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Research Article

Stability of Difference Schemes for Fractional Parabolic PDE with the Dirichlet-Neumann Conditions

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The stable difference schemes for the fractional parabolic equation with Dirichlet and Neumann boundary conditions are presented. Stability estimates and almost coercive stability estimates with $\ln(1/(\tau + |h|))$ for the solution of these difference schemes are obtained. A procedure of modified Gauss elimination method is used for solving these difference schemes of one-dimensional fractional parabolic partial differential equations.

1. Introduction

Theory and applications, methods of solutions of problems for fractional differential equations have been studied extensively by many researchers [1–18]. In this study, initial-boundary-value problem for the fractional parabolic equation

$$\begin{aligned}
&\frac{\partial u(t,x)}{\partial t} + D_t^{1/2} u(t,x) - \sum_{p=1}^{m} \left(a_p(x) u_{x_p} \right)_{x_p} + \sigma u(t,x) = f(t,x), \\
&x = (x_1, \ldots, x_m) \in \Omega, \quad 0 < t < T, \\
&u(t,x)|_{S_1} = 0, \quad \left.\frac{\partial u(t,x)}{\partial \vec{n}}\right|_{S_2} = 0, \quad 0 \le t \le T, \ S_1 \cup S_2 = S = \partial\overline{\Omega}, \\
&u(0,x) = 0, \quad x \in \overline{\Omega},
\end{aligned} \tag{1.1}$$

with Dirichlet and Neumann conditions is considered. Here $D_t^{1/2} = D_{0+}^{1/2}$ is the standard Riemann-Liouville's derivative of order $1/2$ and Ω is the open cube in the m-dimensional Euclidean space

$$R^m : \{x \in \Omega : x = (x_1, \ldots, x_m);\ 0 < x_j < 1,\ 1 \le j \le m\}, \tag{1.2}$$

with boundary $S, \overline{\Omega} = \Omega \cup S, a_p(x)$ $(x \in \Omega)$ and $f(t,x)$ $(t \in (0,T), x \in \Omega)$ are given smooth functions, $a_p(x) \ge a > 0$, $\sigma > 0$, and $\vec{n}$ is the normal vector to Ω.

The first and second orders of accuracy stable difference schemes for the numerical solution of problem (1.1) are presented. Stability estimates and almost coercive stability estimates with $\ln(1/(\tau + |h|))$ for the solution of these difference schemes are obtained. The method is illustrated by numerical examples.

2. The First and Second Orders of Accuracy Stable Difference Schemes and Stability Estimates

The discretization of problem (1.1) is carried out in two steps. In the first step, let us define the grid space

$$\begin{gathered}\overline{\Omega}_h = \{x = x_p = (h_1 p_1, \ldots, h_m p_m),\ p = (p_1, \ldots, p_m),\ 0 \le p_j \le M_j,\ h_j M_j = 1,\ j = 1, \ldots, m\}, \\ \Omega_h = \overline{\Omega}_h \cap \Omega, \qquad S_h = \overline{\Omega}_h \cap S.\end{gathered} \tag{2.1}$$

We introduce the Hilbert space $L_{2h} = L_2(\overline{\Omega}_h)$ of the grid function $\varphi^h(x) = \{\varphi(h_1 p_1, \ldots, h_m p_m)\}$ defined on $\overline{\Omega}$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_2(\overline{\Omega}_h)} = \left(\sum_{x \in \overline{\Omega}_h} \left|\varphi^h(x)\right|^2 h_1 \cdots h_m\right)^{1/2}. \tag{2.2}$$

To the differential operator A^x generated by problem (1.1), we assign the difference operator A_h^x by the formula

$$A_h^x u^h = -\sum_{p=1}^{m} \left(a_p(x) u_{\overline{x}_p}^h\right)_{x_p, j_p} + \sigma u_x^h, \tag{2.3}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x) = 0$ for all $x \in S_h^1$ and $D^h u^h(x) = 0$ for all $x \in S_h^2$. Here $D^h u^h(x)$ is the approximation of $\partial u / \partial \vec{n}$. It is known that

A_h^x is a self-adjoint positive definite operator in $L_2(\overline{\Omega}_h)$. With the help of A_h^x we arrive at the initial-boundary-value problem

$$\begin{gathered}\frac{dv^h(t,x)}{dt} + D_t^{1/2}v^h(t,x) + A_h^x v^h(t,x) = f^h(t,x), \quad 0 < t < T,\ x \in \Omega_h, \\ v^h(0,x) = 0, \quad x \in \overline{\Omega}_h,\end{gathered} \tag{2.4}$$

for a finite system of ordinary fractional differential equations.

In the second step, applying the first order of approximation formula

$$D_\tau^{1/2}u_k = \frac{1}{\sqrt{\pi}}\sum_{r=1}^{k}\frac{\Gamma(k-r+1/2)}{(k-r)!}\left(\frac{u_r - u_{r-1}}{\tau^{1/2}}\right), \quad 1 \le k \le N, \tag{2.5}$$

for

$$D_t^{1/2}u(t_k) = \frac{1}{\Gamma(1/2)}\int_0^{t_k}(t_k - s)^{-1/2}u'(s)ds, \tag{2.6}$$

(see [19]) and using the first order of accuracy stable difference scheme for parabolic equations, one can present the first order of accuracy difference scheme with respect to t

$$\begin{gathered}\frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + D_\tau^{1/2}u_k^h + A_h^x u_k^h(x) = f_k^h(x), \quad x \in \overline{\Omega}_h, \\ f_k^h(x) = f^h(t_k,x), \quad t_k = k\tau,\ 1 \le k \le N,\ N\tau = T, \\ u_0^h(x) = 0, \quad x \in \overline{\Omega}_h,\end{gathered} \tag{2.7}$$

for the approximate solution of problem (2.4). Here

$$\Gamma\left(k - r + \frac{1}{2}\right) = \int_0^\infty t^{k-r+1/2}e^{-t}dt. \tag{2.8}$$

Moreover, applying the second order of approximation formula

$$D_\tau^{1/2}u_k = \begin{cases} \left[\dfrac{-d\sqrt{2}}{3}\right]u_0 + \left[\dfrac{d\sqrt{2}}{3}\right]u_1, & k = 1, \\ \left[\dfrac{2d\sqrt{6}}{5}\right]u_0 + \left[\dfrac{d\sqrt{6}}{5}\right]u_1 + \left[\dfrac{d\sqrt{6}}{5}\right]u_2, & k = 2, \\ d\displaystyle\sum_{m=2}^{k-1}\{[(k-m)b_1(k-m) + b_2(k-m)]u_{m-2} \\ \quad +[(2m-2k-1)b_1(k-m) - 2b_2(k-m)]u_{m-1} \\ \quad +[(k-m+1)b_1(k-m) + b_2(k-m)]u_m\} \\ \quad +\dfrac{d}{6\sqrt{2}}[-u_{k-2} - 4u_{k-1} + 5u_k], & 3 \le k \le N, \end{cases} \tag{2.9}$$

for

$$D_t^{1/2}u\left(t_k - \frac{\tau}{2}\right) = \frac{1}{\Gamma(1/2)}\int_0^{t_k-\tau/2}\left(t_k - \frac{\tau}{2} - s\right)^{-1/2}u'(s)ds \tag{2.10}$$

and the Crank-Nicholson difference scheme for parabolic equations, one can present the second order of accuracy difference scheme with respect to t and to x and

$$\begin{gathered} \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + D_\tau^{1/2}u_k^h(x) + \frac{1}{2}A_h^x\left(u_k^h(x) + u_{k-1}^h(x)\right) = f_k^h(x), \quad x \in \overline{\Omega}_h, \\ f_k^h(x) = f\left(t_k - \frac{\tau}{2}, x\right), \quad t_k = k\tau,\ 1 \le k \le N,\ N\tau = T, \\ u_0^h(x) = 0, \quad x \in \overline{\Omega}_h, \end{gathered} \tag{2.11}$$

for the approximate solution of problem (2.4). Here

$$d = \frac{2}{\sqrt{\pi\tau}}, \qquad b_1(r) = \sqrt{r + \frac{1}{2}} - \sqrt{r - \frac{1}{2}}, \qquad b_2(r) = -\frac{1}{3}\left(\left(r + \frac{1}{2}\right)^{3/2} - \left(r - \frac{1}{2}\right)^{3/2}\right). \tag{2.12}$$

Theorem 2.1. *Let τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ be sufficiently small numbers. Then, the solutions of difference scheme* (2.7) *and* (2.11) *satisfy the following stability estimate:*

$$\max_{1\le k\le N}\left\|u_k^h\right\|_{L_{2h}} \le C_1 \max_{1\le k\le N}\left\|f_k^h\right\|_{L_{2h}}. \tag{2.13}$$

Here C_1 does not depend on τ, h, and f_k^h, $1 \le k \le N$.

Proof. For the solution of difference scheme (2.7), we have the following formulas:

$$u_k^h(x) = \sum_{s=1}^{k} R^{k-s+1} F_s^h(x)\tau, \quad 1 \le k \le N, \tag{2.14}$$

where

$$\begin{gathered} R = \left(I + \tau A_h^x\right)^{-1}, \\ F_k^h(x) = f_k^h(x) - D_\tau^{1/2} u_k^h(x), \\ D_\tau^{1/2} u_k^h(x) = \frac{1}{\sqrt{\pi}} \sum_{m=1}^{k} \frac{\Gamma(k-m+1/2)}{(k-m)!} \tau^{-1/2} \left[-D_\tau^{1/2} u_m^h(x) + f_m^h(x)\right]. \end{gathered} \tag{2.15}$$

The proof of (2.13) for (2.7) is based on (2.14) and estimate

$$\left\| A_h^x R^k \right\|_{L_{2h} \longrightarrow L_{2h}} \le \frac{1}{k\tau}, \quad \left\| R^k \right\|_{L_{h2} \longrightarrow L_{2h}} \le 1, \quad 1 \le k \le N, \tag{2.16}$$

and the triangle inequality.

In the same manner, we can obtain (2.13) for (2.11) using the inequality

$$\left\| A_h^x B^k C^2 \right\|_{L_{2h} \longrightarrow L_{2h}} \le \frac{1}{k\tau}, \quad \left\| B^k \right\|_{L_{2h} \longrightarrow L_{2h}} \le 1, \quad 1 \le k \le N. \tag{2.17}$$

□

Theorem 2.2. *Let τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ be sufficiently small numbers. Then, the solutions of difference scheme* (2.7) *satisfy the following almost coercive stability estimate:*

$$\max_{1\le k\le N} \left\| \frac{u_k^h - u_{k-1}^h}{\tau} \right\|_{L_{2h}} + \max_{1\le k\le N} \sum_{p=1}^{m} \left\| \left(u_k^h\right)_{\overline{x}_p x_p, j_p} \right\|_{L_{2h}} \le C_2 \ln \frac{1}{\tau + |h|} \max_{1\le k\le N} \left\| f_k^h \right\|_{L_{2h}}. \tag{2.18}$$

Here C_2 is independent of τ, h, and f_k^h, $1 \le k \le N$.

Proof. The proof of (2.18) for (2.7) is based on (2.14) and estimate (2.16) and the triangle inequality. □

Theorem 2.3. *Let τ and $|h| = \sqrt{h_1^2 + \cdots + h_m^2}$ be sufficiently small numbers. Then, the solutions of difference scheme* (2.11) *satisfy the following almost coercive stability estimate:*

$$\max_{1\le k\le N} \left\| \frac{u_k^h - u_{k-1}^h}{\tau} \right\|_{L_{2h}} + \max_{1\le k\le N} \frac{1}{2} \sum_{p=1}^{m} \left\| \left(u_k^h + u_{k-1}^h\right)_{\overline{x}_p x_p, j_p} \right\|_{L_{2h}} \le C_3 \ln \frac{1}{\tau + |h|} \max_{1\le k\le N} \left\| f_k^h \right\|_{L_{2h}}. \tag{2.19}$$

Here C_3 does not depend on τ, h, and f_k^h, $1 \le k \le N$.

Proof. The proof of (2.19) for (2.11) is based on (2.14) and estimate (2.17) and the triangle inequality. □

Remark 2.4. The method of proofs of Theorems 2.1–2.3 enables us to obtain the estimate of convergence of difference schemes of the first and second orders of accuracy for approximate solutions of the initial-boundary-value problem

$$\begin{gathered}\frac{\partial u(\partial t,x)}{\partial t}-\sum_{p=1}^{n}a_p(x)u_{x_px_p}+\sum_{p=1}^{n}b_p(x)u_{x_p}+D_t^{\alpha}u(t,x)=f(t,x;u(t,x),u_{x_1}(t,x),\ldots,u_{x_n}(t,x)),\\ x=(x_1,\ldots,x_n)\in\Omega,\quad 0<t<T,\\ u(0,x)=0,\quad x\in\overline{\Omega},\\ u(t,x)|_{S_1}=0,\qquad \left.\frac{\partial u(t,x)}{\partial\vec{n}}\right|_{S_2}=0\end{gathered}\tag{2.20}$$

for semilinear fractional parabolic partial differential equations.

Note that one has not been able to obtain a sharp estimate for the constants figuring in the stability estimates of Theorems 2.1, 2.2, and 2.3. Therefore, our interest in the present paper is studying the difference schemes (2.7) and (2.11) by numerical experiments. Applying these difference schemes, the numerical methods are proposed in the following section for solving the one-dimensional fractional parabolic partial differential equation. The method is illustrated by numerical experiments.

3. Numerical Applications

For numerical results we consider two examples.

Example 3.1. We consider the initial-boundary-value problem

$$\begin{gathered}\frac{\partial u(t,x)}{\partial t}+D_t^{1/2}u(t,x)-\frac{\partial}{\partial x}\left((1+x)\frac{\partial u(t,x)}{\partial x}\right)+u(t,x)=f(t,x),\\ f(t,x)=\left[3+t+\frac{16\sqrt{t}}{5\sqrt{\pi}}+(1+x)2\pi^2t\right]t^2\sin^2\pi x-(1+x)2\pi^2t^3\cos^2\pi x\\ -2\pi t^3\sin\pi x\cos\pi x,\quad 0<t<1,\ 0<x<1,\\ u(t,0)=u_x(t,1)=0,\quad 0\le t\le 1,\\ u(0,x)=0,\quad 0\le x\le 1,\end{gathered}\tag{3.1}$$

for the one-dimensional fractional parabolic partial differential equation. The exact solution of problem (3.1) is

$$u(t,x)=t^3\sin^2\pi x.\tag{3.2}$$

First, applying difference scheme (2.7), we obtain

$$\begin{aligned}
&\frac{u_n^k - u_n^{k-1}}{\tau} + \frac{1}{\sqrt{\pi}} \sum_{r=1}^{k} \frac{\Gamma(k-r+1/2)}{(k-r)!} \frac{u_n^r - u_n^{r-1}}{\tau^{1/2}} \\
&\quad - \frac{1}{h}\left[(1+x_{n+1})\frac{u_{n+1}^k - u_n^k}{h} - (1+x_n)\frac{u_n^k - u_{n-1}^k}{h}\right] + u_n^k = \varphi_n^k, \\
&\varphi_n^k = f(t_k, x_n), \quad t_k = k\tau,\ 1 \le k \le N,\ x_n = nh,\ 1 \le n \le M-1, \\
&u_0^k = 0, \quad u_{M-1}^k = u_M^k, \quad 0 \le k \le N, \\
&u_n^0 = 0, \quad 0 \le n \le M.
\end{aligned} \tag{3.3}$$

We get the system of equations in the matrix form

$$\begin{gathered}
AU_{n+1} + BU_n + CU_{n-1} = D\varphi_n, \quad 1 \le n \le M-1, \\
U_0 = \widetilde{0}, \quad U_{M-1} = U_M,
\end{gathered} \tag{3.4}$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_n & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_n & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_n \end{bmatrix}_{(N+1)\times(N+1)}, \tag{3.5}$$

$$B = \begin{bmatrix} b_{11} & 0 & 0 & \cdots & 0 & 0 \\ b_{21} & b_{22} & 0 & \cdots & 0 & 0 \\ b_{31} & b_{32} & b_{33} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{N1} & b_{N2} & b_{N3} & \cdots & b_{NN} & 0 \\ b_{N+1,1} & b_{N+1,2} & b_{N+1,3} & \cdots & b_{N+1,N} & b_{N+1,N+1} \end{bmatrix}_{(N+1)\times(N+1)}, \tag{3.6}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & c_n & 0 & \cdots & 0 & 0 \\ 0 & 0 & c_n & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & c_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & c_n \end{bmatrix}_{(N+1)\times(N+1)}, \tag{3.7}$$

$$D = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{3.8}$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \varphi_n^2 \\ \vdots \\ \varphi_n^{N-1} \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1}, \quad U_q = \begin{bmatrix} U_q^0 \\ U_q^1 \\ U_q^2 \\ \vdots \\ U_q^{N-1} \\ U_q^N \end{bmatrix}_{(N+1)\times 1}, \quad q = n \pm 1, n, \tag{3.9}$$

$$\begin{aligned}
&a_n = -\frac{1 + x_{n+1}}{h^2}, \qquad c_n = -\frac{1 + x_n}{h^2}, \\
&b_{11} = 1, \qquad b_{21} = -\frac{1}{\sqrt{\tau}} - \frac{1}{\tau}, \qquad b_{22} = \frac{1}{\sqrt{\tau}} + \frac{1}{\tau} + \frac{2 + x_{n+1} + x_n}{h^2} + 1, \\
&b_{31} = -\frac{\Gamma(1 + 1/2)}{\sqrt{\pi\tau}}, \qquad b_{32} = \frac{\Gamma(1 + 1/2) - \Gamma(1/2)}{\sqrt{\pi\tau}} - \frac{1}{\tau}, \qquad b_{33} = \frac{1}{\sqrt{\tau}} + \frac{1}{\tau} + \frac{2 + x_{n+1} + x_n}{h^2} + 1, \\
&b_{ij} = \begin{cases} -\dfrac{\Gamma(i - 2 + 1/2)}{\sqrt{\pi\tau}(i-2)!}, & j = 1, \\ \sqrt{\pi\tau}\left[\dfrac{\Gamma(i - j + 1/2)}{(i-j)!} - \dfrac{\Gamma(i - j - 1 + 1/2)}{(i-j-1)!}\right], & 2 \le j \le i - 2, \\ \sqrt{\pi\tau}\left[\Gamma\left(1 + \dfrac{1}{2}\right) - \Gamma\left(\dfrac{1}{2}\right)\right] - \dfrac{1}{\tau}, & j = i - 1, \\ \dfrac{1}{\sqrt{\tau}} + \dfrac{1}{\tau} + \dfrac{2 + x_{n+1} + x_n}{h^2} + 1, & j = i, \\ 0, & i < j \le N + 1, \end{cases}
\end{aligned} \tag{3.10}$$

for $i = 4, 5, \ldots, N + 1$ and

$$\begin{aligned}
\varphi_n^k = &\left[3 + k\tau + \frac{16\sqrt{k\tau}}{5\sqrt{\pi}} + (1 + nh)2\pi^2 k\tau\right](k\tau)^2 \sin^2(\pi nh) \\
&- (1 + nh)2\pi^2 (k\tau)^3 \cos^2(\pi nh) - 2\pi(k\tau)^3 \sin(\pi nh)\cos(\pi nh).
\end{aligned} \tag{3.11}$$

So, we have the second-order difference equation with respect to n matrix coefficients. This type system was developed by Samarskii and Nikolaev [20]. To solve this difference equation we have applied a procedure for difference equation with respect to k matrix coefficients. Hence, we seek a solution of the matrix equation in the following form:

$$U_j = \alpha_{j+1}U_{j+1} + \beta_{j+1}, \quad U_M = (I - \alpha_M)^{-1}\beta_M, \quad j = M-1, \ldots, 2, 1, \tag{3.12}$$

where α_j $(j = 1, 2, \ldots, M)$ are $(N+1) \times (N+1)$ square matrices and β_j $(j = 1, 2, \ldots, M)$ are $(N+1) \times 1$ column matrices defined by

$$\begin{gathered} \alpha_{j+1} = -\left(B + C\alpha_j\right)^{-1} A, \\ \beta_{j+1} = \left(B + C\alpha_j\right)^{-1}\left(D\varphi_j - C\beta_j\right), \quad j = 1, 2, \ldots, M-1, \end{gathered} \tag{3.13}$$

where $j = 1, 2, \ldots, M-1$, α_1 is the $(N+1) \times (N+1)$ zero matrix and β_1 is the $(N+1) \times 1$ zero matrix.

Second, applying difference scheme (2.11), we obtain the second order of accuracy difference scheme in t and in x

$$\begin{gathered} \frac{u_n^k - u_n^{k-1}}{\tau} + D_\tau^{1/2} u_n^k - \frac{1}{2}\left[(1 + x_n)\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + \frac{u_{n+1}^k - u_{n-1}^k}{2h} - u_n^k + (1 + x_n)\right. \\ \left. \cdot \frac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{h^2} + \frac{u_{n+1}^{k-1} - u_{n-1}^{k-1}}{2h} - u_n^{k-1}\right] = \varphi_n^k, \\ \varphi_n^k = f\left(t_k - \frac{\tau}{2}, x_n\right), \quad t_k = k\tau, \ x_n = nh, \ 1 \le k \le N, \ 1 \le n \le M-1, \\ u_0^k = 0, \quad 3u_M^k - 4u_{M-1}^k + u_{M-2}^k = 0, \quad 0 \le k \le N, \\ u_n^0 = 0, \quad 0 \le n \le M. \end{gathered} \tag{3.14}$$

Here $D_\tau^{1/2} u_n^k$ is defined same as in (2.9). We get the system of equations in the matrix form

$$\begin{gathered} AU_{n+1} + BU_n + CU_{n-1} = D\varphi_n, \quad 1 \le n \le M-1, \\ U_0 = \widetilde{0}, \qquad 3U_M - 4U_{M-1} + U_{M-2} = 0, \end{gathered} \tag{3.15}$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ a_n & a_n & 0 & \cdots & 0 & 0 \\ 0 & a_n & a_n & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_n & 0 \\ 0 & 0 & 0 & \cdots & a_n & a_n \end{bmatrix}_{(N+1)\times(N+1)}, \tag{3.16}$$

$$B = \begin{bmatrix} b_{11} & 0 & 0 & \cdots & 0 & 0 \\ b_{21} & b_{22} & 0 & \cdots & 0 & 0 \\ b_{31} & b_{32} & b_{33} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{N1} & b_{N2} & b_{N3} & \cdots & b_{NN} & 0 \\ b_{N+1,1} & b_{N+1,2} & b_{N+1,3} & \cdots & b_{N+1,N} & b_{N+1,N+1} \end{bmatrix}_{(N+1)\times(N+1)}, \tag{3.17}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ c_n & c_n & 0 & \cdots & 0 & 0 \\ 0 & c_n & c_n & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & c_n & 0 \\ 0 & 0 & 0 & \cdots & c_n & c_n \end{bmatrix}_{(N+1)\times(N+1)}, \tag{3.18}$$

$$D = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{3.19}$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \varphi_n^2 \\ \vdots \\ \varphi_n^{N-1} \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1}, \quad U_q = \begin{bmatrix} U_q^0 \\ U_q^1 \\ U_q^2 \\ \vdots \\ U_q^{N-1} \\ U_q^N \end{bmatrix}_{(N+1)\times 1}, \quad q = n \pm 1, n, \tag{3.20}$$

$$a_n = -\frac{1}{2}\left(\frac{1+x_n}{h^2} + \frac{1}{2h}\right), \qquad c_n = -\frac{1}{2}\left(\frac{1+x_n}{h^2} - \frac{1}{2h}\right), \qquad d = \frac{2}{\sqrt{\pi\tau}},$$

$$b_1(r) = \sqrt{r+\frac{1}{2}} - \sqrt{r-\frac{1}{2}}, \qquad b_2(r) = -\frac{1}{3}\left[\left(r+\frac{1}{2}\right)^{3/2} - \left(r-\frac{1}{2}\right)^{3/2}\right],$$

$$b_{11} = 1, \qquad b_{21} = -\frac{d\sqrt{2}}{3} - \frac{1}{\tau} + \frac{1+x_n}{h^2} + \frac{1}{2}, \qquad b_{22} = \frac{d\sqrt{2}}{3} + \frac{1}{\tau} + \frac{1+x_n}{h^2} + \frac{1}{2},$$

$$b_{31} = \frac{d2\sqrt{6}}{5}, \qquad b_{32} = \frac{d\sqrt{6}}{5} - \frac{1}{\tau} + \frac{1+x_n}{h^2} + \frac{1}{2}, \qquad b_{33} = \frac{d\sqrt{6}}{5} + \frac{1}{\tau} + \frac{1+x_n}{h^2} + \frac{1}{2},$$

$$b_{41} = d[1b_1(1) + b_2(1)], \qquad b_{42} = d[-3b_1(1) - 2b_2(1)] - \frac{d}{6\sqrt{2}},$$

$$b_{43} = d[2b_1(1) + b_2(1)] - 4\frac{d}{6\sqrt{2}} - \frac{1}{\tau} + \frac{1+x_n}{h^2} + \frac{1}{2}, \qquad b_{44} = 5\frac{d}{6\sqrt{2}} + \frac{1}{\tau} + \frac{1+x_n}{h^2} + \frac{1}{2},$$

$$b_{51} = d[2b_1(2) + b_2(2)], \qquad b_{52} = d[-5b_1(2) - 2b_2(2) + 1b_1(1) + b_2(1)],$$

$$b_{53} = d[3b_1(2) + b_2(2) - 3b_1(1) - 2b_2(1)] - \frac{d}{6\sqrt{2}},$$

$$b_{54} = d[2b_1(1) + b_2(1)] - 4\frac{d}{6\sqrt{2}} - \frac{1}{\tau} + \frac{1+x_n}{h^2} + \frac{1}{2}, \qquad b_{55} = 5\frac{d}{6\sqrt{2}} + \frac{1}{\tau} + \frac{1+x_n}{h^2} + \frac{1}{2},$$

$$b_{ij} = \begin{cases} d[(i-3)b_1(i-3) + b_2(i-3)], & j = 1, \\ d[(5-2i)b_1(i-3) - 2b_2(i-3) + (i-4)b_1(i-4) + b_2(i-4)], & j = 2, \\ d\big[(i-j+1)b_1(i-j) + b_2(i-j) + (2j-2i+1)b_1(i-j-1) & \\ \quad -2b_2(i-j-1) + (i-j-2)b_1(i-j-2) + b_2(i-j-2)\big], & 3 \le j \le i-3, \\ d[3b_1(2) + b_2(2) - 3b_1(1) - 2b_2(1)] - \dfrac{d}{6\sqrt{2}}, & j = i-2, \\ d[2b_1(1) + b_2(1)] - 4\dfrac{d}{6\sqrt{2}} - \dfrac{1}{\tau} + \dfrac{1+x_n}{h^2} + \dfrac{1}{2}, & j = i-1, \\ 5\dfrac{d}{6\sqrt{2}} + \dfrac{1}{\tau} + \dfrac{1+x_n}{h^2} + \dfrac{1}{2}, & j = i, \\ 0, & i < j \le N+1, \end{cases} \tag{3.21}$$

for $i = 6, 7, \ldots, N+1$ and

$$\begin{aligned} \varphi_n^k = &\left[3 + k\tau + \frac{16\sqrt{k\tau}}{5\sqrt{\pi}} + (1+nh)2\pi^2 k\tau\right](k\tau)^2 \sin^2(\pi nh) \\ &- (1+nh)2\pi^2(k\tau)^3\cos^2(\pi nh) - 2\pi(k\tau)^3 \sin(\pi nh)\cos(\pi nh). \end{aligned} \tag{3.22}$$

For the solution of the matrix equation (3.15), we use the same algorithm as in the first order of accuracy difference scheme, where $u_M = [3I - 4\alpha_M + \alpha_{M-1}\alpha_M]^{-1} * [(4I - \alpha_{M-1})\beta_M - \beta_{M-1}]$.

Example 3.2. We consider the initial-boundary-value problem

$$\begin{gathered}
\frac{\partial u(t,x)}{\partial t} + D_t^{1/2}u(t,x) - \frac{\partial}{\partial x}\left((1+x)\frac{\partial u(t,x)}{\partial x}\right) + u(t,x) = f(t,x), \\
f(t,x) = \left[3 + t + \frac{16\sqrt{t}}{5\sqrt{\pi}} + (1+x)2\pi^2 t\right] t^2 \sin^2 \pi x - (1+x)2\pi^2 t^3 \cos^2 \pi x \\
-2\pi t^3 \sin \pi x \cos \pi x, \quad 0 < t < 1,\ 0 < x < 1, \\
u_x(t,0) = 0, \quad u(t,1) = 0, \quad 0 \le t \le 1, \\
u(0,x) = 0, \quad 0 \le x \le 1,
\end{gathered} \tag{3.23}$$

for the one-dimensional fractional parabolic partial differential equation. The exact solution of problem (3.23) is

$$u(t,x) = t^3 \sin^2 \pi x. \tag{3.24}$$

First, applying difference scheme (2.7), we obtain

$$\begin{gathered}
\frac{u_n^k - u_n^{k-1}}{\tau} + \frac{1}{\sqrt{\pi}} \sum_{r=1}^{k} \frac{\Gamma(k-r+1/2)}{(k-r)!} \frac{u_n^r - u_n^{r-1}}{\tau^{1/2}} \\
-\frac{1}{h}\left[(1+x_{n+1})\frac{u_{n+1}^k - u_n^k}{h} - (1+x_n)\frac{u_n^k - u_{n-1}^k}{h}\right] + u_n^k = \varphi_n^k, \\
\varphi_n^k = f(t_k, x_n), \quad t_k = k\tau,\ 1 \le k \le N,\ x_n = nh,\ 1 \le n \le M-1, \\
u_0^k = u_1^k, \quad u_M^k = 0, \quad 0 \le k \le N, \\
u_n^0 = 0, \quad 0 \le n \le M.
\end{gathered} \tag{3.25}$$

We get the system of equations in the matrix form

$$\begin{gathered}
AU_{n+1} + BU_n + CU_{n-1} = D\varphi_n, \quad 1 \le n \le M-1, \\
U_0 = U_1, \qquad U_M = \tilde{0},
\end{gathered} \tag{3.26}$$

where matrices A, B, C, D, φ_n, U_q $(q = n \mp 1, n)$ are defined same as in (3.5), (3.6), (3.7), (3.8), (3.9), respectively.

So, we have the second-order difference equation with respect to n matrix coefficients. To solve this difference equation we have applied a procedure for difference equation with

respect to k matrix coefficients. Hence, we seek a solution of the matrix equation in the following form:

$$U_j = \alpha_{j+1} U_{j+1} + \beta_{j+1}, \quad U_M = 0, \quad j = M-1, \ldots, 2, 1, \tag{3.27}$$

where α_j $(j = 1, 2, \ldots, M)$ are $(N+1) \times (N+1)$ square matrices and β_j $(j = 1, 2, \ldots, M)$ are $(N+1) \times 1$ column matrices defined by

$$\begin{gathered} \alpha_{j+1} = -\left(B + C\alpha_j\right)^{-1} A, \\ \beta_{j+1} = \left(B + C\alpha_j\right)^{-1} \left(D\varphi_j - C\beta_j\right), \quad j = 1, 2, \ldots, M-1, \end{gathered} \tag{3.28}$$

where $j = 1, 2, \ldots, M-1$, α_1 is the $(N+1) \times (N+1)$ identity matrix and β_1 is the $(N+1) \times 1$ zero matrix.

Second, applying the formulas

$$\begin{gathered} u_x(t_k, 0) = \frac{u_1^k - u_0^k}{h} - \frac{h}{2} u_{xx}(t_k, 0) + o\left(h^2\right), \quad 0 \le k \le N, \\ u_x(t_k, M) = \frac{3u_M^k - 4u_{M-1}^k + u_{M-2}^k}{2h} + o\left(h^2\right), \quad 0 \le k \le N, \\ u_t(t_k, 0) = \frac{u_0^{k+1} - u_0^{k-1}}{2\tau} + o\left(\tau^2\right), \quad 1 \le k \le N-1, \\ u_t(t_N, 0) = \frac{3u_0^N - 4u_0^{N-1} + u_0^{N-2}}{2\tau} + o\left(\tau^2\right), \quad k = N, \end{gathered} \tag{3.29}$$

and applying difference scheme (2.11), we obtain the second order of accuracy difference scheme in t and in x

$$\begin{gathered} \frac{u_n^k - u_n^{k-1}}{\tau} + D_\tau^{1/2} u_n^k - \frac{1}{2}\left[(1 + x_n)\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + \frac{u_{n+1}^k - u_{n-1}^k}{2h} - u_n^k \right. \\ \left. + (1 + x_n)\frac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{h^2} + \frac{u_{n+1}^{k-1} - u_{n-1}^{k-1}}{2h} - u_n^{k-1}\right] = \varphi_n^k, \\ \varphi_n^k = f\left(t_k - \frac{\tau}{2}, x_n\right), \quad t_k = k\tau, \ x_n = nh, \ 1 \le k \le N, \ 1 \le n \le M-1, \\ u_0^0 = 0, \qquad k = 0, \\ -\frac{h}{4\tau} u_0^{k-1} + \left[\frac{1}{h} + \frac{h}{2} D_t^{1/2} + \frac{h}{2}\right] u_0^k + \frac{h}{4\tau} u_0^{k+1} = \frac{1}{h} u_1^k + \frac{h}{2} \varphi_0^k, \quad 1 \le k \le N-1, \\ \frac{h}{4\tau} u_0^{N-2} - \frac{h}{\tau} u_0^{N-1} + \left[\frac{1}{h} + \frac{3h}{4\tau} + \frac{h}{2} D_t^{1/2} + \frac{h}{2}\right] u_0^N = \frac{1}{h} u_1^N + \frac{h}{2} \varphi_0^N, \quad k = N, \\ u_M^k = 0, \quad 0 \le k \le N, \\ u_n^0 = 0, \quad 0 \le n \le M. \end{gathered} \tag{3.30}$$

Here $D_\tau^{1/2}u_n^k$ is defined similar to (2.9). We get the system of equations in the matrix form

$$\begin{aligned} &AU_{n+1} + BU_n + CU_{n-1} = D\varphi_n, \quad 1 \le n \le M-1, \\ &EU_0 = FU_1 + R\varphi_0, \qquad U_M = \widetilde{0}, \end{aligned} \tag{3.31}$$

where matrices $A,\ B,\ C,\ D,\ \varphi_n,\ U_q\ (q = n\mp 1, n)$ are defined same as in (3.16), (3.17), (3.18), (3.19), (3.20), respectively.

For the solution of the matrix equation (3.31), we use the same algorithm as in the first order of accuracy difference scheme, where

$$u_M = \widetilde{0}, \qquad \alpha_1 = E^{-1}F, \qquad \beta_1 = E^{-1}R\varphi_0,$$

$$F = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{h} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{h} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{1}{h} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{h} \end{bmatrix}_{(N+1)\times(N+1)}, \qquad R = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$E = \begin{bmatrix} e_{11} & 0 & 0 & \cdots & 0 & 0 \\ e_{21} & e_{22} & 0 & \cdots & 0 & 0 \\ e_{31} & e_{32} & e_{33} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e_{N1} & e_{N2} & e_{N3} & \cdots & e_{NN} & 0 \\ e_{N+1,1} & e_{N+1,2} & e_{N+1,3} & \cdots & e_{N+1,N} & e_{N+1,N+1} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$e_{11} = 1, \qquad e_{21} = -\frac{h}{4\tau} - \frac{4h}{3\sqrt{\pi\tau}}, \qquad e_{22} = \frac{1}{h} + \frac{h}{2} + \frac{4h}{3\sqrt{\pi\tau}}, \qquad e_{23} = \frac{h}{4\tau},$$

$$e_{31} = \frac{2\sqrt{2}h}{15\sqrt{\pi\tau}}, \qquad e_{32} = \frac{-16\sqrt{2}h}{15\sqrt{\pi\tau}} - \frac{h}{4\tau}, \qquad e_{33} = \frac{1}{h} + \frac{h}{2} + \frac{14\sqrt{2}h}{15\sqrt{\pi\tau}}, \qquad e_{34} = \frac{h}{4\tau},$$

$$e_{41} = \frac{dh}{2}\left[\left(1 + \frac{1}{2}\right)b_1(1) + b_2(1)\right],$$

$$e_{42} = \frac{dh}{2}\left[-4b_1(1) - 2b_2(1) + \frac{1}{2b_1}(0) + b_2(0)\right],$$

$$e_{43} = \frac{dh}{2}\left[\left(2 + \frac{1}{2}\right)b_1(1) + b_2(1) - 2 - 2\left(-\frac{1}{3}\right)\right] - \frac{h}{4\tau},$$

$$e_{44} = \frac{1}{h} + \frac{h}{2} + \frac{dh}{2}\left[\left(1 + \frac{1}{2}\right)b_1(0) + b_2(0)\right], \qquad e_{45} = \frac{h}{4\tau},$$

$$e_{51} = \frac{dh}{2}\left[\left(2+\frac{1}{2}\right)b_1(2)+b_2(2)\right],$$

$$e_{52} = \frac{dh}{2}\left[-23b_1(2)-2b_2(2)+\left(1+\frac{1}{2}\right)b_1(1)+b_2(1)\right],$$

$$e_{53} = \frac{dh}{2}\left[\left(2+1+\frac{1}{2}\right)b_1(2)+b_2(2)-22b_1(1)-2b_2(1)+\frac{1}{2b_1}(0)+b_2(0)\right],$$

$$e_{54} = \frac{dh}{2}\left[\left(1+1+\frac{1}{2}\right)b_1(1)+b_2(1)-2b_1(0)-2b_2(0)\right]-\frac{h}{4\tau},$$

$$e_{55} = \frac{1}{h}+\frac{h}{2}+\frac{dh}{2}\left[\left(1+\frac{1}{2}\right)b_1(0)+b_2(0)\right], \qquad e_{56} = \frac{h}{4\tau},$$

$$e_{ij} = \begin{cases} \frac{dh}{2}\left[\left(i-3+\frac{1}{2}\right)\ b_1(i-3)+b_2(i-3)\right], & j=1, \\ \frac{dh}{2}\left[-2(i-2)\ b_1(i-3)-2b_2(i-3)+\left(i-4+\frac{1}{2}\right)\ b_1(i-4)+b_2(i-4)\right], & j=2, \\ \frac{dh}{2}\left[\left(i-j+1+\frac{1}{2}\right)\ b_1(i-j)+b_2(i-j)-2(i-j)\ b_1(i-j-1)\right. & \\ \qquad \left.-2b_2(i-j-1)+\left(i-j-2+\frac{1}{2}\right)\ b_1(i-j-2)+b_2(i-j-2)\right], & 3\le j\le i-2, \\ -\frac{h}{4\tau}+\frac{dh}{2}\left[\left(2+\frac{1}{2}\right)\ b_1(1)+\ b_2(1)-2\ b_1(0)-2\ b_2(0)\right], & j=i-1, \\ \frac{1}{h}+\frac{h}{2}+\frac{dh}{2}\left[\left(1+\frac{1}{2}\right)\ b_1(0)+\ b_2(0)\right], & j=i, \\ \frac{h}{4\tau}, & j=i+1, \\ \frac{h}{4\tau}+\frac{dh}{2}\left[\left(i-N+2+\frac{1}{2}\right)\ b_1(i-N+1)+\ b_2(i-N+1)\right. & \\ \qquad -2(i-N+1)\ b_1(i-N)-2\ b_2(i-N) & \\ \qquad \left.+\left(i-N-1+\frac{1}{2}\right)\ b_1(i-N-1)+\ b_2(i-N-1)\right], & j=N-1, \\ -\frac{h}{\tau}+\frac{dh}{2}\left[\left(2+\frac{1}{2}\right)\ b_1(1)+\ b_2(1)-2\ b_1(0)-2\ b_2(0)\right], & j=N, \\ \frac{1}{h}+\frac{h}{2}+\frac{3h}{4\tau}+\frac{dh}{2}\left[\left(1+\frac{1}{2}\right)\ b_1(0)+\ b_2(0)\right], & j=N+1, \\ 0, & j>i+1, \end{cases} \tag{3.32}$$

for $i = 6, 7, \ldots, N+1$ and

$$\varphi_0^k = -2\pi^2(k\tau)^3. \tag{3.33}$$

Table 1: Error analysis of first and second order of accuracy difference schemes for Example 3.1.

Method	$N = M = 25$	$N = M = 50$	$N = M = 100$
1st order of accuracy	0.2553	0.1256	0.0622
2nd order of accuracy	0.0062	9.771×10^{-4}	1.963×10^{-4}

Table 2: Error analysis of first and second order of accuracy difference schemes for Example 3.2.

Method	$N = M = 25$	$N = M = 50$	$N = M = 100$
1st order of accuracy	0.1653	0.0807	0.0399
2nd order of accuracy	0.0025	5.943×10^{-4}	1.453×10^{-4}

3.1. Error Analysis

Finally, we give the results of the numerical analysis. The error is computed by the following formula:

$$E_M^N = \max_{1\le k\le N, 1\le n\le M-1} \left| u(t_k, x_n) - u_n^k \right|, \tag{3.34}$$

where $u(t_k, x_n)$ represents the exact solution and u_n^k represents the numerical solutions of these difference schemes at (t_k, x_n). The numerical solutions are recorded for different values of N and M. Tables 1 and 2 are constructed for $N = M = 25,\ 50$, and 100, respectively.

The results in Tables 1 and 2 show that, by using the Crank-Nicholson difference scheme, more accurate approximate results can be obtained rather than the first order of accuracy difference scheme.

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