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Research Article

A Multiplayer Pursuit Differential Game on a Closed Convex Set with Integral Constraints

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We study a simple motion pursuit differential game of many pursuers and many evaders on a nonempty convex subset of $\mathbb{R}^n$. In process of the game, all players must not leave the given set. Control functions of players are subjected to integral constraints. Pursuit is said to be completed if the position of each evader y_j, $j \in \{1, 2, ...k\}$, coincides with the position of a pursuer x_i, $i \in \{1, ..., m\}$, at some time t_j, that is, $x_i(t_j) = y_j(t_j)$. We show that if the total resource of the pursuers is greater than that of the evaders, then pursuit can be completed. Moreover, we construct strategies for the pursuers. According to these strategies, we define a finite number of time intervals $[\theta_{i-1}, \theta_i]$ and on each interval only one of the pursuers pursues an evader, and other pursuers do not move. We derive inequalities for the resources of these pursuer and evader and, moreover, show that the total resource of the pursuers remains greater than that of the evaders.

1. Related Work

Linear differential games with integral constraints on controls were examined in many works, for example, [1–16].

Satimov et al. [10] studied a linear pursuit differential game of many pursuers and one evader with integral constraints on controls of players in $\mathbb{R}^n$. Game is described by the following equations:

$$\dot{z}_i = C_i z_i + u_i - v, \qquad z_i(t_0) = z_i^0, \quad i = 1, \ldots, m, \tag{1.1}$$

where u_i is the control parameter of the ith pursuer and v is that of the evader. The eigenvalues of the matrices C_i are assumed to be real numbers. They proved that if the total

resource of controls of the pursuers is greater than that of the evader, then under certain conditions pursuit can be completed.

Ibragimov [15] examined a pursuit differential game of m pursuers and k evaders with integral constraints described by the following systems of differential equations

$$\begin{gathered}\dot{z}_{ij} = C_{ij}z_{ij} + u_i - v_j, \qquad z_{ij}(t_0) = z_{ij}^0, \quad i = 1,\dots,m,\ j = 1,\dots,k,\\ \int_0^\infty |u_i(s)|^2 ds \le \rho_i{}^2, \quad i = 1,\dots,m, \qquad \int_0^\infty \left|v_j(s)\right|^2 ds \le \sigma_j{}^2, \quad j = 1,\dots,k,\end{gathered} \tag{1.2}$$

where u_i is the control parameter of the ith pursuer and v_j is that of the jth evader. Different from the previous work, here eigenvalues of matrices C_{ij} are not necessarily real, and, moreover, the number of evaders can be any. If the total resource of controls of the pursuers is greater than that of the evaders, that is,

$$\rho_1^2 + \rho_2^2 + \cdots + \rho_m^2 > \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_k^2, \tag{1.3}$$

and real parts of all eigenvalues of the matrices C_{ij} are nonpositive, then it was proved that pursuit can be completed from any initial position. Here, game was considered in $\mathbb{R}^n$ without any state constraint.

Ivanov [17] considered generalized Lion and Man problem in the case of geometric constraints. All players have equal dynamic possibilities. Motions of the players are described by the following equations:

$$\dot{x}_i = u_i, \qquad x_i(0) = x_{i0}, \qquad |u_i| \le 1,\ i = 0,1,\dots,m, \tag{1.4}$$

where $x_i \in \mathbb{R}^n$, u_i, $i = 1,\dots,m$, are control parameters of the pursuers and u_0 is control parameter of the evader. During the game, all players may not leave a given compact subset N of $\mathbb{R}^n$. It was shown that if the number of pursuers m does not exceed the dimension of the space n, then evasion is possible; otherwise pursuit can be completed. In other words, Ivanov [17] derived necessary and sufficient condition of evasion for the multiple Lion and Man game in $\mathbb{R}^n$.

Ibragimov [16] studied a differential game problem of one pursuer and one evader with integral constraints. Game occurs on a closed convex subset S of $\mathbb{R}^n$, and movements of the players are described by the following equations:

$$\begin{gathered}\dot{x} = \alpha(t)u, \qquad x(0) = x_0, \qquad \int_0^\infty |u(s)|^2 ds \le \rho^2,\\ \dot{y} = \alpha(t)v, \qquad y(0) = y_0, \qquad \int_0^\infty |v(s)|^2 ds \le \sigma^2.\end{gathered} \tag{1.5}$$

Evasion and pursuit problems were investigated. In the latter case, a formula for optimal pursuit time was found.

Leong and Ibragimov [14] studied simple motion pursuit differential game of m pursuers and one evader on a closed convex subset of the Hilbert space l_2. Control functions of

Table 1: Brief summary of obtained results.

Author(s)	Number of pursuers	Number of evaders	The set where the pursuit occurs	Constraint on controls
Satimov et al. (1983) [10]	m	1	$\mathbb{R}^n$	Integral
Ibragimov (2004) [15]	m	k	$\mathbb{R}^n$	Integral
Ivanov (1980) [17]	m	1	Closed convex subset N of $\mathbb{R}^n$	Geometric
Ibragimov (2002) [16]	1	1	Closed convex subset N of $\mathbb{R}^n$	Integral
Leong and Ibragimov (2008) [14]	m	1	Closed convex subset N of l_2	Integral
Present paper	m	k	Closed convex subset N of $\mathbb{R}^n$	Integral

the players are subjected to integral constraints. The total resource of the pursuers is assumed to be greater than that of the evader. Strategies of pursuers were constructed to complete the pursuit from any initial position.

In the present paper, we study a pursuit differential game of many pursuers and many evaders on a nonempty convex subset N of $\mathbb{R}^n$, $n \geq 2$. In process of the game, all players must not leave the set N. Control functions of the players are subjected to integral constraints. We will show that if the total resource of the pursuers is greater than that of the evaders, then pursuit can be completed. In Table 1, we can compare the cases studied in the works of the previous researches and the present paper.

2. Statement of the Problem

We consider a differential game described by the following equations:

$$\begin{aligned} \dot{x}_i &= \varphi(t)u_i, \qquad x_i(0) = x_{i0}, \qquad i = 1, \ldots, m, \\ \dot{y}_j &= \varphi(t)v_i, \qquad y_j(0) = y_{j0}, \qquad j = 1, \ldots, k, \end{aligned} \tag{2.1}$$

where x_i, u_i, y_j, $v_j \in \mathbb{R}^n$, u_i is control parameter of the pursuer x_i, $i = 1, \ldots, m$, v_j is that of the evader y_j, $j = 1, \ldots, k$, and $\varphi(t)$ is a scalar measurable function that satisfies the following conditions:

$$a(\tau) \triangleq \left(\int_0^\tau \varphi^2(t)dt \right)^{1/2} < \infty, \qquad \tau > 0, \qquad \lim_{\tau \to \infty} a(\tau) = \infty. \tag{2.2}$$

Definition 2.1. A measurable function $u_i(t) = (u_{i1}(t), \ldots, u_{in}(t))$, $t \geq 0$, is called admissible control of the pursuer x_i if

$$\int_0^\infty |u_i(s)|^2 ds \leq \rho_i^2, \tag{2.3}$$

where ρ_i, $i = 1, \ldots, m$, are given positive numbers. We denote the set of all admissible controls of the pursuer x_i by $S(\rho_i)$.

Definition 2.2. A measurable function $v_j(t) = (v_{j1}(t), \dots, v_{jk}(t))$, $t \geq 0$, is called an admissible control of the evader if

$$\int_0^\infty |v_j(s)|^2 ds \leq \sigma_j^2, \tag{2.4}$$

where σ_j, $j = 1, \dots, k$, are given positive numbers.

Definition 2.3. A Borel measurable function $U_i(x_i, y_1, \dots, y_k, v_1, \dots, v_k)$, $U_i : \mathbb{R}^{(2k+1)n} \to \mathbb{R}^n$, is called a strategy of the pursuer x_i if for any control of the evader $v(t)$, $t \geq 0$, the initial value problem

$$\begin{gathered} \dot{x}_i = \varphi(t) U_i(x_i, y_1, \dots, y_k, v_1(t), \dots, v_k(t)), \qquad x_i(0) = x_{i0}, \\ \dot{y}_j = \varphi(t) v_i(t), \qquad y_j(0) = y_{j0}, \quad j = 1, \dots, k, \end{gathered} \tag{2.5}$$

has a unique solution $(x_i(t), y_1(t), \dots, y_k(t))$ and the inequality

$$\int_0^\infty |U_i(x_i(s), y_1(s), \dots, y_k(s), v_1(s), \dots, v_k(s))|^2 ds \leq \rho_i^2 \tag{2.6}$$

holds.

Definition 2.4. We say that pursuit can be completed from the initial position $\{x_{10}, \dots, x_{m0}, y_{10}, \dots, y_{k0}\}$ for the time T in the game (2.1)–(2.4), if there exist strategies U_i, $i = 1, \dots, m$, of the pursuers such that for any controls $v_1 = v_1(\cdot), \dots, v_k = v_k(\cdot)$ of the evaders and numbers $j = 1, 2, \dots k$, the equality $x_i(t_j) = y_j(t_j)$ holds for some $i \in \{1, \dots, k\}$ at some time $t_j \in [0, T]$.

Given nonempty convex subset N of $\mathbb{R}^n$, according to the rule of the game all players must not leave the set N, that is, $x_{i0}, x_i(t), y_{j0}, y_j(t) \in N$, $t \geq 0$, $i = 1, \dots, m$, $j = 1, \dots, k$. This information describes a differential game of many players with integral constraints on control functions of players.

Problem 1. Find a sufficient condition of completing pursuit in the game (2.1)–(2.4).

3. The Main Result

Since the control parameters of the players can take the values of opposite signs, without loss of generality, we may assume that $\varphi(t) \geq 0$. Moreover, without loss of generality, we may assume that $\varphi(t)$ is not identically zero on every open interval. Otherwise, if for instance $\varphi(t) = 0, t \in (t_1, t_2)$, then instead of $\varphi(t)$ we consider

$$\varphi_1(t) = \begin{cases} \varphi(t), & 0 \leq t \leq t_1, \\ \varphi(t + t_2 - t_1), & t > t_1. \end{cases} \tag{3.1}$$

We now formulate the main result of the paper.

Theorem 3.1. *If*

$$\rho_1^2 + \rho_2^2 + \cdots + \rho_m^2 > \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_k^2, \tag{3.2}$$

then pursuit can be completed for a finite time T in the game (2.1)–(2.4) *from any initial position.*

Proof. (1^0) An auxiliary differential game: To prove this theorem, we first study an auxiliary differential game of one pursuer x and one evader y, which is described by the following equations:

$$\begin{aligned} \dot{x} &= \varphi(t)u, \qquad x(0) = x_0, \\ \dot{y} &= \varphi(t)v, \qquad y(0) = y_0, \end{aligned} \tag{3.3}$$

where $x, y \in \mathbb{R}^n$, u is control parameter of the pursuer x, and v is that of the evader y. Assume that $u = u(\cdot) \in S(\rho)$, $v = v(\cdot) \in S(\sigma)$. Pursuit is completed if $x(t') = y(t')$ at some time $t' \geq 0$. Here the players move in $\mathbb{R}^n$ without any state constraint.

Set

$$u(t) = \frac{\varphi(t)}{a^2(\theta)}(y_0 - x_0) + v(t), \quad t \geq 0, \tag{3.4}$$

where θ is an arbitrary fixed number satisfying

$$a^2(\theta) \geq \frac{|y_0 - x_0|^2}{(\rho - \sigma)^2}. \tag{3.5}$$

Let

$$\begin{gathered} \rho^2(t) = \rho^2 - \int_o^t |u(s)|^2 ds, \quad \sigma^2(t) = \sigma^2 - \int_o^t |v(s)|^2 ds, \\ K(\theta, x_0, y_0) = \frac{1}{a(\theta)}|y_0 - x_0|^2 + 2\sigma|y_0 - x_0|. \end{gathered} \tag{3.6}$$

We will now prove the following statement. □

Lemma 3.2. *Let the pursuer use the strategy* (3.4). *(i) If $\rho > \sigma$ then pursuit can be completed in the game* (3.3) *for the time θ, and, moreover,*

$$\rho^2(\theta) \geq \rho^2 - \sigma^2 - \frac{K(\theta, x_0, y_0)}{a(\theta)}. \tag{3.7}$$

(ii) If $\rho \le \sigma$, then either $x(\theta) = y(\theta)$ or

$$\sigma^2(\theta) \le \sigma^2 - \rho^2 + \frac{K(\theta, x_0, y_0)}{a(\theta)}. \tag{3.8}$$

Proof of the Lemma. Let $\rho > \sigma$. In this case, we show that the control (3.4) is admissible and ensures the equality $x(\theta) = y(\theta)$. Indeed, clearly

$$x(\theta) = x_0 + \int_0^\theta \varphi(t)\left(\frac{\varphi(t)}{a^2(\theta)}(y_0 - x_0) + v(t)\right)dt = y_0 + \int_0^\theta \varphi(t)v(t)dt = y(\theta). \tag{3.9}$$

To show the admissibility of the strategy (3.4), we use the Cauchy-Schwartz inequality as follows:

$$(y_0 - x_0)\int_0^\theta \varphi(t)v(t)dt \le |y_0 - x_0| \int_0^\theta \varphi(t)|v(t)|\, dt \le |y_0 - x_0| a(\theta)\sigma. \tag{3.10}$$

Then by this inequality we have

$$\begin{aligned}
\int_0^\theta |u(t)|^2 dt &= \frac{1}{a^2(\theta)}|y_0 - x_0|^2 + 2\frac{1}{a^2(\theta)}(y_0 - x_0)\int_0^\theta \varphi(t)v(t)dt + \int_0^\theta |v(t)|^2 dt \\
&\le \frac{1}{a^2(\theta)}|y_0 - x_0|^2 + \frac{2}{a(\theta)}|y_0 - x_0|\sigma + \int_0^\theta |v(t)|^2 dt \\
&= \frac{K(\theta, x_0, y_0)}{a(\theta)} + \int_0^\theta |v(t)|^2 dt.
\end{aligned} \tag{3.11}$$

According to (3.5) and the condition $v(\cdot) \in S(\sigma)$, the right-hand side of (3.11) is less than or equal to

$$(\sigma - \rho)^2 + 2(\rho - \sigma)\sigma + \sigma^2 = \rho^2. \tag{3.12}$$

At this point, we have proved admissibility of the control (3.4).

In particular, using (3.11) we obtain

$$\rho^2(\theta) = \rho^2 - \int_0^\theta |u(t)|^2 dt \ge \rho^2 - \sigma^2 - \frac{K(\theta, x_0, y_0)}{a(\theta)}. \tag{3.13}$$

Thus, (3.7) holds. We now turn to the part (ii) of the Lemma 3.2. Let $\rho \le \sigma$ and the pursuer use the strategy (3.4) on the interval $[0, \theta]$. If for a control of the evader $v(t)$, $t \in [0, \theta]$, the following inequality is satisfied:

$$\int_0^\theta |u(t)|^2 dt = \int_0^\theta \left|\frac{\varphi(t)}{a^2(\theta)}(y_0 - x_0) + v(t)\right|^2 dt \le \rho^2, \tag{3.14}$$

then, clearly, the control (3.4) is admissible, and similar to (3.9) we obtain that $x(\theta) = y(\theta)$ and the proof of the lemma follows. Hence, if $x(\theta) \neq y(\theta)$, then for the control (3.4) we must have

$$\int_0^\theta \left| \frac{\varphi(t)}{a^2(\theta)} (y_0 - x_0) + v(t) \right|^2 dt > \rho^2. \tag{3.15}$$

From this inequality by using calculations similar to (3.11), we then obtain

$$\int_0^\theta |v(t)|^2 dt > \rho^2 - \frac{K(\theta, x_0, y_0)}{a(\theta)}, \tag{3.16}$$

and hence

$$\sigma^2(\theta) = \sigma^2 - \int_0^\theta |v(t)|^2 dt < \sigma^2 - \rho^2 + \frac{K(\theta, x_0, y_0)}{a(\theta)}. \tag{3.17}$$

This completes the proof of the lemma. □

The last inequality can be interpreted as follows. Though $x(\theta) \neq y(\theta)$, the pursuer can force to expend the evader's energy more than $\rho^2 - (K(\theta, x_0, y_0)/a(\theta))$. At the same time, using the strategy (3.4), the pursuer will spend all his energy by a time τ

$$\int_0^\tau |u(t)|^2 dt = \rho^2, \qquad 0 < \tau < \theta. \tag{3.18}$$

Then, of course, the pursuer cannot move anymore and automatically $u(t) \equiv 0$, $t \geq \tau$.

(2^0) Fictitious pursuers (FP): We now prove the theorem. To this purpose, introduce fictitious pursuers $z_1, \ldots, z_m$ by equations

$$\dot{z}_i = \varphi(t) w_i, \qquad z_i(0) = x_{i0}, \qquad w_i \in S(\rho_i),\ i = 1, \ldots, m, \tag{3.19}$$

where w_i is control parameter of the pursuer z_i. FPs may go out of the set N. They move without any state constraint in $\mathbb{R}^n$. The aim of the FPs is to complete the pursuit as earlier as possible. Set

$$w_m(t) = \frac{\varphi(t)}{a^2(\theta_1)} (y_{k0} - x_{m0}) + v_k(t), \quad 0 \leq t \leq \theta_1, \tag{3.20}$$

$$w_i(t) \equiv 0, \quad i = 1, \ldots, m-1,\ 0 \leq t \leq \theta_1, \tag{3.21}$$

where θ_1 is any number satisfying inequalities

$$a^2(\theta_1) \geq \frac{|y_{k0} - x_{m0}|^2}{(\rho_m - \sigma_k)^2}, \tag{3.22}$$

$$\rho_1^2 + \rho_2^2 + \cdots + \rho_m^2 > \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_k^2 + \frac{K(\theta_1, x_{m0}, y_{k0})}{a(\theta_1)}. \tag{3.23}$$

Since by the assumption $a(t) \to \infty$ as $t \to \infty$, it follows from (3.2) that such θ_1 exists. Equations (3.20), (3.21) mean that all FPs $z_1, \ldots, z_{m-1}$ do not move on the time interval $[0, \theta_1]$, and only one FP z_m moves according to (3.20).

We will now show that if (3.2) holds and the FPs use the strategies (3.20), (3.21), then the pursuit problem with the pursuers $z_1, \ldots, z_m$ and evaders $y_1, \ldots, y_k$ is reduced to a pursuit problem with the pursuers $z_1, \ldots, z_p$ and evaders $y_1, \ldots, y_q$, for which $\rho_1^2(\theta_1) + \cdots + \rho_p^2(\theta_1) > \sigma_1^2(\theta_1) + \cdots + \sigma_q^2(\theta_1)$ and $p+q < m+k$. Hence, by the time θ_1, the number of pursuers is reduced to $p + q$.

Indeed, we consider two possible cases: (i) $\rho_m \leq \sigma_k$; (ii) $\rho_m > \sigma_k$. In the former case, that is, $\rho_m \leq \sigma_k$, if the equality $z_m(\theta_1) = y_k(\theta_1)$ holds, then we eliminate the pursuer z_m and the evader y_k and then consider the pursuit problem with pursuers $z_1, \ldots, z_{m-1}$ and evaders $y_1, \ldots, y_{k-1}$ under the condition

$$\rho_1^2 + \cdots + \rho_{m-1}^2 > \sigma_1^2 + \cdots + \sigma_{k-1}^2 \quad (p = m - 1,\ q = k - 1). \tag{3.24}$$

In case of $z_m(\theta_1) \neq y_k(\theta_1)$, according to (3.8), we obtain that

$$\sigma_k^2(\theta_1) \leq \sigma_k^2 - \rho_m^2 + \frac{K(\theta_1, x_{m0}, y_{k0})}{a(\theta_1)}. \tag{3.25}$$

Then in view of (3.23) we obtain

$$\rho_1^2 + \cdots + \rho_{m-1}^2 > \sigma_1^2 + \cdots + \sigma_k^2(\theta_1) \quad (p = m - 1, q = k), \tag{3.26}$$

and at the time θ_1 we consider the pursuit problem with the pursuers $z_1, \ldots, z_{m-1}$ and evaders $y_1, \ldots, y_k$.

We now turn to the case (ii), that is, $\rho_m > \sigma_k$. Then the pursuer z_m certainly ensures the equality $z_m(\theta_1) = y_k(\theta_1)$ and according to Lemma 3.2 (see, (3.7))

$$\rho_m^2(\theta_1) \geq \rho_m^2 - \sigma_k^2 - \frac{K(\theta_1, x_{m0}, y_{k0})}{a(\theta_1)}. \tag{3.27}$$

Then with the aid of (3.23) we obtain

$$\rho_1^2 + \cdots + \rho_{m-1}^2 + \rho_m^2(\theta_1) > \sigma_1^2 + \cdots + \sigma_{k-1}^2 \quad (p = m, q = k - 1), \tag{3.28}$$

and therefore at the time θ_1 we arrive at the pursuit problem with the pursuers $z_1, \ldots, z_m$ and evaders $y_1, \ldots, y_{k-1}$.

Let now θ_2 be an arbitrary fixed number satisfying inequalities

$$\begin{gathered}\theta_2 > \theta_1, \quad a^2(\theta_2 - \theta_1) \geq \frac{\left|y_q(\theta_1) - z_p(\theta_1)\right|^2}{\left(\rho_p(\theta_1) - \sigma_q(\theta_1)\right)^2}, \\ \rho_1^2(\theta_1) + \cdots + \rho_p^2(\theta_1) > \sigma_1^2(\theta_1) + \cdots + \sigma_q^2(\theta_1) + \frac{K\left(\theta_2 - \theta_1, y_q(\theta_1), z_q(\theta_1)\right)}{a(\theta_2 - \theta_1)},\end{gathered} \tag{3.29}$$

where p and q are the numbers of pursuers and evaders, respectively, which take part in the pursuit problem at time θ_1. Set

$$\begin{gathered}w_p(t) = \frac{\varphi(t)}{a^2(\theta_2 - \theta_1)}\left(y_q(\theta_1) - z_p(\theta_1)\right) + v_q(t), \quad \theta_1 < t \leq \theta_2, \\ w_i(t) \equiv 0, \quad \theta_1 < t \leq \theta_2,\ i \in \{1, \ldots, m\} \setminus \{p\}.\end{gathered} \tag{3.30}$$

Observe that according to (3.30) all pursuers except for z_p will not move on the time interval $(\theta_1, \theta_2]$.

Let the pursuers use the strategies (3.30). Applying the same arguments above, we arrive at the following conclusion.

(1) If $\rho_p(\theta_1) \leq \sigma_q(\theta_1)$ and $z_p(\theta_2) = y_q(\theta_2)$, then starting from the time θ_2 we consider a pursuit problem with the pursuers $z_1, \ldots, z_{p-1}$ and evaders $y_1, \ldots, y_{q-1}$ under the condition

$$\rho_1^2(\theta_2) + \cdots + \rho_{p-1}^2(\theta_2) > \sigma_1^2(\theta_2) + \cdots + \sigma_{q-1}^2(\theta_2). \tag{3.31}$$

(2) If $\rho_p(\theta_1) \leq \sigma_q(\theta_1)$ and $z_p(\theta_2) \neq y_q(\theta_2)$, then starting from the time θ_2 we consider the pursuit problem with the pursuers $z_1, \ldots, z_{p-1}$ and evaders $y_1, \ldots, y_q$ under the condition

$$\rho_1^2(\theta_2) + \cdots + \rho_{p-1}^2(\theta_2) > \sigma_1^2(\theta_2) + \cdots + \sigma_q^2(\theta_2). \tag{3.32}$$

(3) If $\rho_p(\theta_1) > \sigma_q(\theta_1)$, then by Lemma 3.2 the equality $z_p(\theta_2) = y_q(\theta_2)$ holds. In this case, starting from the time θ_2 the pursuit problem with the pursuers $z_1, \ldots, z_p$ and evaders $y_1, \ldots, y_{q-1}$ is considered under the condition

$$\rho_1^2(\theta_2) + \cdots + \rho_p^2(\theta_2) > \sigma_1^2(\theta_2) + \cdots + \rho_{q-1}^2(\theta_2). \tag{3.33}$$

Repeated application of this procedure enables us to complete the pursuit for some finite time T since the number of players is decreasing. Thus, we have proved that FPs can complete the pursuit.

(2^0) Completion of the proof of the Theorem: We will now show that the actual pursuers also can complete the pursuit. Define the controls $u_1, \ldots, u_m$ of the pursuers $x_1, \ldots, x_m$ by the controls of the FPs $w_1, \ldots, w_m$. We denote by $F_N(x)$ the projection of the point $x \in \mathbb{R}^n$ on the set N, that is,

$$\min_{y \in N} |x - y| = |x - F_N(x)|. \tag{3.34}$$

Note that $F_N(x) = x$ if $x \in N$. It is familiar that for any point $x \in \mathbb{R}^n$ there exists a unique point $F_N(x)$. Moreover, for any $x, y \in \mathbb{R}^n$,

$$|F_N(x) - F_N(y)| \leq |x - y|, \tag{3.35}$$

and hence the operator $F_N(x)$ relates any absolute continuous function $z(t),\ 0 \leq t \leq T$, to an absolute continuous function

$$x(t) = F_N(z(t)), \quad 0 \leq t \leq T, \tag{3.36}$$

where T is the time in which pursuit can be completed by FPs. Define the control $u_i(t)$ to satisfy

$$x_i(t) = F_N(z_i(t)), \quad 0 \leq t \leq T,\ i = 1, \ldots, m. \tag{3.37}$$

We first show admissibility of such defined control $u_i(t)$. Indeed, from (3.35),

$$\begin{aligned} \varphi(t)|u_i(t)| = |\dot{x}_i(t)| &= \lim_{h \to 0} \frac{|x_i(t+h) - x(t)|}{|h|} \\ &= \lim_{h \to 0} \frac{|F_N(z(t+h)) - F_N(z(t))|}{|h|} \\ &\leq \lim_{h \to 0} \frac{|z_i(t+h) - z_i(t)|}{|h|} = |\dot{z}_i(t)| = \varphi(t)|w_i(t)|. \end{aligned} \tag{3.38}$$

Hence the inequality $|u_i(t)| \leq |w_i(t)|$ holds almost everywhere and therefore

$$\int_0^T |u_i(t)|^2 dt \leq \int_0^T |w_i(t)|^2 dt \leq \rho_i^2. \tag{3.39}$$

Since for any evader $y_i,\ i \in \{1, \ldots, k\}$, the equality $z_{n_i}(t_i) = y_i(t_i)$ holds for some $t_i \leq T$ and $n_i \in \{1, \ldots, m\}$, and the evader $y_i(t)$ is in N for any $t \geq 0$; in particular, $y_i(t_i) \in N$, then FP $z_{n_i}(t_i)$ is also in N. Consequently,

$$x_{n_i}(t_i) = F_N(z_{n_i}(t_i)) = z_{n_i}(t_i) = y_i(t_i). \tag{3.40}$$

This means differential game (2.1)–(2.4) can be completed for the time T. The proof of the theorem is complete.

We give an illustrative example.

Example 3.3. We consider a differential game described by the following equations:

$$\begin{aligned} \dot{x}_i &= \lambda x_i + u_i, \qquad x_i(0) = x_{i0},\ i = 1,\dots,m,\ \ \lambda \le 0, \\ \dot{y}_j &= \lambda y_j + v_j, \qquad y_j(0) = y_{j0},\ j = 1,\dots,k, \end{aligned} \tag{3.41}$$

It is assumed that $x_{i0} \neq y_{j0}$ for all $i = 1,\dots,m$, $j = 1,\dots,k$. The control functions satisfy the integral constraints (2.3) and (2.4). Pursuit is said to be completed if for any numbers $j = 1,2,\dots k$ the equality $x_i(t_j) = y_j(t_j)$ holds for some $i \in \{1,\dots,k\}$ at some time $t_j \ge 0$. Players may not leave the half space $N = \{x \mid (x,l) \ge 0, x \in \mathbb{R}^n\}$, where $l \in \mathbb{R}^n$ is a nonzero vector. Since

$$x_i(t) = e^{\lambda t}\overline{x}_i(t), \qquad y_i(t) = e^{\lambda t}\overline{y}_i(t), \tag{3.42}$$

where

$$\overline{x}_i(t) = x_{i0} + \int_0^t e^{-\lambda s}u_i(s)ds, \qquad \overline{y}_i(t) = y_{j0} + \int_0^t e^{-\lambda s}v_j(s)ds, \tag{3.43}$$

then the equality $x_i(t_j) = y_j(t_j)$ is equivalent to the one $\overline{x}_i(t_j) = \overline{y}_j(t_j)$ and inclusions $x_i(t) \in N$ and $y_i(t) \in N$ are equivalent to ones $\overline{x}_i(t) \in N$ and $\overline{y}_i(t) \in N$, respectively. Therefore, differential game described by (3.41) is equivalent to the game described by the following equations:

$$\begin{aligned} \overline{x}_i(t_j) &= e^{-\lambda t}u_i, \qquad \overline{x}_i(0) = x_{i0}, \quad i = 1,\dots,m,\ \ \lambda \le 0, \\ \overline{y}_j(t_j) &= e^{-\lambda t}v_j, \qquad \overline{y}_j(0) = y_{j0}, \quad j = 1,\dots,k. \end{aligned} \tag{3.44}$$

Denoting $\varphi(t) = e^{-\lambda t}$, we obtain a differential game of the form (2.1).

4. Conclusion

We have obtained a sufficient condition (3.2) to complete the pursuit in the differential game of m pursuers and k evaders with integral and state constraints. We have constructed strategies of the pursuers and showed that pursuit can be completed from any initial position in N. In the case of $N = \mathbb{R}^n$ and $k = 1$ (one evader), the condition (3.2) of the theorem takes the form $\rho_1^2 + \rho_2^2 + \cdots + \rho_m^2 > \sigma_1^2$. This condition is sharp since if $\rho_1^2 + \rho_2^2 + \cdots + \rho_m^2 \le \sigma_1^2$, then evasion is possible [13].

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