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Research Article

Strong and Weak Convergence Theorems for Equilibrium Problems and Weak Relatively Uniformly Nonexpansive Multivalued Mappings in Banach Spaces

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Equilibrium problem and fixed point problem are considered. A general iterative algorithm is introduced for finding a common element of the set of solutions to the equilibrium problem and the common set of fixed points of two weak relatively uniformly nonexpansive multivalued mappings. Furthermore, strong and weak convergence results for the common element in the two sets mentioned above are established in some Banach space.

1. Introduction

Let E be a smooth Banach space, and let C be a nonempty closed convex subset of E. In the sequel, we denote by 2^C the family of all nonempty subsets of C. We use $\phi : E \times E \to \mathbb{R}$ to denote the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{1.1}$$

We know several fundamental properties of ϕ as follows: $\phi(x, y) \geq 0$ for all $x, y \in E$. For a sequence $\{y_n\} \subset E$ and $x \in E$, $\{y_n\}$ is bounded if and only if $\{\phi(x, y_n)\}$ is bounded.

Let $T : C \to 2^C$ be a multivalued mapping. We denote by $F(T)$ the set of fixed points of T, that is,

$$F(T) = \{x \in C : x \in Tx\}. \tag{1.2}$$

For a multivalued mapping T, we define an asymptotic fixed point and a strong asymptotic fixed point of T as follows.

Definition 1.1 (see [1]). Let $T : C \to 2^C$ be a multivalued mapping.

(1) A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p and there exists a sequence $\{y_n\}$ such that $y_n \in Tx_n$, $\lim_{n\to\infty}\|y_n - x_n\| = 0$. The set of asymptotic fixed point of T will be denoted by $\widehat{F}(T)$.

(2) A point p in C is said to be a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges strongly to p and there exists a sequence $\{y_n\}$ such that $y_n \in Tx_n$, $\lim_{n\to\infty}\|y_n - x_n\| = 0$. The set of strong asymptotic fixed point of T will be denoted by $\widetilde{F}(T)$.

Definition 1.2 (see [1]). A multivalued mapping $T : C \to 2^C$ is called relatively nonexpansive multivalued mapping (weak relatively nonexpansive multivalued mapping) if the following conditions are satisfied:

(1) $F(T)$ is nonempty;

(2) $\phi(u, v) \le \phi(u, x)$, for all $u \in F(T)$, for all $x \in C$, $\exists v \in Tx$;

(3) $\widehat{F}(T) = F(T)$ $(\widetilde{F}(T) = F(T))$.

Definition 1.3 (see [1]). A multivalued mapping $T : C \to 2^C$ is called relatively uniformly nonexpansive multivalued mapping (weak relatively uniformly nonexpansive multivalued mapping) if the following conditions are satisfied:

(1) $F(T)$ is nonempty;

(2) $\phi(u, v) \le \phi(u, x)$, for all $u \in F(T)$, for all $x \in C$, for all $v \in Tx$;

(3) $\widehat{F}(T) = F(T)$ $(\widetilde{F}(T) = F(T))$.

Remark 1.4. By comparing condition (2) of Definitions 1.2 and 1.3, one easily draws the following conclusions:

(1) the class of relatively nonexpansive multivalued mappings contains the class of relatively uniformly nonexpansive multivalued mappings as a subclass, but the converse may be not true;

(2) the class of weak relatively nonexpansive multivalued mappings contains the class of weak relatively uniformly nonexpansive multivalued mappings as a subclass, but the converse may be not true.

For any operator T, $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$ is held. So we have the following remark.

Remark 1.5. From Definitions 1.2 and 1.3, the following conclusions can easily be drawn:

(1) the class of weak relatively nonexpansive multivalued mappings contains the class of relatively nonexpansive multivalued mappings as a subclass, but the converse may be not true;

(2) the class of weak relatively uniformly nonexpansive multivalued mappings contains the class of relatively uniformly nonexpansive multivalued mappings as a subclass, but the converse may be not true.

Remark 1.6. The examples of weak relatively uniformly nonexpansive multivalued mapping can be found in Su [1] and Homaeipour and Razani [2].

Let E be a real Banach space, and let E^* be the dual space of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$. The equilibrium problem is to find

$$\widehat{x} \in C \quad \text{such that } f(\widehat{x}, y) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $EP(f)$. Given a mapping $T : C \to E^*$, let $f(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then $\widehat{x} \in EP(f)$ if and only if $\langle T\widehat{x}, y - \widehat{x} \rangle \geq 0$ for all $y \in C$, that is, $\widehat{x}$ is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.3). Some methods have been proposed to solve the equilibrium problem in Hilbert spaces, see [3–5] for details.

In recent years, iterative methods for approximating fixed points of multivalued mappings in Banach spaces have been studied by many authors, see [2, 6–9] for details. In 2011, Homaeipour and Razani [2] introduced the concept of relatively nonexpansive multivalued mappings and proved some weak and strong convergence theorems to approximate a fixed point for a single relatively nonexpansive multivalued mapping in a uniformly convex and uniformly smooth Banach space E which improved and extended the corresponding results of Matsushita and Takahashi [10]. Very recently, Su [1] not only redefined relatively nonexpansive multivalued mappings, which was different from Homaeipour and Razani [2]'s definition, but also introduced some interesting examples about the multivalued mappings. On the other hand, in 2009, Qin et al. [11] introduced an iterative algorithm for the equilibrium problem (1.3) and relatively nonexpansive mappings. Moreover, they proved a weak convergence theorem for finding a common element of the set of solutions to the equilibrium problem (1.3) and the common set of fixed points of two relatively nonexpansive mappings, which improved and extended the corresponding results of Takahashi and Zembayashi [12].

Motivated and inspired by the above facts, the purpose of this paper will introduce an iterative algorithm for the equilibrium problem (1.3) and two weak relatively uniformly nonexpansive multivalued mappings. Furthermore, a weak convergence theorem will given for finding a common element of the set of solutions to the equilibrium problem (1.3) and the common set of fixed points of two weak relatively uniformly nonexpansive multivalued mappings in some Banach space. Our results improve and extend the corresponding results of Qin et al. [11] and Takahashi and Zembayashi [12].

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$, and let J be the normalized duality mapping from E into 2^{E^*} given by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}, \tag{2.1}$$

for all $x \in E$, where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ the generalized duality pairing between E and E^*. It is well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E.

As we all know that if C is a nonempty closed convex subset of a Hilbert space H, and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces, and, consequently, it is not available in more general Banach spaces. In this connection, Alber [13] introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces. The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the Lyapunov functional $\phi(x, y)$, that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem;

$$\phi(\overline{x}, x) = \inf_{y \in C} \phi(y, x). \tag{2.2}$$

The existence and uniqueness of the operator Π_C follow from the properties of the Lyapunov functional $\phi(x, y)$ and strict monotonicity of the mapping J, see, for example, [13, 14]. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$\left(\|y\| - \|x\|\right)^2 \leq \phi(y, x) \leq \left(\|y\| - \|x\|\right)^2, \quad \forall x, y \in E. \tag{2.3}$$

A Banach space E is said to be strictly convex if $\|(x + y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|(x_n + y_n)/2\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided by

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}, \tag{2.4}$$

which exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

We also need the following lemmas for the proof of our main results.

Lemma 2.1 (see [2]). *Let E be a strictly convex and smooth Banach space, then $\phi(x, y) = 0$ if and only if $x = y$.*

Lemma 2.2 (see [2]). *Let E be a uniformly convex and smooth Banach space and $r > 0$. Then,*

$$g(\|y - z\|) \leq \phi(y, z), \tag{2.5}$$

for all $y, z \in B_r(0) = \{x \in E : \|x\| \leq r\}$, where $g : [0, \infty) \to [0, \infty)$ is a continuous, strictly increasing, and convex function with $g(0) = 0$.

Lemma 2.3 (see [11]). *Let C be a nonempty closed convex subset of a reflexive, strictly convex, and smooth Banach space E. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \quad \forall x \in C \text{ and } y \in E. \tag{2.6}$$

Lemma 2.4 (see [11]). *Let E be a uniformly convex Banach space and $B_r(0)$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x-y\|), \tag{2.7}$$

for all $x, y, z \in B_r(0) = \{x \in E : \|x\| \leq r\}$ and $\lambda, \mu, \gamma \in [0,1]$ with $\lambda + \mu + \gamma = 1$.

Lemma 2.5. *Let E be a strictly convex and smooth Banach space, and let C be a closed convex subset of E. Suppose $T : C \to 2^C$ is a weak relatively uniformly nonexpansive multivalued mapping. Then, $F(T)$ is closed and convex.*

Proof. First, we show that $F(T)$ is closed. Let $\{p_n\}$ be a sequence in $F(T)$ such that $p_n \to p$ as $n \to \infty$. Since the multivalued operator T is uniformly weak relatively nonexpansive, one has

$$\phi(p_n, \tilde{p}) \leq \phi(p_n, p), \tag{2.8}$$

for all $\tilde{p} \in Tp$ and for all $n \in \mathbb{N}$. Therefore,

$$\phi(p, \tilde{p}) = \lim_{n\to\infty} \phi(p_n, \tilde{p}) \leq \lim_{n\to\infty} \phi(p_n, p) = \phi(p, p). \tag{2.9}$$

Applying Lemma 2.1, one gets $p = \tilde{p}$. Hence $Tp = \{p\}$. Therefore, $p \in F(T)$.

Next, we show that $F(T)$ is convex. To this end, for arbitrary $p_1, p_2 \in F(T)$, $t \in (0,1)$. Putting $p = tp_1 + (1-t)p_2$, we prove that $Tp = \{p\}$. Let $q \in Tp$, we have

$$\begin{aligned}
\phi(p, q) &= \|p\|^2 - 2\langle p, Jq\rangle + \|q\|^2 \\
&= \|p\|^2 - 2\langle tp_1 + (1-t)p_2, Jq\rangle + \|q\|^2 \\
&= \|p\|^2 - 2t\langle p_1, Jq\rangle - 2(1-t)\langle p_2, Jq\rangle + \|q\|^2 \\
&= \|p\|^2 + t\phi(p_1, q) + (1-t)\phi(p_2, q) - t\|p_1\|^2 - (1-t)\|p_2\|^2 \\
&\leq \|p\|^2 + t\phi(p_1, p) + (1-t)\phi(p_2, p) - t\|p_1\|^2 - (1-t)\|p_2\|^2 \\
&= \|p\|^2 - 2\langle tp_1 + (1-t)p_2, Jp\rangle + \|p\|^2 \\
&= \|p\|^2 - 2\langle p, Jp\rangle + \|p\|^2 \\
&= 0.
\end{aligned} \tag{2.10}$$

Using Lemma 2.1 again, we also obtain $p = q$. Hence, $T(p) = \{p\}$, that is, $p \in F(T)$. Therefore, $F(T)$ is convex. □

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:

(A_1) $f(x, x) = 0$ for all $x \in C$;

(A_2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A_3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y); \tag{2.11}$$

(A_4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semicontinuous.

Lemma 2.6 (see [12]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to R satisfying (A_1)–(A_4), and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C. \tag{2.12}$$

Lemma 2.7 (see [12]). *Let C be a closed subset of a strictly convex, uniformly smooth, and reflexive Banach space E, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4). For all $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \ \forall y \in C \right\}, \tag{2.13}$$

for all $x \in E$. Then, the following hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive mapping, that is, for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle; \tag{2.14}$$

(3) *$F(T_r) = EP(f)$*

(4) *$EP(f)$ is closed and convex.*

Lemma 2.8 (see [12]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4), and let $r > 0$ and $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x). \tag{2.15}$$

Lemma 2.9 (see [12]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let $x \in E$, and let $z \in C$. Then*

$$z = \Pi_C x \Longleftrightarrow \langle y - z, Jx - Jz\rangle \leq 0, \quad \forall y \in C. \tag{2.16}$$

3. Main Results

In this section, we prove a weak convergence theorem for finding a common element of the set of solutions for an equilibrium problem and the set of fixed points of two weak relatively uniformly nonexpansive multivalued mappings in a Banach space. Before proving the result, we need the following theorem.

Theorem 3.1. *Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to R satisfying (A_1)–(A_4) and let $T,\ S : C \to C$ be two weak relatively uniformly nonexpansive multivalued mappings such that $\mathcal{F} = F(T) \cap F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{gathered} x_n \in C \quad \text{such that } f(x_n, y) + \frac{1}{r_n}\langle y - x_n,\ Jx_n - Ju_n\rangle \geq 0, \quad \forall y \in C, \\ u_{n+1} = J^{-1}(\alpha_n Jx_n + \beta_n Jy_n + \gamma_n Jz_n), \end{gathered} \tag{3.1}$$

where $y_n \in Tx_n$, $z_n \in Sx_n$, and J are the duality mapping on E. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[0,1]$ satisfying the following conditions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\liminf_{n\to\infty}\alpha_n\beta_n > 0$, $\liminf_{n\to\infty}\alpha_n\gamma_n > 0$;

(c) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.

Then $\{\Pi_{\mathcal{F}} x_n\}$ converges strongly to $z \in \mathcal{F}$, where $\Pi_{\mathcal{F}}$ is the generalized projection of E onto $\mathcal{F}$.

Proof. Let $p \in \mathcal{F}$. Putting $x_n = T_{r_n}u_n$ for all $n \in \mathbb{N}$, it is well known that T_{r_n} is relatively nonexpansive, one has

$$\begin{aligned} \phi(p, x_{n+1}) &= \phi(p, T_{r_n}u_{n+1}) \\ &\leq \phi(p, u_{n+1}) \\ &= \phi\Big(p, J^{-1}(\alpha_n Jx_n + \beta_n Jy_n + \gamma_n Jz_n)\Big) \\ &= \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2\beta_n\langle p, Jy_n\rangle - 2\gamma_n\langle p, Jz_n\rangle + \|\alpha_n Jx_n + \beta_n Jy_n + \gamma_n Jz_n\|^2 \\ &\leq \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2\beta_n\langle p, Jy_n\rangle - 2\gamma_n\langle p, Jz_n\rangle + \alpha_n\|Jx_n\|^2 + \beta_n\|Jy_n\|^2 \\ &\quad + \gamma_n\|Jz_n\|^2 \\ &= \phi(p, x_n) + \beta_n\phi(p, y_n) + \gamma_n\phi(p, z_n) \\ &\leq \phi(p, x_n). \end{aligned} \tag{3.2}$$

Therefore, $\lim_{n\to\infty}\phi(p, x_n)$ exists. Since $\phi(p, x_n)$ is bounded, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are bounded.

Define $v_n = \Pi_{\mathcal{F}} x_n$ for all $n \in \mathbb{N}$. Then, from $v_n \in \mathcal{F}$ and (3.2), one gets

$$\phi(v_n, x_{n+1}) \leq \phi(v_n, x_n). \tag{3.3}$$

Since $\Pi_{\mathcal{F}}$ is the generalized projection, from Lemma 2.3, one sees

$$\begin{aligned}\phi(v_{n+1}, x_{n+1}) &= \phi(\Pi_{\mathcal{F}} x_{n+1}, x_{n+1}) \\ &\le \phi(v_n, x_{n+1}) - \phi(v_n, \Pi_{\mathcal{F}} x_{n+1}) \\ &= \phi(v_n, x_{n+1}) - \phi(v_n, v_{n+1}) \\ &\le \phi(v_n, x_{n+1}).\end{aligned} \tag{3.4}$$

Hence, from (3.3), one has

$$\phi(v_{n+1}, x_{n+1}) \le \phi(v_n, x_n). \tag{3.5}$$

Therefore, $\{\phi(v_n, x_n)\}$ is a convergent sequence. Applying (3.3) again, one also obtains that, for all $m \in \mathbb{N}$,

$$\phi(v_n, x_{n+m}) \le \phi(v_n, x_n). \tag{3.6}$$

From $v_{n+m} = \Pi_{\mathcal{F}} x_{n+m}$ and Lemma 2.3, one has

$$\phi(v_n, v_{n+m}) + \phi(v_{n+m}, x_{n+m}) \le \phi(v_n, x_{n+m}) \le \phi(v_n, x_n), \tag{3.7}$$

and hence

$$\phi(v_n, v_{n+m}) \le \phi(v_n, x_n) - \phi(v_{n+m}, x_{n+m}). \tag{3.8}$$

Let $r = \sup_{n\in\mathbb{N}}\|v_n\|$. From Lemma 2.2, there exists a continuous, strictly increasing, and convex function g with $g(0) = 0$ such that

$$g(\|x - y\|) \le \phi(x, y) \quad \text{for } x, y \in B_r. \tag{3.9}$$

Therefore, one has

$$g(\|v_n - v_{n+m}\|) \le \phi(v_n, v_{n+m}) \le \phi(v_n, x_n) - \phi(v_{n+m}, x_{n+m}). \tag{3.10}$$

Since $\{\phi(v_n, x_n)\}$ is a convergent sequence, from the property of g, one obtains that $\{v_n\}$ is a Cauchy sequence. Since $\mathcal{F}$ is closed, $\{v_n\}$ converges strongly to $z \in \mathcal{F}$. This completes the proof of Theorem 3.1. □

In the following, we give our weak convergence result in this paper.

Theorem 3.2. *Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to R satisfying (A_1)–(A_4) and let*

$T, S : C \to C$ be two weak relatively uniformly nonexpansive multivalued mappings such that $\mathcal{F} = F(T) \cap F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:

$$\begin{gathered} x_n \in C \quad \text{such that } f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n \rangle \geq 0, \quad \forall y \in C, \\ u_{n+1} = J^{-1}(\alpha_n J x_n + \beta_n J y_n + \gamma_n J z_n), \end{gathered} \tag{3.11}$$

where $y_n \in Tx_n$, $z_n \in Sx_n$, and J are the duality mapping on E. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the following conditions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\liminf_{n\to\infty} \alpha_n \beta_n > 0$, $\liminf_{n\to\infty} \alpha_n \gamma_n > 0$;

(c) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.

If J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to $z \in \mathcal{F}$, where $z = \lim_{n\to\infty} \Pi_{\mathcal{F}} x_n$.

Proof. In view of the proof Theorem 3.1, one has that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are bounded. Let

$$r = \sup_{n \geq 0}\{\|x_n\|, \|y_n\|, \|z_n\|\}, \tag{3.12}$$

from Lemma 2.4, for $p \in \mathcal{F}$, one has

$$\begin{aligned} \phi(p, x_{n+1}) &= \phi(p, T_{r_n} u_{n+1}) \\ &\leq \phi(p, u_{n+1}) \\ &= \phi\Big(p, J^{-1}(\alpha_n J x_n + \beta_n J y_n + \gamma_n J z_n)\Big) \\ &= \|p\|^2 - 2\alpha_n \langle p, J x_n \rangle - 2\beta_n \langle p, J y_n \rangle - 2\gamma_n \langle p, J z_n \rangle \\ &\quad + \|\alpha_n J x_n + \beta_n J y_n + \gamma_n J z_n\|^2 \leq \|p\|^2 - 2\alpha_n \langle p, J x_n \rangle - 2\beta_n \langle p, J y_n \rangle - 2\gamma_n \langle p, J z_n \rangle \\ &\quad + \alpha_n \|J x_n\|^2 + \beta_n \|J y_n\|^2 + \gamma_n \|J z_n\|^2 - \alpha_n \beta_n g(\|J y_n - J x_n\|) \\ &= \phi(p, x_n) + \beta_n \phi(p, y_n) + \gamma_n \phi(p, z_n) - \alpha_n \beta_n g(\|J y_n - J x_n\|) \\ &\leq \phi(p, x_n) - \alpha_n \beta_n g(\|J y_n - J x_n\|). \end{aligned} \tag{3.13}$$

It follows that

$$\alpha_n \beta_n g(\|J y_n - J x_n\|) \leq \phi(p, x_n) - \phi(p, x_{n+1}). \tag{3.14}$$

In view of $\liminf_{n\to\infty} \alpha_n \beta_n > 0$, by taking the limit in (3.14), one sees

$$g(\|J y_n - J x_n\|) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.15}$$

From the property of g, one has

$$\|Jy_n - Jx_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.16}$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, one obtains

$$\|y_n - x_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.17}$$

Similarly, one could obtain

$$\|z_n - x_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.18}$$

Noticing that $\{x_n\}$ is bounded, one gets that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that x_{n_k} converges weakly to $x^* \in C$. From (3.17) and (3.18), $\widetilde{F}(T) = F(T)$ and $\widetilde{F}(S) = F(S)$, one has

$$x^* \in \widetilde{F}(T) \cap \widetilde{F}(S) = F(T) \cap F(S). \tag{3.19}$$

Next, we show that $x^* \in EP(f)$. Let $s = \sup_{n\geq 1}\{\|x_n\|, \|u_n\|\}$. From Lemma 2.2, there exists a continuous, strictly increasing, and convex function g_1 with $g_1(0) = 0$ such that

$$g_1(\|x - y\|) \leq \phi(x, y), \quad \forall x, y \in B_s. \tag{3.20}$$

Noticing that $x_n = T_{r_n}u_n$ and from Lemma 2.8 and (3.13), for $p \in \mathcal{F}$, one has

$$g_1(\|x_n - u_n\|) \leq \phi(x_n, u_n) \leq \phi(p, u_n) - \phi(p, x_n) \leq \phi(p, x_{n-1}) - \phi(p, x_n). \tag{3.21}$$

Noticing that $\lim_{n\to\infty}\phi(p, x_n)$ exists, one gets

$$\lim_{n\to\infty} g_1(\|x_n - u_n\|) = 0. \tag{3.22}$$

It follows from the property of g_1 that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.23}$$

Since J is uniformly norm-to-norm continuous on bounded sets, one has

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{3.24}$$

From the assumption $r_n \geq a$, one obtains

$$\lim_{n\to\infty} \frac{\|Jx_n - Ju_n\|}{r_n} = 0. \tag{3.25}$$

Since $x_n = T_{r_n} u_n$, one has

$$f(x_n, y) + \frac{1}{r_n}\langle y - x_n,\ Jx_n - Ju_n\rangle \geq 0, \quad \forall y \in C. \tag{3.26}$$

By replacing n by n_k and from (A_2), one sees

$$\begin{aligned} \|y - x_{n_k}\| \frac{\|Jx_{n_k} - Ju_{n_k}\|}{r_{n_k}} &\geq \frac{1}{r_{n_k}}\langle y - x_{n_k},\ Jx_{n_k} - Ju_{n_k}\rangle \\ &\geq -f(x_{n_k}, y) \\ &\geq f(y, x_{n_k}), \quad \forall y \in C. \end{aligned} \tag{3.27}$$

Taking $k \to \infty$ in the above inequality and from (A_4), one has

$$f(y, x^*) \leq 0, \quad \forall y \in C. \tag{3.28}$$

For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)x^*$. Since $y, x^* \in C$, one gets $y_t \in C$, which yields that $f(y_t, x^*) \leq 0$. It follows from (A_1) that

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, x^*) \leq tf(y_t, y), \tag{3.29}$$

that is,

$$f(y_t, y) \geq 0. \tag{3.30}$$

Let $t \downarrow 0$; from (A_3), we obtain $f(x^*, y) \geq 0$ for all $y \in C$. This implies that $x^* \in EP(f)$. Therefore, $x^* \in \mathcal{F}$.

On the other hand, let $v_n = \Pi_{\mathcal{F}} x_n$; from Lemma 2.9 and $x^* \in \mathcal{F}$, we have

$$\langle v_{n_k} - x^*,\ Jx_{n_k} - Jv_{n_k}\rangle \geq 0. \tag{3.31}$$

From Theorem 3.1, one knows that $\{v_n\}$ converges strongly to $z \in \mathcal{F}$. Since J is weakly sequentially continuous, one has

$$\langle z - x^*,\ Jx^* - Jz\rangle \geq 0, \tag{3.32}$$

as $k \to \infty$. On the other hand, since J is monotone, one has

$$\langle z - x^*,\ Jx^* - Jz\rangle \leq 0, \tag{3.33}$$

as $k \to \infty$. Also, since E is uniformly convex, one has $z = x^*$. This completes the proof of Theorem 3.2. □

If we only consider one operator T, the following corollary can been established by Theorem 3.2.

Corollary 3.3. *Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to R satisfying (A_1)–(A_4), and let $T : C \to C$ be a weak relatively uniformly nonexpansive multivalued mapping such that $\mathcal{F} = F(T) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$x_n \in C \quad \text{such that } f(x_n, y) + \frac{1}{r_n}\langle y - x_n, \ Jx_n - Ju_n\rangle \geq 0, \quad \forall y \in C,$$
$$u_{n+1} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jy_n), \tag{3.34}$$

where $y_n \in Tx_n$, and J is the duality mapping on E. Assume that $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying the following conditions:

(a) $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$;

(b) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.

If J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to $z \in \mathcal{F}$, where $z = \lim_{n\to\infty}\Pi_{\mathcal{F}}x_n$.

If T and S are two relatively uniformly nonexpansive multivalued mappings, from Definitions 1.1 and 1.3, it is easy to know that the class of weak relatively uniformly nonexpansive multivalued mappings contains the class of relatively uniformly nonexpansive multivalued mappings as a subclass. Therefore, the following corollary can be easily obtained by Theorem 3.2.

Corollary 3.4. *Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to R satisfying (A_1)–(A_4) and let $T, S : C \to C$ be two relatively uniformly nonexpansive multivalued mappings such that $\mathcal{F} = F(T) \cap F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$x_n \in C \quad \text{such that } f(x_n, y) + \frac{1}{r_n}\langle y - x_n, \ Jx_n - Ju_n\rangle \geq 0, \quad \forall y \in C,$$
$$u_{n+1} = J^{-1}(\alpha_n Jx_n + \beta_n Jy_n + \gamma_n Jz_n), \tag{3.35}$$

where $y_n \in Tx_n$, $z_n \in Sx_n$, and J are the duality mapping on E. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[0,1]$ satisfying the following conditions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\liminf_{n\to\infty}\alpha_n\beta_n > 0$, $\liminf_{n\to\infty}\alpha_n\gamma_n > 0$;

(c) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.

If J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to $z \in \mathcal{F}$, where $z = \lim_{n\to\infty}\Pi_{\mathcal{F}}x_n$.

Remark 3.5. Our results improve Theorem 4.1. of Takahashi and Zembayashi [12] and Theorem 3.5. of Qin et al. [11] in the following senses:

(1) from single-valued mappings to multivalued ones;

(2) from relatively nonexpansive single-valued mappings (the definition can be found in [1, 11, 12]) to weak relatively uniformly nonexpansive multivalued ones.

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