathbf{z}}^n$ are given. The figures of convergence rate are shown by Figure 1. The convergence rate of $U^n - u^n$ in Figure 1(a) is one order, and in Figure 1(b) the convergence rate of $\overline{\mathbf{Z}}^n - \overline{\mathbf{z}}^n$ is a little smaller than one order at the beginning. But the convergence rate $\overline{\mathbf{Z}}^n - \overline{\mathbf{z}}^n$ will be close to one order when h minishes, which is consistent with the analysis in this paper.

For Case 4, we compare the exact solution $\{u, \tilde{\mathbf{z}}\}$ with the approximate solution $\{U, \tilde{\mathbf{Z}}\}$ at time $t = 0.25$, 0.5, and 0.75, respectively; see Figure 2. From Figures 2(a) and 2(b), we can see that the approximate solutions are very close to the exact solutions.



(a) Convergence rates of u (b) Convergence rates of $\tilde{\mathbf{z}}$

Figure 1: Convergence rate figures.



(a) Comparing u with U (b) Comparing $\tilde{\mathbf{z}}$ with $\tilde{\mathbf{Z}}$

Figure 2: Compare figures for Case 4 at time $t = 0.25$, 0.5, and 0.75.

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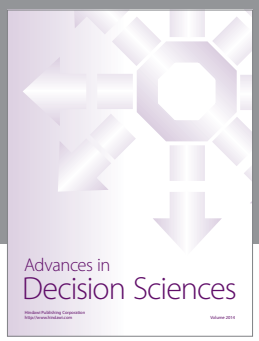

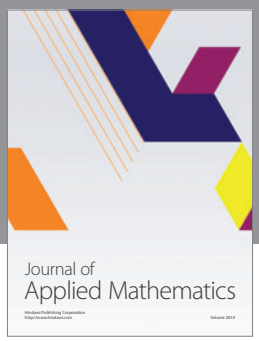

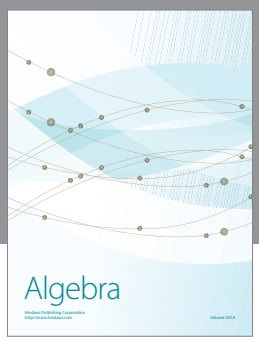

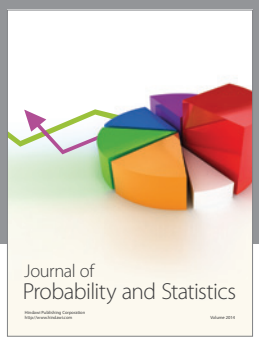

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