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Research Article

Forward Euler Solutions and Weakly Invariant Time-Delayed Systems

Norma L. Ortiz-Robinson[1] and Vinicio R. Ríos[2]

[1] *Department of Mathematics and Applied Mathematics, VA Commonwealth University, Richmond, Virginia 23284, USA*
[2] *Departamento de Matemáticas, Facultad Experimental de Ciencias, Universidad del Zulia, Apartado 526, Maracaibo, Edo Zulia, Venezuela*

Correspondence should be addressed to Norma L. Ortiz-Robinson, nlortiz@vcu.edu

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This paper presents a necessary and sufficient condition for the weak invariance property of a time-delayed system parametrized by a differential inclusion. The aforementioned condition generalizes the well-known Hamilton-Jacobi inequality that characterizes weakly invariant systems in the nondelay setting. The forward Euler approximation scheme used in the theory of discontinuous differential equations is extended to the time-delayed context by incorporating the delay and tail functions featuring the dynamics. Accordingly, an existence theorem of weakly invariant trajectories is established under the extended forward Euler approach.

1. Introduction

The main goal of this paper is to establish a characterization of the so-called weakly invariant property of a closed set $S \in \mathbb{R}^n$ with respect to the set of solutions of the time-delay differential inclusion

$$\dot{x}(t) \in F(t, x(t), x(t - \delta(t))), \quad t \in I. \tag{1.1}$$

Throughout, the prescribed delay function $\delta(\cdot)$ is nonnegative and continuous on the interval $I := [0, +\infty)$. The multifunction $F : I \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is endowed with the following structural hypotheses (SH).

(i) $F(t, x, y)$ is a nonempty, convex, and compact set for every (t, x, y).

(ii) There are constants $\gamma, c > 0$ such that for each (t, x, y)

$$\sup\{\|v\| : v \in F(t, x, y)\} \leq \gamma(\|x\| + \|y\|) + c. \tag{1.2}$$

(iii) $F(\cdot, \cdot, \cdot)$ is almost upper semicontinuous, which in presence of the previous assumptions, is equivalent to the following condition: for every compact interval $\widetilde{I} \subset I$ and each $\varepsilon > 0$, there exists a closed set $\mathcal{N}_\varepsilon \subseteq \widetilde{I}$ with Lebesgue measure $\mu(\widetilde{I} \setminus \mathcal{N}_\varepsilon) < \varepsilon$, so that the graph

$$\mathcal{G}(F) := \{(t, x, y, v) : v \in F(t, x, y)\} \tag{1.3}$$

is closed in $\mathcal{N}_\varepsilon \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$.

A solution (or trajectory) to (1.1) is an absolutely continuous function x satisfying (1.1) for almost all $t \in [t_0, \infty)$, for some $t_0 \geq 0$. A more precise description and assumptions on the elements defining the data are provided in the next section.

Time delay has been shown to appear in many deterministic dynamical systems (see for instance [1–4]). When this feature is present, its study is essential to understand important qualitative and quantitative aspects of the dynamics. Furthermore, ignoring an existing time delay may result in an ill-posed model [5]. Based upon the nature of the state space, time-delay systems can be investigated by means of either infinite ([6–8]) or finite dimensional models ([1, 9, 10]). In the present work, the considered dynamics evolve according to the differential inclusion (1.1), which fits in the category of finite dimensional models, since the delay component of its right-hand side is being identified with the pointwise history of its trajectories. This setting, we believe, will prove to be satisfactory to further study strong invariance properties and their possible interconnections with Hamilton-Jacobi theory for problems involving time-delayed differential inclusions on finite dimensional spaces.

A complete exposition on the invariance concept used in this paper for the nondelayed setting can be consulted in [6, 11]. We also refer the reader to the comprehensive work [12] where notions of approximate invariance are introduced and a complete set of criteria for the weak and strong invariance properties are obtained in infinite dimensional Hilbert spaces. Here we briefly recall the weak invariance terminology in terms of the time-delayed differential inclusion (1.1). Given $\Delta > 0$, a closed set $S \subseteq \mathbb{R}^n$ and a multifunction F, the pair (S, F) is said to be *weakly invariant* if for every initial condition $(t_0, x_0) \in I \times S$ and every *tail function* $\phi : [-\Delta, t_0] \to \mathbb{R}^n$ with $\phi(t_0) = x_0$, there is a solution x to (1.1) satisfying $x(t) = \phi(t)$ on $[-\Delta, t_0]$ and $x(t) \in S$ for all $t \geq t_0$.

The original motivation to study invariance properties rises from the qualitative theory of dynamical systems under ordinary differential equations. Nevertheless, their applicability branched out to other areas, including partial differential equations, dynamic optimization, and control theory (see [6, 11, 13–16]).

The literature centered upon the invariance topic itself is abundant. We highlight the primary works [17–20] given in the context of vector field differential equations and the references [6–8, 13, 21–24] in the differential inclusion setting. All of the aforementioned works provide either sufficient or necessary conditions for the weak invariance property. However, it is noteworthy to mention that to the best of the authors' knowledge, weak invariance for dynamics exhibiting a time-delay behavior have only been discussed in the references [6–8], with functional differential inclusions being employed to model the underlined

dynamics. The conditions for weak invariance provided in [6–8] are of tangential type only and roughly speaking they assert that weak invariance is equivalent to having, at each $x \in S$, the corresponding set of velocities overlapping the *Bouligand cone of* S *at* x:

$$T_S^B(x) := \left\{ \lim_{i \to \infty} \frac{x_i - x}{t_i} : x_i \xrightarrow{S} x, t_i \downarrow 0 \right\}, \tag{1.4}$$

where $x_i \xrightarrow{S} x$ means that $x_i \in S$ converges to the point x, which is identified as the initial condition of the weakly invariant trajectory of (1.1). The closed cone (1.4) coincides with the tangent space if S is a smooth manifold, but of course $T_S^B(x)$ encloses a more general tangency concept.

A condition that is dual with the previous and that has been revisited by several authors (see [6, 11, 12, 15–17, 22, 24, 25]) to characterize the weak invariance property in the nondelayed framework is the following (we recall here the autonomous version). The pair (S, F) is weakly invariant if and only if

$$h_F(x, \xi) \leq 0, \quad \forall x \in S, \ \forall \xi \in N_S^P(x), \tag{1.5}$$

where

$$h_F(x, \xi) := \inf\{\langle v, \xi \rangle : v \in F(x)\}, \tag{1.6}$$

is the *Hamiltonian* of the multifunction $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ and

$$N_S^P(x) := \left\{ \xi : \exists \sigma > 0 \text{ such that } \langle x' - x, \xi \rangle \leq \sigma \|x' - x\|^2 \ \forall x' \in S \right\} \tag{1.7}$$

is the *Proximal Normal Cone* to S at x.

The Hamiltonian inequality (1.5) has proven to be an important link between some of the invariance properties of nondelayed systems (S, F) and the theory of generalized solutions of Hamilton-Jacobi equations (see [11, 16, 18, 22, 26, 27]). As it was pointed out previously, this Hamiltonian interpretation is not available for time-delayed systems. To our knowledge, the only results deriving a Hamilton-Jacobi equation for time-delayed systems involve a game-theoretic approach in which the dynamics are prescribed by a continuous differential equation [28]. However, it is well understood that a great deal of applications stemming from optimal control theory, including discontinuous differential equations models, can be reformulated in terms of the more general paradigm (1.1). Therefore, the investigation about the form and the vality of (1.5) under the context of the time-delay differential inclusion (1.1) is of natural interest and represents in particular the main goal of this article.

The remaining material of this paper has been organized in three additional sections. Section 2 introduces the extension of the forward Euler solution concept given in [29] to Cauchy problems with time-delay components and establishes the corresponding existence theorem. Section 3 reveals the connection between the new solution concept and the trajectories of the time-delayed differential inclusion (1.1). The final section (Section 4) contains our main result, which is a Hamiltonian characterization of the weak invariance property for time-delayed systems modeled by (1.1).

2. Delayed Forward Euler Solutions

Motivated by the problem of asymptotic stabilization, Bressan introduced the forward Euler approximation scheme in [29] to provide a suitable class of generalized solutions for non-delayed discontinuous differential equations. The new category of solutions subsumes the one generated by the classic Euler polygonal approach and provides some topological advantages when compared with the classes of Krasovskii and Filippov solutions that are used in the theory of discontinuous feedback controls [30, 31]. The aim of this section is to extend Bressan's solution concept to a more general setting where the dynamics exhibit a time delay behavior. This will serve as the initial step to show that weakly invariant trajectories of (1.1) can be identified within a wider class of solutions of certain initial value problems given in terms of time-delayed feedback selections. In this context, we consider for $T > 0$ and every $0 \leq t_0 < T$ the following *time-delayed Cauchy problem*:

$$\begin{gathered} \dot{x}(t) = f(t, x(t), x(t - \delta(t))), \quad t \in [t_0, T] \\ x(t) = \phi(t), \quad t \in [-\Delta, t_0], \end{gathered} \tag{2.1}$$

where the *tail function* $\phi : [-\Delta, t_0] \to \mathbb{R}^n$ is Lebesgue measurable and bounded, the *delay function* $\delta : [0, T] \to [0, \Delta]$ is continuous and for some positive constants γ and c, the vector field $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following linear growth condition:

$$\|f(t, x, y)\| \leq \gamma(\|x\| + \|y\|) + c, \quad \forall (t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n. \tag{2.2}$$

We start with a partition of the interval $[t_0, T]$

$$\pi = \{t_0, t_1, \ldots, t_k\}, \tag{2.3}$$

where $t_k = T$. As usual, the diameter of π is defined by $\mu_\pi = \max\{t_i - t_{i-1} : 1 \leq i \leq k\}$. Given an initial point x_0 and a set of outer perturbations $q_0, q_1, \ldots, q_k$, all of them in $\mathbb{R}^n$, we construct a *delayed forward Euler polygonal* as follows:

$$\psi_\pi(t) = \phi(t) \quad \text{if} - \Delta \leq t < t_0, \tag{2.4}$$

$$\begin{gathered} \psi_\pi(t_0) = x_0 \\ \psi_\pi(t) = \psi_\pi(t_i) + \left[f\left(t_i, \psi_\pi(t_i), \psi_\pi(t_i - \delta(t_i))\right) + q_i\right](t - t_i) \quad \text{for } 0 \leq i \leq k-1,\ t \in [t_i, t_{i+1}]. \end{gathered} \tag{2.5}$$

A function $x : [-\Delta, T] \to \mathbb{R}^n$ is a *delayed forward Euler solution* (DFES) to the initial value problem (2.1) if there are sequences $x_0^m \to \phi(t_0)$, $q_i^m \to 0$, for all $i = 0, 1, \ldots, k$, such that the corresponding sequence of delayed forward Euler polygonals ψ_{π_m} converges uniformly to x, as $\mu_{\pi_m} \to 0$.

The existence of a DFES is established by proving the stabilization of the previous scheme for certain subsequences of delayed forward Euler polygonals. This is done with the help of the following two discrete versions of Gronwall's inequality.

Lemma 2.1. *Assume x_i, y_i are sequences of nonnegative real numbers having the property that for some $\alpha \geq 0$*

$$y_i \leq \alpha + \sum_{0 \leq j < i} y_j x_j, \quad \forall i \geq 0. \tag{2.6}$$

Then the following estimate holds:

$$y_i \leq \alpha e^{\left(\sum_{0 \leq j < i} x_j\right)}, \quad \forall i \geq 0. \tag{2.7}$$

Proof. To prove the lemma we first show by induction that if (2.6) holds, then

$$y_i \leq \alpha \left(1 + \sum_{0 \leq j < i} x_j \prod_{j < k < i} (1 + x_k) \right), \quad \forall i \geq 0. \tag{2.8}$$

In fact, since a sum with zero terms is defined as 0 and the product of zero factors is defined as 1, inequality (2.6) implies $y_0 \leq \alpha = \alpha(1 + 0)$, which is (2.8) for $i = 0$. Let $i > 0$ and assume that (2.8) holds for $j < i$. Using hypothesis (2.6) we can write

$$\begin{aligned} y_i &\leq \alpha + \sum_{0 \leq j < i} x_j y_j \\ &\leq \alpha + \sum_{0 \leq j < i} x_j \alpha \left(1 + \sum_{0 \leq k < j} x_k \prod_{k < r < j} (1 + x_r) \right) \\ &= \alpha \left(1 + \sum_{0 \leq j < i} x_j + \sum_{0 \leq j < i} x_j \sum_{0 \leq k < j} x_k \prod_{k < r < j} (1 + x_r) \right). \end{aligned} \tag{2.9}$$

On the other hand, notice that (using induction again on i)

$$\sum_{0 \leq j < i} x_j \sum_{0 \leq k < j} x_k \prod_{k < r < j} (1 + x_r) = \sum_{0 \leq k < i} x_k \sum_{k < j < i} x_j \prod_{k < r < j} (1 + x_r). \tag{2.10}$$

The expansion of the last sum and product on the right hand side of (2.10) yields

$$\begin{aligned} &1 + \sum_{k < j < i} x_j \prod_{k < r < j} (1 + x_r) \\ &\quad = (1 + x_{k+1}) + x_{k+2}(1 + x_{k+1}) + \cdots + x_{i-1}(1 + x_{k+1}) \cdots (1 + x_{i-2}) \\ &\quad = (1 + x_{k+1})(1 + x_{k+2} + \cdots + x_{i-1}(1 + x_{k+2}) \cdots (1 + x_{i-2})) \\ &\quad \vdots \end{aligned}$$

$$
\begin{aligned}
&= (1 + x_{k+1})(1 + x_{k+2}) \cdots (1 + x_{i-1}) \\
&= \prod_{k<r<i} (1 + x_r).
\end{aligned}
\tag{2.11}
$$

The following estimate follows from the tautology $\sum_{0\le j<i} x_j = \sum_{0\le k<i} x_k$ and from taking (2.10) into (2.9) and then using (2.11):

$$
\begin{aligned}
y_i &\le \alpha\left(1 + \sum_{0\le k<i} x_k + \sum_{0\le k<i} x_k \sum_{k<j<i} x_j \prod_{k<r<j} (1 + x_r)\right) \\
&= \alpha\left(1 + \sum_{0\le k<i} x_k \left(1 + \sum_{k<j<i} x_j \prod_{k<r<j} (1 + x_r)\right)\right) \\
&= \alpha\left(1 + \sum_{0\le k<i} x_k \prod_{k<r<i} (1 + x_r)\right).
\end{aligned}
\tag{2.12}
$$

The last inequality completes the induction. Finally, we use (2.8) to obtain the exponential estimate (2.7). For every r it is clear that $1 + x_r \le e^{x_r}$. Therefore,

$$
\begin{aligned}
y_i &\le \alpha\left(1 + \sum_{0\le k<i} (1 + x_k - 1) \prod_{k<r<i} (1 + x_r)\right) \\
&= \alpha\left(1 + \sum_{0\le k<i} \left(\prod_{k\le r<i} (1 + x_r) - \prod_{k<r<i} (1 + x_r)\right)\right) \\
&= \alpha\left(1 + \prod_{0\le r<i} (1 + x_r) - \prod_{i-1<r<i} (1 + x_r)\right) \\
&= \alpha\left(1 + \prod_{0\le r<i} (1 + x_r) - 1\right) \\
&= \alpha \prod_{0\le r<i} (1 + x_r) \\
&\le \alpha e^{(\sum_{0\le r<i} x_r)},
\end{aligned}
\tag{2.13}
$$

which is precisely the sought-after inequality. □

Lemma 2.2. *Let $r_0, r_1, \ldots, r_N$ be nonnegative numbers satisfying*

$$
r_{i+1} \le (1 + d_i) r_i + D_i, \quad i = 0, 1, \ldots, N - 1, \tag{2.14}
$$

where $d_i \ge 0$, $D_i \ge 0$ and $r_0 = 0$. Then

$$
r_N \le e^{(\sum_{i=0}^{N-1} d_i)} \sum_{i=0}^{N-1} D_i. \tag{2.15}
$$

Proof. This lemma is an exercise proposed in page 183 of [11]. □

It is noteworthy to mention that the existence of forward Euler solutions has been clarified by Bressan in the context of nondelay patchy vector fields by comparing the new set of solutions with the set of Carathéodory solutions (see Theorem 3, part *(iv)* in [29]). The following existence result contrasts with Bressan's approach by following the ideas from Theorem 1.7 in ([11], Chapter 4) and extending it to the time-delayed framework under the more general class of vector fields satisfying (2.2).

Theorem 2.3. *Suppose that f satisfies the linear growth condition* (2.2) *and is otherwise arbitrary. Then, at least one DFES exists for the time-delayed Cauchy problem* (2.1) *and the restriction of any DFES on $[t_0, T]$ is Lipschitz.*

Proof. To relax the notation, we assume without loss of generality that $t_0 = 0$. Let N be a natural number. For the uniform partition of $[0, T]$

$$\pi := \left\{ t_i = \frac{T}{N} i : i = 0, 1, \ldots, N \right\}, \tag{2.16}$$

let φ_π be the corresponding delayed forward Euler polygonal with outer perturbations q_i, nodes $x_i = \varphi_\pi(t_i)$, and delayed nodes $y_i = \varphi_\pi(t_i - \delta(t_i))$, for $i = 0, 1, \ldots, N$. We assume $\max\{\|q_i\| : 0 \le i \le N\} \le 1/N$ and $\|x_0 - \phi(0)\| < 1/N$. Let $i \in \{0, 1, \ldots, N\}$ be fixed. Due to the hypothesis on f, for every $t \in (t_i, t_{i+1})$ we have

$$\|\dot{\varphi}_\pi(t)\| = \|f(t_i, x_i, y_i) + q_i\| \le \gamma(\|x_i\| + \|y_i\|) + (1 + c). \tag{2.17}$$

Let $\beta = \max\{\|\phi(t)\| : t \in [-\Delta, 0]\}$ and choose an index $p_i \in \{0, 1, \ldots, i-1\}$ such that $\|x_{p_i}\| = \max\{\|x_p\| : 0 \le p \le i-1\}$. The definition of y_i implies $\|y_i\| \le \|x_{p_i}\| + \beta$. Using this observation, (2.4), and (2.17) we can write

$$\begin{aligned}
\|x_{i+1} - \phi(0)\| &\le \|x_{i+1} - x_i\| + \|x_i - \phi(0)\| \\
&\le [\gamma(\|x_i\| + \|y_i\|) + c + 1]\frac{T}{N} + \|x_i - \phi(0)\| \\
&\le [\gamma(\|x_i - \phi(0)\| + \|\phi(0)\| + \|y_i\|) + c + 1]\frac{T}{N} + \|x_i - \phi(0)\| \\
&\le \left(\gamma\frac{T}{N} + 1\right)\|x_i - \phi(0)\| + [\gamma(\beta + \|y_i\|) + c + 1]\frac{T}{N} \\
&\le \left(\gamma\frac{T}{N} + 1\right)\|x_i - \phi(0)\| + (3\gamma\beta + \gamma\|x_{p_i} - \phi(0)\| + c + 1)\frac{T}{N}.
\end{aligned} \tag{2.18}$$

Let $M := 3\gamma\beta + c + 1$. Using the identifications

$$r_i = \|x_i - \phi(0)\|, \qquad d_i = \gamma\frac{T}{N}, \qquad D_i = (M + \gamma\|x_{p_i} - \phi(0)\|)\frac{T}{N} \tag{2.19}$$

and applying Lemma 2.2 to the last inequality, we have

$$\begin{aligned}\|x_{i+1}-\phi(0)\| &\le e^{\gamma T}\sum_{j=0}^{i}\left[M+\gamma\left\|x_{p_j}-\phi(0)\right\|\right]\frac{T}{N}\\ &= e^{\gamma T}M\sum_{j=0}^{i}\frac{T}{N}+\gamma e^{\gamma T}\sum_{j=0}^{i}\left\|x_{p_j}-\phi(0)\right\|\frac{T}{N}\\ &\le e^{\gamma T}MT+\gamma e^{\gamma T}\frac{T}{N}\sum_{j=0}^{i}\|x_j-\phi(0)\|.\end{aligned} \tag{2.20}$$

If we set $\alpha = e^{\gamma T}MT$ and $\rho = \gamma e^{\gamma T}T$, the application of Lemma 2.1 on (2.20) leads to

$$\|x_i-\phi(0)\| \le \alpha e^{\rho}, \tag{2.21}$$

and thus, all nodes x_i lie in $\overline{B}(\phi(0),\alpha e^{\rho})$. By convexity the delayed forward Euler polygonal, including its tail ϕ, must lie completely within the ball $\overline{B}(\phi(0),\alpha e^{\rho}+2\beta)$. For $t \in (t_i,t_{i+i})$ inequality (2.17) then implies

$$\|\dot{\psi}_{\pi}(t)\| \le \gamma(\|x_i\|+\|y_i\|)+\|q_i\|+c \le \gamma(2\alpha e^{\rho}+3\beta)+c+1 =: k. \tag{2.22}$$

Therefore, the restriction of ψ_{π} on $[0,T]$ is Lipschitz of rank k. If π_m is a sequence of uniform partitions such that $\mu_{\pi_m} \to 0$ as $m \to \infty$, we can apply the theorem of Arzela-Ascoli to the family $\{\psi_{\pi_m}\}$ restricted on $[0,T]$, which is equicontinuous and uniformly bounded, to obtain a subsequence that converges uniformly to a continuous function x on $[0,T]$. Notice that not only does x inherit the Lipschitz constant k on $[0,T]$, but also $x(0) = \lim_{m\to\infty}\psi_{\pi_m}(0)$. Finally, a DFES of (2.1) is obtained by appending x and ϕ. □

3. Trajectories of Time-Delay Differential Inclusions

This section identifies a particular set of trajectories of the time-delay differential inclusion problem (TDDI)

$$\begin{gathered}\dot{x}(t) \in F(t,x(t),x(t-\delta(t))) \quad \text{a.e. } t \in [t_0,\infty),\\ x(s) = \phi(s) \quad \text{for } s \in [-\Delta,t_0],\end{gathered} \tag{3.1}$$

where the data above consists of the interval $I = [0,+\infty)$, a multifunction $F : I\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ satisfying SH, a positive constant Δ, an initial time $t_0 \in I$, a continuous delay function $\delta : I \to [0,\Delta]$, and a tail function $\phi : [-\Delta,t_0] \to \mathbb{R}^n$ that is assumed to be Lebesgue measurable and bounded. As in the nondelay case [11], we show that any DFES associated to selections of the multifunction F are indeed trajectories of TDDI. The following lemma plays a key role in this regard. The proof is a slight modification of Lemma 1.1 in [32] and therefore it is omitted.

Lemma 3.1. *Suppose x is a continuous function on $[t_0, T]$ and satisfies*

$$\begin{gathered}\|x(t)\| \le a + b\int_{t_0}^{t}\|x(s)\|ds + c\int_{t_0}^{t}\|x(s-\delta(s))\|ds, \quad t \in [t_0, T],\\ x(t) = \phi(t), \quad t \in [-\Delta, t_0],\end{gathered} \tag{3.2}$$

where $a, b, c \ge 0$ and $\phi(\cdot)$ is measurable and bounded by $\lambda > 0$ on $[-\Delta, t_0]$. Then the following inequality holds:

$$\|x(t)\| \le (a + c(t-t_0)\lambda)e^{(b+c)(t-t_0)}, \quad t \in [t_0, T]. \tag{3.3}$$

The next theorem is the main result of this section. It generalizes Corollary 1.12 of ([11], Chapter 4) to the context of a time-delayed differential inclusion with trajectories defined on a semi-infinite interval. For this purpose, it is necessary to extend the concept of a DFES to semi-infinite intervals by considering selections of F. Recall that a selection is a function $f : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying $f(t, x, y) \in F(t, x, y)$ for almost all $t \in I$ and all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Under this extended domain, we agree that a function x is a delayed forward Euler solution of the corresponding Cauchy problem for f if for every $T > 0$ the restriction of x on $[-\Delta, T]$ is a DFES for f in the regular sense. This definition becomes coherent under the same argument that is used to extend nondelayed Euler solutions: an appropriate sequence of delayed forward Euler polygonals on $[-\Delta, \infty)$ is constructed, then a convergent subsequence on $[-\Delta, T]$ is obtained for a fixed $T > 0$, a further convergent subsequence on $[-\Delta, 2T]$, and so on (see [11]).

Theorem 3.2. *Let f be a selection of F as above. If x is a DFES for the Cauchy problem* (2.1) *on $[-\Delta, \infty)$, then x is a trajectory of F.*

Proof. Let ψ_m be a sequence of delayed forward Euler polygonals that converge uniformly to x and that coincide with ϕ on $[-\Delta, t_0]$. Choose $T > t_0$. Let $\pi_m = \{t_0^m, t_1^m, \ldots, t_N^m\}$ be the partition of $[t_0, T]$ associated with ψ_m, and for $t \in (t_i^m, t_{i+1}^m)$ we define the following functions:

$$\begin{gathered}\tau_m(t) = t_i^m,\\ w_m(t) = \psi_m(\tau_m(t)) - \psi_m(t),\\ z_m(t) = \psi_m(\tau_m(t) - \delta(\tau_m(t))) - \psi_m(t - \delta(t)).\end{gathered} \tag{3.4}$$

For any $\alpha > 0$, it is readily seen that

$$\tau_m^{-1}(\alpha, \infty) := \begin{cases} \bigcup_{i=0}^{N}(t_i^m, t_{i+1}^m), & \text{if } \alpha < t_0^m,\\ \bigcup_{i=j+1}^{N}(t_i^m, t_{i+1}^m), & \text{if } t_j^m \le \alpha < t_{j+1}^m,\ j \in \{0, 1, \ldots, N-1\},\\ \emptyset, & \text{if } t_{N-1}^m \le \alpha, \end{cases} \tag{3.5}$$

which guarantees the measurability of τ_m. Moreover, φ_m and δ are continuous functions, whence the composites $\varphi_m \circ \tau_m$ and $\delta \circ \tau_m$ turn out to be measurable, and so also do the linear combinations w_m and z_m.

If k is the Lipschitz constant for φ_m on $[t_0, T]$, then the following estimates hold:

$$\begin{gathered} \|w_m(t)\| \leq k\mu_{\pi_m}, \\ \|\tau_m(t) - t\| \leq \mu_{\pi_m}, \\ \|z_m(t)\| \leq k\big(\mu_{\pi_m} + \|\delta(t) - \delta(\tau_m(t))\|\big). \end{gathered} \tag{3.6}$$

Thus w_m, z_m both converge to zero in $L^2[t_0, T]$ and $\tau_m(t)$ converges to t a.e. Since f is a selection of F, at every nonpartition point t we have

$$\dot{\varphi}_m(t) \in F\big(\tau_m(t), \varphi_m(t) + w_m(t), \varphi_m(t - \delta(t)) + z_m(t)\big). \tag{3.7}$$

Therefore, for almost all $t \in [t_0, T]$ the linear growth condition on F implies

$$\|\dot{\varphi}_m(t)\| \leq \gamma\big(\|\varphi_m(t)\| + \|\varphi_m(t - \delta(t))\|\big) + \gamma(\|w_m(t)\| + \|z_m(t)\|) + c. \tag{3.8}$$

We now apply Gronwall's inequality ([11], page 179) on (3.8), followed by Lemma 3.1 with $a = e^{\gamma(T-t_0)}\|\varphi_m(0)\| + \gamma e^{\gamma(T-t_0)}(\|w_m\|_1 + \|z_m\|_1 + c(T - t_0)))$, $b = 0$, and $c = \gamma e^{\gamma(T-t_0)}$ to obtain

$$\|\varphi_m(t)\| \leq \Big(a + \gamma e^{\gamma(t-t_0)}(t - t_0)\lambda\Big)e^{\gamma(t-t_0)e^{\gamma(t-t_0)}}, \quad t \in [t_0, T]. \tag{3.9}$$

The last inequality leads to a uniform bound for $\|\varphi_m\|$ and hence for $\|\dot{\varphi}_m\|_2$ due to (3.8). Given the theorem of Arzela-Ascoli and the weak compactness in $L^2_n[t_0, T]$, we may choose converging subsequences of $\{\varphi_m\}$ and $\{\dot{\varphi}_m\}$, respectively (without relabeling), so that

$$\varphi_m(t) = \varphi_m(t_0) + \int_{t_0}^{t} \dot{\varphi}_m(s)ds \longrightarrow \phi(t_0) + \int_{t_0}^{t} \dot{x}(s)ds = x(t), \quad \text{as } m \longrightarrow \infty. \tag{3.10}$$

A simple variant of the sequential compactness result given in Chapter 3 of [11] that holds under SH can be applied to obtain that x is a trajectory of F on $[t_0, T]$. Repeating the argument described above on $[t_0, 2T]$ with the subsequence $\{\varphi_m\}$ shows that x is a trajectory of F on $[t_0, 2T]$. We may continue in this fashion to establish the desired property for x on I. □

4. The Main Result

The following observation provides a sufficient condition for delayed forward Euler solutions of (2.1) to approach a closed set $S \subset \mathbb{R}^n$. It plays in particular a crucial role in the main contribution of this article (Theorem 4.2 below) by linking the existence result given in Theorem 3.2 and the weak invariance property of TDDI.

Theorem 4.1. *Suppose $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies (2.2) and let x be a DFES for f on $[-\Delta, T]$. Let Ω be an open set containing $x(t)$ for all $t \in [-\Delta, T]$ and suppose that for every $(t,z,y) \in [t_0,T] \times \Omega \times \Omega$ there exists $s \in proj_S(z)$ such that $\langle f(t,z,y), z-s\rangle \le 0$. Then one has*

$$d_S(x(t)) \le d_S(x(t_0)) \quad \forall t \in [t_0, T]. \tag{4.1}$$

Proof. Let ψ_π be one element from the sequence of delayed forward Euler polygonals converging uniformly to x and let $\pi = \{t_0, t_1, \ldots, t_N\}$ be its associated partition. Let $q_0, q_1, \ldots, q_N$ be the external perturbations defining ψ_π and as in the previous result, we denote by x_i and y_i their associated nodes and delayed nodes. Given $\varepsilon > 0$, due to the uniform convergence we may assume that ψ_π lies completely in Ω and $\|q_i\| < \varepsilon/2$ for $i = 0, 1, \ldots, N$. The hypothesis asserts that for each i there is a point $s_i \in \mathrm{proj}_S(x_i)$ such that $\langle f(t_i, x_i, y_i), x_i - s_i\rangle \le 0$. If k is as in (2.22), then the following estimates hold:

$$\begin{aligned}
d_S^2(x_{i+1}) &\le \|x_{i+1} - s_i\|^2 \quad (\text{since } s_i \in S) \\
&= \|x_{i+1} - x_i\|^2 + \|x_i - s_i\|^2 + 2\langle x_{i+1} - x_i, x_i - s_i\rangle \\
&\le k^2(t_{i+1} - t_i)^2 + d_S^2(x_i) + 2\int_{t_i}^{t_{i+i}} \langle f(t_i, x_i, y_i) + q_i, x_i - s_i\rangle dt \\
&\le k^2(t_{i+1} - t_i)^2 + d_S^2(x_i) + 2\int_{t_i}^{t_{i+i}} \|q_i\| \|x_i - s_i\| dt \\
&\le k^2(t_{i+1} - t_i)^2 + d_S^2(x_i) + \varepsilon(t_{i+1} - t_i) d_S(x_i).
\end{aligned} \tag{4.2}$$

The recursive application of last inequality yields

$$\begin{aligned}
d_S^2(x_{i+1}) &\le d_S^2(x_0) + \sum_{j=0}^{i} k^2(t_{j+1} - t_j)^2 + \sum_{j=0}^{i} \varepsilon(t_{j+1} - t_j) d_S(x_j) \\
&\le d_S^2(x_0) + k^2\mu_\pi \sum_{j=0}^{N-1} (t_{j+1} - t_j) + \sum_{j=0}^{i} \varepsilon(t_{j+1} - t_j) d_S(x_j) \\
&\le d_S^2(x_0) + k^2\mu_\pi (T - t_0) + \sum_{j=0}^{i} \varepsilon(t_{j+1} - t_j) d_S(x_j).
\end{aligned} \tag{4.3}$$

Let $\Gamma := \{j : d_S(x_j) < 1\}$. If $j \notin \Gamma$, then $d_S(x_j) \le d_S^2(x_j)$. From this last and (4.3) we have

$$\begin{aligned}
d_S^2(x_{i+1}) &\le d_S^2(x_0) + k^2\mu_\pi (T - t_0) + \sum_{j\in\Gamma} \varepsilon(t_{j+1} - t_j) + \sum_{j\notin\Gamma} \varepsilon(t_{j+1} - t_j) d_S^2(x_j) \\
&\le d_S^2(x_0) + \left(k^2\mu_\pi + \varepsilon\right)(T - t_0) + \sum_{j=0}^{i} \varepsilon(t_{j+1} - t_j) d_S^2(x_j).
\end{aligned} \tag{4.4}$$

We now apply Lemma 2.1 with $\alpha = d_S^2(x_0) + (k^2\mu_\pi + \varepsilon)(T - t_0)$ to obtain

$$\begin{aligned} d_S^2(x_i) &\le \left(d_S^2(x_0) + \left(k^2\mu_\pi + \varepsilon\right)(T - t_0)\right)e^{\sum_{j=0}^{i}\varepsilon(t_{j+1}-t_j)} \\ &\le \left(d_S^2(x_0) + \left(k^2\mu_\pi + \varepsilon\right)(T - t_0)\right)e^{\varepsilon(T-t_0)}. \end{aligned} \tag{4.5}$$

The previous estimates holds at any node and then after passing to the limit when $\mu_\pi \to 0$ we obtain

$$d_S^2(x(t)) \le \left(d_S^2(x_0) + \varepsilon(T - t_0)\right)e^{\varepsilon(T-t_0)}, \quad \forall t \in [0, T], \tag{4.6}$$

which in turn implies

$$d_S(x(t)) \le d_S(x_0), \quad \forall t \in [t_0, T], \tag{4.7}$$

since ε is arbitrary. □

In view of the previous result, weakly invariant trajectories for TDDI are naturally identified as delayed forward Euler solutions of particular selections of F. Accordingly, the Hamiltonian (1.5) plays an important role in the design of these selections and features the following weakly invariant criterion, which is the main contribution of this paper.

Theorem 4.2. *Let $S \subset \mathbb{R}^n$ be closed. Under the assumptions of TDDI, the pair (S, F) is weakly invariant if and only if*

$$h(t, x, y, \zeta) \le 0, \quad \text{a.e., } t \in I,\ \forall (x, y) \in S \times \mathbb{R}^n,\ \forall \zeta \in N_S^P(x). \tag{4.8}$$

In this case, (4.8) *holds at all points of density of a certain countable family of pairwise disjoint closed sets $I_k \subset I$ $(k = 0, 1, \ldots)$, for which $F(\cdot, \cdot, \cdot)$ is upper semicontinuous on each $I_k \times \mathbb{R}^n \times \mathbb{R}^n$.*

Remarks 1. Recall that a point $t \in \mathbb{R}$ is a point of density of a measurable set $\mathcal{M} \subset \mathbb{R}$ if

$$\lim_{\delta \to 0} \frac{\mu([t - \delta, t + \delta] \cap \mathcal{M})}{2\delta} = 1, \tag{4.9}$$

where μ denotes the Lebesgue measure in $\mathbb{R}$. The set of all points of density of $\mathcal{M}$ is denoted by $\widetilde{\mathcal{M}}$ and has full measure in $\mathcal{M}$. Furthermore, if $\mathcal{U} \subset \widetilde{\mathcal{M}}$, then the equality $\mathrm{cl}(\mathcal{U}) = \mathrm{cl}(\mathcal{U} \setminus \mathcal{N})$ holds for any null set $\mathcal{N} \subset \mathbb{R}$. It is clear that if $\mathcal{M}$ is closed, then $\widetilde{\mathcal{M}} \subset \mathcal{M}$ (see [33]).

Proof. Let $A \subset I$ be the set of full measure such that (4.8) holds for all $t \in A$. For every $(t, x, y) \in A \times \mathbb{R}^n \times \mathbb{R}^n$ we choose an element $s_x \in \mathrm{proj}_S(x)$ and then select $f(t, x, y) \in F(t, s_x, y)$ so that

$$h(t, s_x, y, x - s_x) = \langle f(t, x, y), x - s_x \rangle. \tag{4.10}$$

We extend f, if necessary, by setting $f(t,x,y) := 0$ for all $(t,x,y) \in (I \setminus A) \times \mathbb{R}^n \times \mathbb{R}^n$. It follows that the multifunction F induces the linear growth property (2.2) on f and also $\langle f(t,x,y), x - s_x \rangle \leq 0$ for all $(t,x,y) \in I \times \mathbb{R}^n \times \mathbb{R}^n$. Consequently, if x is a DFES of the problem $\dot{x}(t) = f(t, x(t), x(t-\delta(t)))$ a.e., $t \in [t_0, T]$, with $\phi(t_0) \in S$, Theorem 4.1 implies $x(t) \in S$ for all $t \in [t_0, T]$. We may extend x on $I = [t_0, \infty)$ by considering the previous Cauchy problem on the interval $[t_0, 2T]$ and so on. It turns out that the resulting arc is a solution to TDDI. In fact, define

$$F_s(t,x,y) := \operatorname{co}\{F(t,s,y) : s \in \operatorname{proj}_S(x)\}, \quad \forall (t,x,y) \in I \times \mathbb{R}^n \times \mathbb{R}^n. \tag{4.11}$$

Then F_s satisfies the standing hypotheses and we have $F_s(t,x,y) = F(t,x,y)$ when $x \in S$. It is clear that $f(t,x,y) \in F_s(t,x,y)$ for almost all $t \in I$ and all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. Therefore, Theorem 3.2 implies that x is a trajectory of F_s. Since $x(t) \in S$ for all $t \geq t_0$, it follows that $F_s(t, x(t), x(t-\delta(t))) = F(t, x(t), x(t-\delta(t)))$ and hence x is also a trajectory of F.

To prove the converse, let us consider $\widetilde{I} = [0,1]$ and recall that the almost upper semicontinuity of F implies the existence of a countable family of pairwise disjoint closed sets $I_k \subset \widetilde{I}$ such that $F(\cdot,\cdot,\cdot)$ is upper semicontinuous on $I_k \times \mathbb{R}^n \times \mathbb{R}^n$ for all k and the union $\bigcup_{k\geq 1} I_k$ has full measure in $\widetilde{I}$ (see page 29 in [34] or page 24 in [35]). Let $\widetilde{I}_k$ denote the points of density of I_k, and define $A := \widetilde{I} \setminus \bigcup_{k\geq 1} \widetilde{I}_k$, which has null measure, and take $(t_0, x_0, y_0) \in \widetilde{I}_k \times S \times \mathbb{R}^n$ for some k. Let $\{t_j\}$ be a sequence in $[t_0, \infty) \bigcap I_k$ satisfying $t_j \to t_0$ and set $h_j = t_j - t_0$. By hypothesis, there is a weakly invariant solution x to TDDI such that $x(t_0) = x_0$. The absolute continuity of x, the compact-convex valuedness of F and the upper semicontinuity of F on $I_k \times \mathbb{R}^n \times \mathbb{R}^n$, lead to the existence of a subsequence $\{t_{j_k}\}$, which we relabel as $\{t_j\}$, such that

$$v := \lim_{j\to\infty} \frac{x(t_j) - x(t_0)}{h_j} = \lim_{j\to\infty} \frac{1}{h_j} \int_{t_0}^{t_j} \dot{x}(s)ds \in F(t_0, x_0, y_0). \tag{4.12}$$

On the other hand, recall that $x(t_j) \in S$ for all j. Therefore, for any $\xi \in N_S^P(x_0)$ the definition (1.7) asserts the existence of $\sigma > 0$ such that

$$\begin{aligned} \langle v, \xi \rangle &= \lim_{j\to\infty} \left\langle \frac{x(t_j) - x(t_0)}{h_j}, \xi \right\rangle \\ &\leq \sigma \lim_{j\to\infty} \left\| \frac{x(t_j) - x(t_0)}{h_j} \right\| \left\| x(t_j) - x(t_0) \right\| = 0, \end{aligned} \tag{4.13}$$

which implies the Hamiltonian inequality in the statement at the point (t_0, x_0, y_0). Finally, we repeat the previous argument on each of the intervals $[1,2], [2,3], \ldots$ to obtain that (4.8) is satisfied almost everywhere on the whole interval I. □

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