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Research Article

Convergence Theorems for Infinite Family of Multivalued Quasi-Nonexpansive Mappings in Uniformly Convex Banach Spaces

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We introduce an iterative method for finding a common fixed point of a countable family of multivalued quasi-nonexpansive mapping $\{T_i\}$ in a uniformly convex Banach space. We prove that under certain control conditions, the iterative sequence generated by our method is an approximating fixed point sequence of each T_i. Some strong convergence theorems of the proposed method are also obtained for the following cases: all T_i are continuous and one of T_i is hemicompact, and the domain K is compact.

1. Introduction

Let X be a real Banach space. A subset K of X is called *proximinal* if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x,k) = d(x,K), \tag{1.1}$$

where $d(x,K) = \inf\{\|x-y\| : y \in K\}$ is the distance from the point x to the set K. It is clear that every closed convex subset of a uniformly convex Banach space is proximinal.

Let X be a uniformly convex real Banach space, K be a nonempty closed convex subset of X, $CB(K)$ be a family of nonempty closed bounded subsets of K, and $P(K)$ be

a nonempty proximinal bounded subsets of K. The *Hausdorff metric* on $CB(X)$ is defined by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}, \tag{1.2}$$

for all $A, B \in CB(X)$.

An element $p \in K$ is called a *fixed point* of a single valued mapping T if $p = Tp$ and of a multivalued mapping T if $p \in Tp$. The set of fixed points of T is denoted by $F(T)$.

A single valued mapping $T : K \to K$ is said to be *quasi-nonexpansive* if $\|Tx - p\| \leq \|x - p\|$ for all $x \in K$ and $p \in F(T)$.

A multivalued mapping $T : K \to CB(K)$ is said to be:

(i) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq \|x - p\|$ for all $x \in K$ and $p \in F(T)$,

(ii) *nonexpansive* if $H(Tx, Ty) \leq \|x - y\|$ for all $x, y \in K$.

It is well known that every nonexpansive multivalued mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive. But there exist quasi-nonexpansive mappings that are not nonexpansive. It is clear that if T is a quasi-nonexpansive multivalued mapping, then $F(T)$ is closed.

A map $T : K \to CB(K)$ is called *hemicompact* if, for any sequence $\{x_n\}$ in K such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$. We note that if K is compact, then every multivalued mapping T is hemicompact.

In 1969, Nadler [1] proved a fixed point theorem for multivalued contraction mappings and convergence of a sequence. They extended theorems on the stability of fixed points of single-valued mappings and also given a counterexample to a theorem about (ε, λ)-uniformly locally expansive (single-valued) mappings. Later in 1997, Hu et al. [2] obtained common fixed point of two nonexpansive multivalued mappings satisfying certain contractive condition.

In 2005, Sastry and Babu [3] proved the convergence theorem of multivalued maps by defining Ishikawa and Mann iterates and gave example which shows that the limit of the sequence of Ishikawa iterates depends on the choice of the fixed point p and the initial choice of x_0. In 2007, there is paper which generalized results of Sastry and Babu [3] to uniformly convex Banach spaces by Panyanak [4] and proved a convergence theorem of Mann iterates for a mapping defined on a noncompact domain.

Later in 2008, Song and Wang [5] shown that strong convergence for Mann and Ishikawa iterates of multivalued nonexpansive mapping T under some appropriate conditions. In 2009, Shahzad and Zegeye [6] proved strong convergence theorems of quasi-nonexpansive multivalued mapping for the Ishikawa iteration. They also constructed an iteration scheme which removes the restriction of T with $Tp = \{p\}$ for any $p \in F(T)$ which relaxed compactness of the domain of T.

Recently, Abbas et al. [7] established weak- and strong-convergence theorems of two multivalued nonexpansive mappings in a real uniformly convex Banach space by one-step iterative process to approximate common fixed points under some basic boundary conditions.

A fixed points and common fixed points theorem of multivalued maps in uniformly convex Banach space or in complete metric spaces or in convex metric spaces have been intensively studied by many authors; for instance, see [8–23].

In this paper, we generalize and modify the iteration of Abbas et al. [7] from two mapping to the infinite family mappings $\{T_i : i \in \mathbb{N}\}$ of multivalued quasi-nonexpansive mapping in a uniformly convex Banach space.

Let $\{T_i\}$ be a countable family of multivalued quasi-nonexpansive mapping from a bounded and closed convex subset K of a Banach space into $P(K)$ with $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and let $p \in F$. For $x_1 \in K$, we define

$$x_{n+1} = \alpha_{n,0} x_n + \sum_{i=1}^{\infty} \alpha_{n,i} x_{n,i}, \tag{1.3}$$

where the sequences $\{\alpha_{n,i}\} \subset [0,1)$ satisfying $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ and $x_{n,i} \in T_i x_n$ such that $d(p, x_{n,i}) = d(p, T_i x_n)$ for $i \in \mathbb{N}$. The main purpose of this paper is to prove strong convergence of the iterative scheme (1.3) to a common fixed point of T_i.

2. Preliminaries

Before to say the main theorem, we need the following lemmas.

Lemma 2.1 (see [24]). *Suppose that X is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of X such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$, and $\lim_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = r$ hold for some $r \geq 0$. Then $\limsup_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.2 (see [25]). *Let E be a uniformly convex Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : \|x\| \leq r\}$. Then, for any given sequence $\{x_n\}_{n=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers such that $\sum_{i=1}^{\infty} \lambda_i = 1$, there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \longrightarrow \mathbb{R}, \quad g(0) = 0, \tag{2.1}$$

such that for any positive integers i, j with $i < j$, the following inequality holds:

$$\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\|^2 \leq \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|). \tag{2.2}$$

3. Main Results

We first prove that the sequence $\{x_n\}$ generated by (1.3) is an approximating fixed point sequence of each T_i $(i \in \mathbb{N})$.

Lemma 3.1. *Let K be a nonempty bounded and closed convex subset of a uniformly convex Banach space X. For $i \in \mathbb{N}$, let $\{T_i\}$ be a sequence of multivalued quasi-nonexpansive mappings from K into $P(K)$ with $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $p \in F$. Let $\{x_n\}$ be a sequence defined by* (1.3). *Then*

(i) $\|x_{n+1} - p\| \leq \|x_n - p\|$,

(ii) $\lim_{n\to\infty} \|x_n - p\|$ *exists.*

Proof. By (1.3), we have

$$\begin{aligned}
\|x_{n+1}-p\| &\le \alpha_{n,0}\|x_n-p\| + \sum_{i=1}^{\infty}\alpha_{n,i}\|x_{n,i}-p\| \\
&= \alpha_{n,0}\|x_n-p\| + \sum_{i=1}^{\infty}\alpha_{n,i}d(T_ix_n,p) \\
&\le \alpha_{n,0}\|x_n-p\| + \sum_{i=1}^{\infty}\alpha_{n,i}H(T_ix_n,T_ip) \\
&\le \alpha_{n,0}\|x_n-p\| + \sum_{i=1}^{\infty}\alpha_{n,i}\|x_n-p\| \\
&= \|x_n-p\|.
\end{aligned} \tag{3.1}$$

So (i) is obtained. (ii) follows from (i). □

Theorem 3.2. *Let K be a nonempty bounded and closed convex subset of a uniformly convex Banach space X. For $i \in \mathbb{N}$, let $\{T_i\}$ be a sequence of multivalued quasi-nonexpansive mappings from K into $P(K)$ with $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $p \in F$. Let $\{x_n\}$ be a sequence defined by* (1.3) *with* $\liminf_{n\to\infty}\alpha_{n,0}\alpha_{n,j} > 0$ *for all $j \in \mathbb{N}$. Then* $\lim_{n\to\infty} d(x_n, T_ix_n) = 0$ *for all $i \in \mathbb{N}$.*

Proof. For $j \in \mathbb{N}$, by Lemma 2.2, we get

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \left\|\alpha_{n,0}(x_n-p) + \sum_{i=1}^{\infty}\alpha_{n,i}(x_{n,i}-p)\right\|^2 \\
&\le \alpha_{n,0}\|x_n-p\|^2 + \sum_{i=1}^{\infty}\alpha_{n,i}\|x_{n,i}-p\|^2 - \alpha_{n,0}\alpha_{n,j}g(\|x_n-x_{n,j}\|) \\
&= \alpha_{n,0}\|x_n-p\|^2 + \sum_{i=1}^{\infty}\alpha_{n,i}(d(T_ix_n,p))^2 - \alpha_{n,0}\alpha_{n,j}g(\|x_n-x_{n,j}\|) \\
&\le \alpha_{n,0}\|x_n-p\|^2 + \sum_{i=1}^{\infty}\alpha_{n,i}(H(T_ix_n,T_ip))^2 - \alpha_{n,0}\alpha_{n,j}g(\|x_n-x_{n,j}\|) \\
&\le \alpha_{n,0}\|x_n-p\|^2 + \sum_{i=1}^{\infty}\alpha_{n,i}\|x_n-p\|^2 - \alpha_{n,0}\alpha_{n,j}g(\|x_n-x_{n,j}\|) \\
&= \|x_n-p\|^2 - \alpha_{n,0}\alpha_{n,j}g(\|x_n-x_{n,j}\|).
\end{aligned} \tag{3.2}$$

Thus, $0 < \alpha_{n,0}\alpha_{n,j}g(\|x_n-x_{n,j}\|) \le \|x_n-p\|^2 - \|x_{n+1}-p\|^2$. It follows that $\lim_{n\to\infty} g(\|x_n-x_{n,j}\|) = 0$. By property of g, we have $\lim_{n\to\infty}\|x_n - x_{n,j}\| = 0$. Thus $\lim_{n\to\infty} d(x_n, T_jx_n) = 0$ for $i \in \mathbb{N}$. □

Theorem 3.3. *Let X be a uniformly convex real Banach space and K be a bounded and closed convex subset of X. For $i \in \mathbb{N}$, let $\{T_i\}$ be a sequence of multivalued quasi-nonexpansive and continuous mappings from K into $P(K)$ with $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $p \in F$. Let $\{x_n\}$ be a sequence defined by*

(1.3) *with* $\liminf_{n\to\infty}\alpha_{n,0}\alpha_{n,j} > 0$ *for all* $j \in \mathbb{N}$. *Assume that one of* T_i *is hemicompact. Then* $\{x_n\}$ *converges strongly to a common fixed point of* $\{T_i\}$.

Proof. Suppose that T_i is hemicompact for some $i_0 \in \mathbb{N}$. By Theorem 3.2, we have $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for all $i \in \mathbb{N}$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = q \in K$. From continuity of T_i, we get $d(x_{n_k}, T_i x_{n_k}) \to d(q, T_i q)$. This implies that $d(q, T_i q) = 0$ and $q \in F$. Since $\lim_{n\to\infty}\|x_n - q\|$ exists, it follows that $\{x_n\}$ converges strongly to q. □

Theorem 3.4. *Let* X *be a uniformly convex real Banach space and* K *be a compact convex subset of* X. *For* $i \in \mathbb{N}$, *let* $\{T_i\}$ *be a sequence of multivalued quasi-nonexpansive mappings from* K *into* $P(K)$ *with* $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ *and* $p \in F$. *Let* $\{x_n\}$ *be a sequence defined by* (1.3) *with* $\liminf_{n\to\infty}\alpha_{n,0}\alpha_{n,j} > 0$ *for all* $j \in \mathbb{N}$. *Then* $\{x_n\}$ *converges strongly to a common fixed point of* $\{T_i\}$.

Proof. From the compactness of K, there exists a subsequence $\{x_{n_k}\}_{n=k}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{k\to\infty}\|x_{n_k} - q\| = 0$ for some $q \in K$. Thus, it follows by Theorem 3.2 that,

$$\begin{aligned} d(q, T_i q) &\leq d(q, x_{n_k}) + d(x_{n_k}, T_i x_{n_k}) + \mathrm{H}(T_i x_{n_k}, T_i q) \\ &\leq 2\|x_{n_k} - q\| + d(x_{n_k}, T_i x_{n_k}) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \end{aligned} \tag{3.3}$$

Hence $q \in F$. By Lemma 3.1(ii), $\lim_{n\to\infty}\|x_n - q\|$ exists. Hence $\lim_{n\to\infty} x_n = q$. The proof is complete. □

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