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Research Article

Fixed Point Theory for Cyclic Generalized Weak ϕ-Contraction on Partial Metric Spaces

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A new fixed point theorem is obtained for the class of cyclic weak ϕ-contractions on partially metric spaces. It is proved that a self-mapping T on a complete partial metric space X has a fixed point if it satisfies the cyclic weak ϕ-contraction principle.

1. Introduction and Preliminaries

Fixed point theorems analyze the conditions on maps (single or multivalued) under which the existence of a solution for the equation $d(x,Tx) = 0$ can be guaranteed. The Banach contraction mapping principle [1] is one of the earliest, widely known, and important results in this direction. After this pivotal principle, a number of remarkable results started to appear in the literature published by many authors (see, e.g., [2–36]). In this trend, one of the distinguished contributions was announced by Kirk et al. [15] in 2003. They introduced the notion of cyclic contraction, which is defined as follows.

Let A and B be two subsets of a metric space (X,d). A self-mapping T on $A\cup B$ is called cyclic provided that $T(A) \subset B$ and $T(B) \subset A$. In addition, if for some $k \in (0,1)$, the mapping T satisfies the inequality

$$d(Tx,Ty) \leq kd(x,y) + (1-k)\,\mathrm{dist}(A,B) \quad \forall x \in A,\ y \in B, \tag{1.1}$$

then it is called cyclic contraction where $\mathrm{dist}(A,B) = \inf\{d(x,y) : x \in A,\ y \in B\}$. In their outstanding paper [15], the authors also proved that if A and B are closed subsets of a complete metric space with $A \cap B \neq \emptyset$, then the cyclic contraction T has a unique fixed point in $A \cap B$.

The notion of Φ-contraction was defined by Boyd and Wong [11]: A self-mapping T on a metric space X is called Φ-contraction if there exists an upper semi-continuous function $\Phi : [0,\infty) \to [0,\infty)$ such that

$$d(Tx,Ty) \leq \Phi(d(x,y)) \quad \forall x,y \in X. \tag{1.2}$$

This concept was generalized by Alber and Guerre-Delabriere [7], by introducing weak ϕ-contraction. A self-mapping T on a metric space X is called weak ϕ-contraction if $\phi : [0,\infty) \to [0,\infty)$ is a strictly increasing map with $\phi(0) = 0$ and

$$d(Tx,Ty) \leq d(x,y) - \phi(d(x,y)), \quad \forall x,y \in X. \tag{1.3}$$

Păcurar and Rus [19] gave a characterization of Φ-contraction mappings in the context of cyclic operators and proved some fixed point results for such mappings on a complete metric space. As a natural next step, Karapınar [13] generalized the results in [19] by replacing the notion of cyclic φ-contraction mappings with cyclic weak ϕ-contraction mappings. For more results for cyclic mapping analysis we refer to [3, 7, 12, 14, 16, 17, 21–23] and the references therein.

Recently, in fixed point theory, one of the celebrated subjects is the partial metric spaces. The notion of a partial metric space was defined by Matthews [37] in 1992 as a generalization of usual metric spaces. The motivation behind the theory of partial metric spaces is to transfer mathematical techniques into computer science to develop the branches of computer science such as domain theory and semantics (see, e.g., [35, 38–46]).

Definition 1.1 (see [37, 47]). A partial metric on a nonempty set X is a function $p : X \times X \to [0,\infty)$ such that for all $x,y,z \in X$

(PM1) $p(x,y) = p(y,x)$ (symmetry),

(PM2) if $p(x,x) = p(x,y) = p(y,y)$, then $x = y$ (equality),

(PM3) $p(x,x) \leq p(x,y)$ (small self-distances),

(PM4) $p(x,z) + p(y,y) \leq p(x,y) + p(y,z)$ (triangularity).

The pair (X,p) is called a partial metric space (abbreviated by PMS).

Remark 1.2. If $x = y$, then $p(x,y)$ may not be 0.

Example 1.3 (see [47]). Let $X = \{[a,b] : a,b,\in \mathbb{R},\ a \leq b\}$, and define $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. Then, (X,p) is a partial metric space.

Example 1.4 (see [47]). Let $X := [0,1] \cup [2,3]$, and define $p : X \times X \to [0,\infty)$ by

$$p(x,y) = \begin{cases} \max\{x,y\} & \text{if } \{x,y\} \cap [2,3] \neq \emptyset, \\ |x-y| & \text{if } \{x,y\} \subset [0,1]. \end{cases} \tag{1.4}$$

Then, (X,p) is a complete partial metric space.

Example 1.5 (see [48]). Let (X,d) and (X,p) be a metric space and a partial metric space, respectively. Mappings $\rho_i : X \times X \to \mathbb{R}^+$ $(i \in \{1,2,3\})$ defined by

$$\begin{aligned} \rho_1(x,y) &= d(x,y) + p(x,y) \\ \rho_2(x,y) &= d(x,y) + \max\{\omega(x),\omega(y)\} \\ \rho_3(x,y) &= d(x,y) + a \end{aligned} \tag{1.5}$$

induce partial metrics on X, where $\omega : X \to \mathbb{R}^+$ is an arbitrary function and $a \geq 0$.

Each partial metric p on X generates a T_0 topology τ_p on X which has the family of open p-balls $\{B_p(x,\varepsilon), x \in X, \varepsilon > 0\}$ as a base, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the function $d_p : X \times X \to [0,\infty)$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{1.6}$$

is a metric on X. Furthermore, it is possible to observe that the following

$$d_m(x,y) = \max\{p(x,y) - p(x,x), p(x,y) - p(y,y)\}, \tag{1.7}$$

also defines a metric on X. In fact, d_p and d_m are equivalent (see, e.g., [9]).

Example 1.6 (see, e.g. [5, 27, 30, 47]). Consider $X = [0,\infty)$ with $p(x,y) = \max\{x,y\}$. Then, (X,p) is a partial metric space. It is clear that p is not a (usual) metric. Note that in this case $d_p(x,y) = |x-y| = d_m(x,y)$.

For our purposes, we need to recall some basic topological concepts in partial metric spaces (for details see, e.g., [5, 8, 27, 30, 37, 47]).

Definition 1.7. (1) A sequence $\{x_n\}$ in the PMS (X,p) converges to a limit point x if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

(2) A sequence $\{x_n\}$ in the PMS (X,p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.

(3) A PMS (X,p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

(4) A mapping $F : X \to X$ is said to be continuous at $x_0 \in X$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $F(B_p(x_0,\delta)) \subseteq B_p(Fx_0,\epsilon)$.

Lemma 1.8. *(1) A sequence $\{x_n\}$ is a Cauchy sequence in the PMS (X,p) if and only if it is a Cauchy sequence in the metric space (X,d_p).*

(2) A PMS (X,p) is complete if and only if the metric space (X,d_p) is complete. Moreover,

$$\lim_{n\to\infty} d_p(x,x_n) = 0 \Longleftrightarrow p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m). \tag{1.8}$$

In [32], Romaguera introduced the concepts of a 0-Cauchy sequence in a partial metric space and of a 0-complete partial metric space as follows.

Definition 1.9. A sequence $\{x_n\}$ in a partial metric space (X,p) is called 0-Cauchy if $\lim_{n,m\to\infty} p(x_n,x_m) = 0$. A partial metric space (X,p) is said to be 0-complete if every 0-Cauchy sequence in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = 0$. In this case, p is said to be a 0-complete partial metric on X.

Remark 1.10 (see [32]). Each 0-Cauchy sequence in (X,p) is a Cauchy sequence in (X,p), and each complete partial metric space is 0-complete.

The following example shows that there exists a 0-complete partial metric space that is not complete.

Example 1.11 (see [32]). Let $(\mathbb{Q} \cap [0,\infty), p)$ be a partial metric space, where $\mathbb{Q}$ denotes the set of rational numbers and the partial metric p is given by $p(x,y) = \max\{x,y\}$.

Now, we state a number of simple but useful lemmas that can be derived by manipulating the properties $(P1) - (P4)$.

Lemma 1.12. *Let (X,p) be a PMS. Then, the following statements hold true:*

(A) *if $\lim_{n\to\infty} p(x_n,z) = p(z,z)$ and $p(z,z) = 0$, then $\lim_{n\to\infty} p(x_n,y) = p(z,y)$ for every $y \in X$ (see [5, 27, 30]),*

(B) *if $\lim_{n\to\infty} p(x_n,y) = p(y,y)$, $\lim_{n\to\infty} p(x_n,z) = p(z,z)$, and $p(y,y) = p(z,z)$, then $y = z$ (see [31]),*

(C) *if $p(x,y) = 0$ then $x = y$ (see [27, 30]),*

(D) *if $x \neq y$, then $p(x,y) > 0$ (see [27, 30]).*

One of the implications of Lemma 1.12 can be stated as follows.

Remark 1.13. If $\{x_n\}$ converges to x in (X,p), then $\lim_{n\to\infty} p(x_n,y) \leq p(x,y)$ for all $y \in X$.

We would like to point out that the topology induced by a partial metric differs from the topology induced by a metric in certain aspects. The following remark and example highlight one of these aspects.

Remark 1.14. Limit of a sequence $\{x_n\}$ in a partial metric space (X,p) may not be unique.

Example 1.15. Consider $X = [0,\infty)$ with $p(x,y) = \max\{x,y\}$. Then, (X,p) is a partial metric space. Clearly, p is not a metric. Observe that the sequence $\{1 + (1/1+n)\}$ converges to both $x = 5$ and $y = 6$, for example. Therefore, there is no uniqueness of the limit in this partial metric space.

2. Main Results

In this section we aim to state and prove our main results. These results are more general in the sense that they are applicable to chains of cyclic representations and related generalized weak ϕ-contractions introduced in the following two definitions.

Definition 2.1 (see, e.g., [13]). Let X be a nonempty set, m, a positive integer, and $T : X \to X$ an operator. The union $X = \cup_{i=1}^{m} X_i$ is called a cyclic representation of X with respect to T if

(1) each X_i is a nonempty set for $i = 1, \ldots, m$,

(2) $T(X_1) \subset X_2, \ldots, T(X_{m-1}) \subset X_m, T(X_m) \subset X_1$.

Definition 2.2 (cf. [13]). Let (X, p) be a partial metric space, m be a positive integer, $A_1, \ldots, A_m$ closed nonempty subsets of X and $Y = \cup_{i=1}^{m} A_i$. An operator $T : Y \to Y$ is called a cyclic generalized weak ϕ-contraction if

(1) $\cup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T,

(2) there exists a continuous, nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(t) > 0$ for every $t \in (0, \infty)$ and $\phi(0) = 0$, such that

$$p(Tx, Ty) \leq M(x, y) - \phi(M(x, y)), \tag{2.1}$$

where

$$M(x, y) = \max\{p(x, y), p(Tx, x), p(y, Ty)\}, \tag{2.2}$$

for any $x \in A_i$, $y \in A_{i+1}$ for $i = 1, 2, \ldots, m$ with $A_{m+1} = A_1$.

Our main theorem in this work is stated below.

Theorem 2.3. *Let (X, p) be a 0-complete partial metric space, m a positive integer, and $A_1, \ldots, A_m$ be closed non-empty subsets of X. Let $Y = \cup_{i=1}^{m} A_i$ be a cyclic representation of Y with respect to T. Suppose that T is a cyclic generalized weak ϕ-contraction. Then, T has a unique fixed point $z \in \cap_{i=1}^{m} A_i$.*

Proof. First we show that if T has a fixed point, then it is unique. Suppose on the contrary that there exist $z, w \in \cap_{i=1}^{m} A_i$ such that $Tz = z$ and $w = Tw$ with $z \neq w$. Due to (2.1), we have

$$p(z, w) = p(Tz, Tw) \leq M(z, w) - \phi(M(z, w)), \tag{2.3}$$

which is equivalent to

$$p(z, w) = p(Tz, Tw) \leq p(z, w) - \phi(p(z, w)) \tag{2.4}$$

by (PM3) in Definition 1.1. This is a contradiction because $p(z, w) > 0$ by Lemma 1.12 and $\phi(t) > 0$ for all $t > 0$. Hence, T has a unique fixed point.

Next we prove the existence of a fixed point of T. Let $x_0 \in Y = \cup_{i=1}^{m} A_i$ and $x_{n+1} = Tx_n$ be the Picard iteration. If there exists n_0 such that $x_{n_0} = x_{n_0+1}$, then the theorem follows. Indeed, we get $x_{n_0} = x_{n_0+1} = Tx_{n_0}$ and, therefore, x_0 is a fixed point of T. Thus, we may assume that $x_n \neq x_{n+1}$ for all n where we have

$$p(x_n, x_{n+1}) > 0, \tag{2.5}$$

by Lemma 1.12.

Since T is cyclic, for any $n \geq 0$, there is $i_n \in \{1, 2, \ldots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$. Then, by (2.1), we have

$$p(x_{n+1}, x_{n+2}) = p(Tx_n, Tx_{n+1}) \leq M(x_n, x_{n+1}) - \phi(M(x_n, x_{n+1})), \tag{2.6}$$

where

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\{p(x_n, x_{n+1}), p(Tx_n, x_n), p(x_{n+1}, Tx_{n+1})\} \\ &= \max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\}. \end{aligned} \tag{2.7}$$

If we assume $M(x_n, x_{n+1}) = p(x_{n+1}, x_{n+2})$, then (2.6) turns into

$$p(x_{n+1}, x_{n+2}) \leq p(x_{n+1}, x_{n+2}) - \phi(p(x_{n+1}, x_{n+2})), \tag{2.8}$$

which is a contradiction. Because when we let $t = p(x_{n+1}, x_{n+2})$, which is positive by (2.5), we get $\phi(t) = \phi(p(x_{n+1}, x_{n+2})) > 0$, we need to take $M(x_n, x_{n+1}) = p(x_n, x_{n+1})$. Define $t_n = p(x_n, x_{n+1})$. Then, the inequality in (2.6) turns into

$$t_{n+1} \leq t_n - \phi(t_n) \leq t_n. \tag{2.9}$$

Therefore, $\{t_n\}$ is a nonnegative nonincreasing sequence. Hence, $\{t_n\}$ converges to $L \geq 0$. We aim to show that $L = 0$. Suppose on the contrary that $L > 0$. Letting $n \to \infty$ in (2.9), we get that

$$L \leq L - \lim_{n\to\infty} \phi(t_n) \leq L. \tag{2.10}$$

Thus, we obtain that

$$\lim_{n\to\infty} \phi(t_n) = 0. \tag{2.11}$$

Since $L = \inf\{t_n = p(x_n, x_{n+1}) : n \in \mathbb{N}\}$, then $0 < L \leq t_n$ for $n = 0, 1, 2, \ldots$. Taking $n \to \infty$ in the previous inequality, we derive

$$0 < \phi(L) \leq \lim_{n\to\infty} \phi(t_n), \tag{2.12}$$

which contradicts with (2.11). Thus, we have $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$.

We claim that $\{t_n\}$ is a 0-Cauchy sequence. In order to prove this assertion, we first prove the following statement.

(A) For every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that if $s, q \geq n_0$ with $s - q \equiv 1(m)$, then $p(x_s, x_q) < \epsilon$.

Assume the contrary that there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$ we can find $s > q \geq n$ with $s - q \equiv 1(m)$ satisfying the inequality $p(x_s, x_q) \geq \epsilon$. Then, we have

$$\begin{aligned} \epsilon \leq p(x_s, x_q) &\leq p(x_s, x_{s+1}) + p(x_{s+1}, x_{q+1}) + p(x_{q+1}, x_q) \\ &= p(x_s, x_{s+1}) + p(Tx_s, Tx_q) + p(x_{q+1}, x_q), \end{aligned} \tag{2.13}$$

by the triangular inequality. Since x_s and x_q lie in different adjacently labeled sets A_i and A_{i+1} for certain $1 \leq i \leq m$, from (2.13) we get

$$\begin{aligned} \epsilon \leq p(x_s, x_q) &\leq p(x_s, x_{s+1}) + M(x_s, x_q) - \phi(M(x_s, x_q)) + p(x_{q+1}, x_q) \\ &\leq p(x_s, x_{s+1}) + M(x_s, x_q) - \phi(p(x_s, x_q)) + p(x_{q+1}, x_q) \end{aligned} \tag{2.14}$$

by using the contractive condition in (2.1), where

$$M(x_s, x_q) = \max\{p(x_s, x_q), p(x_s, x_{s+1}), p(x_{q+1}, x_q)\}. \tag{2.15}$$

If $M(x_s, x_q)$ is equal to either $p(x_{q+1}, x_q)$ or $p(x_s, x_{s+1})$, then letting $s, q \to \infty$ in (2.14) yields that

$$\epsilon \leq \lim_{s,q\to\infty} p(x_s, x_q) \leq -\lim_{s,q\to\infty} \phi(p(x_s, x_q)) \tag{2.16}$$

is a contradiction. Hence, $M(x_s, x_q) = p(x_s, x_q)$ and inequality (2.14) give

$$\phi(p(x_s, x_q)) \leq p(x_s, x_{s+1}) + p(x_{q+1}, x_q). \tag{2.17}$$

As $\epsilon \leq p(x_s, x_q)$ and ϕ is nondecreasing, we have

$$0 < \phi(\epsilon) \leq \phi(p(x_s, x_q)) \tag{2.18}$$

Inequalities (2.17) and (2.18) give us

$$0 < \phi(\epsilon) \leq \phi(p(x_s, x_q)) \leq p(x_s, x_{s+1}) + p(x_{q+1}, x_q). \tag{2.19}$$

Letting $s, q \to \infty$ with $s - q \equiv 1(m)$ and since $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$, we have

$$\begin{gathered} 0 < \phi(\epsilon) \leq \lim_{s,q\to\infty} \phi(p(x_s, x_q)) \leq 0 \\ s - q \equiv 1(m), \end{gathered} \tag{2.20}$$

which is a contradiction. Therefore, (A) is proved.

Now, in order to prove that $\{x_n\}$ is a Cauchy sequence, we fix an $\epsilon > 0$. Using (A), we find $n_0 \in \mathbb{N}$ such that if $s, q \geq n_0$ with $s - q \equiv 1(m)$,

$$p(x_s, x_q) \leq \frac{\epsilon}{m}. \tag{2.21}$$

On the other hand, since $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$, we also find $n_1 \in \mathbb{N}$ such that

$$p(x_n, x_{n+1}) \leq \frac{\epsilon}{m}, \tag{2.22}$$

for any $n \geq n_1$. We take $k, l \geq \max\{n_0, n_1\}$ with $l > k$. Then, there exists $K \in \{1, 2, \ldots, m\}$ such that $l - k \equiv K(m)$. Therefore, $l - k + j \equiv 1(m)$ for $j = m - K + 1$, and so

$$p(x_k, x_l) \leq p(x_k, x_{l+j}) + p(x_{l+j}, x_{l+j-1}) + \cdots + p(x_{l+1}, x_l). \tag{2.23}$$

From (2.21) and (2.22) and the last inequality,

$$p(x_k, x_l) \leq \frac{\epsilon}{m} + (j-1)\frac{\epsilon}{m} = j\frac{\epsilon}{m} \leq \epsilon. \tag{2.24}$$

Hence,

$$\lim_{k,l\to\infty} p(x_k, x_l) = 0. \tag{2.25}$$

This prove that $\{t_n\}$ is a 0-Cauchy sequence.

Since X is a 0-complete partial metric space, there exists $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = p(x, x)$. Now, we will prove that x is a fixed point of T. In fact, since $\{x_n\}$ converges to x and $Y = \cup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T, the sequence $\{x_n\}$ has infinite number of terms in each A_i for $i \in \{1, 2, \ldots, m\}$. Regarding that A_i is closed for $i \in \{1, 2, \ldots, m\}$, we have $x \in \cap_{i=1}^{m} A_i$. Now fix $i \in \{1, 2, \ldots, m\}$ such that $x \in A_i$ and $Tx \in A_{i+1}$. We take a subsequence $\{x_{n_k}\}$ of $\{x_m\}$ with $x_{n_k} \in A_{i-1}$ (the existence of this subsequence is guaranteed by the mentioned comment above). Using the triangular inequality and the contractive conditions, we can obtain

$$\begin{aligned} p(x, Tx) &\leq p(x, x_{n_{k+1}}) + p(x_{n_{k+1}}, Tx) \\ &= p(x, x_{n_{k+1}}) + p(Tx_{n_k}, Tx) \\ &\leq p(x, x_{n_{k+1}}) + M(x_{n_k}, x) - \phi(M(x_{n_k}, x)), \end{aligned} \tag{2.26}$$

where $M(x, x_{n_k}) = \{p(x, Tx), p(x_{n_k}, x_{n_k+1}), p(x, x_{n_k})\}$. If $M(x, x_{n_k})$ is equal to either $p(x_{n_k}, x_{n_k+1})$ or $p(x, x_{n_k})$, then by letting $k \to \infty$, inequality (2.26) implies

$$p(x, Tx) \leq 0. \tag{2.27}$$

Therefore, $p(x,Tx) = 0$, that is, x is a fixed point of T by Lemma 1.12. If $M(x,x_{n_k}) = p(x,Tx)$ then (2.26) turns into

$$p(x,Tx) \le p(x,x_{n_{k+1}}) + p(x,Tx) - \phi(p(x,Tx)) \Longleftrightarrow \phi(p(x,Tx)) \le p(x,x_{n_{k+1}}). \tag{2.28}$$

Letting $k \to \infty$, we get that $\lim_{k\to\infty}\phi(p(x,Tx)) = 0$. Hence, by the properties of ϕ we have $p(x,Tx) = 0$. □

Theorem 2.4. *Let $T : Y \to Y$ be a self-mapping as in Theorem 2.3. Then, the fixed point problem for T is well posed, that is, if there exists a sequence $\{y_n\}$ in Y with $p(y_n,Ty_n) \to 0$, as $n \to \infty$, then $p(y_n,z) \to 0$ as $n \to \infty$.*

Proof. Due to Theorem 2.3, we know that for any initial value $y \in Y$, $z \in \cap_{i=1}^{m} A_i$ is the unique fixed point of T. Thus, $p(y_n,z)$ is well defined. Consider

$$p(y_n,z) \le p(y_n,Ty_n) + p(Ty_n,Tz) \le p(y_n,Ty_n) + M(y_n,z) - \phi(M(y_n,z)), \tag{2.29}$$

where $M(y_n,z) = \{p(y_n,z), p(y_n,Ty_n), p(z,Tz)\}$. There are three cases to consider: if $M(y_n,z) = p(y_n,z)$, then (2.29) is equivalent to

$$0 \le p(y_n,Ty_n) - \phi(p(y_n,z)) \Longleftrightarrow \phi(p(y_n,z)) \le p(y_n,Ty_n). \tag{2.30}$$

Since we are given $p(y_n,Ty_n) \to 0$ as $n \to \infty$, we derive that $\phi(p(y_n,z)) \to 0$. Regarding the definition of ϕ, we obtain that $p(y_n,z) \to 0$. The theorem follows.

If it is the case that $M(y_n,z) = p(y_n,Ty_n)$, then the right-hand side of (2.29) tends to 0 as n tends to infinity. Hence, the theorem follows again.

As the last case, we have $M(y_n,z) = p(z,Tz)$. Similarly we conclude that the right-hand side of (2.29) tends to 0 as n tends to infinity because we know that $p(z,Tz) = 0$ by Theorem 2.3 and $p(y_n,Ty_n) \to 0$, which completes the proof. □

Theorem 2.5. *Let $T : Y \to Y$ be a self-mapping as in Theorem 2.3. Then, T has the limit shadowing property, that is, if there exists a convergent sequence $\{y_n\}$ in Y with $p(y_{n+1},Ty_n) \to 0$ and $p(y_n,Ty_n) \to 0$, as $n \to \infty$, then there exists $x \in Y$ such that $p(y_n,T^n x) \to 0$ as $n \to \infty$.*

Proof. As in the proof of Theorem 2.4, we observe that for any initial value $x \in Y$, $z \in \cap_{i=1}^{m} A_i$ is the unique fixed point of T. Thus, $p(y_n,z), p(y_{n+1},z)$ are well defined. Set y as a limit of a convergent sequence $\{y_n\}$ in Y. Consider

$$p(y_{n+1},z) \le p(y_{n+1},Ty_n) + p(Ty_n,Tz) \le p(y_{n+1},Ty_n) + M(y_n,z) - \phi(M(y_n,z)), \tag{2.31}$$

where $M(y_n,z) = \{p(y_n,z), p(y_n,Ty_n), p(z,Tz)\}$. There are three cases to consider: If $M(y_n,z) = p(y_n,z)$, then (2.31) is equivalent to

$$p(y,z) \le p(y,z) - \phi(p(y,z)), \tag{2.32}$$

as $n \to \infty$. This is possible only if $\phi(p(y,z)) = 0$, which implies that $p(y,z) = 0$ and thus $y = z$. Thus, we have $p(y_n,T^n x) \to p(y,z) = 0$, as $n \to \infty$.

If it is the case that $M(y_n, z) = p(y_n, Ty_n)$, then the right-hand side of (2.31) tends to 0 as n tends to infinity. Hence, the theorem follows again.

As the last case, we have $M(y_n, z) = p(z, Tz)$. Similarly we conclude that the right-hand side of (2.31) tends to 0 as n tends to infinity because we know that $p(z, Tz) = 0$ by Theorem 2.3 and $p(y_{n+1}, Ty_n) \to 0$, which completes the proof. □

Theorem 2.6. *Let X be a non-empty set, (X, p) and (X, ρ) partial metric spaces, m a positive integer, and $A_1, A_2, \ldots, A_m$ closed non-empty subsets of X and $Y = \cup_{i=1}^{m} A_i$. Suppose that*

(1) *$Y = \cup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T.*

(2) *$p(x, y) \leq \rho(x, y)$ for all $x, y \in Y$,*

(3) *(Y, p) is a complete partial metric space,*

(4) *$T : (Y, p) \to (Y, \rho)$ is continuous,*

(5) *$T : (Y, p) \to (Y, \rho)$ is a cyclic weak ϕ-contraction where $\phi : [0, \infty) \to [0, \infty)$ with $\phi(t) > 0$ is a lower semicontinuous function for $t \in (0, \infty)$ and $\phi(0) = 0$.*

Then, $\{T^n x_0\}$ converges to z in (Y, p) for any $x_0 \in Y$ and z is the unique fixed point of T.

Proof. Let $x_0 \in Y$. As in Theorem 2.3, assumption (5) implies that $\{T^n x_0\}$ is a Cauchy sequence in (Y, ρ). Taking (2) into account, $\{T^n x_0\}$ is a Cauchy sequence in (Y, d), and due to (3) it converges to z in (Y, ρ) for any $x_0 \in Y$. Condition (4) implies the uniqueness of z. □

Definition 2.7 (cf. [2]). Let (X, p) be a partial metric space, m a positive integer, and $A_1, \ldots, A_m$ closed non-empty subsets of X and $Y = \cup_{i=1}^{m} A_i$. An operator $T : Y \to Y$ is called a cyclic generalized φ-contraction if

(1) $\cup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T, and

(2) there exists a continuous, non-decreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(t) > 0$ for every $t \in (0, \infty)$ and $\varphi(0) = 0$, such that

$$p(Tx, Ty) \leq \varphi(M(x, y)), \tag{2.33}$$

where

$$M(x, y) = \max\{p(x, y), p(Tx, x), p(y, Ty)\} \tag{2.34}$$

for any $x \in A_i, y \in A_{i+1}$ for $i = 1, 2, \ldots, m$ with $A_{m+1} = A_1$.

The following is an analog of the main theorem in [2].

Corollary 2.8. *Let (X, p) be a 0-complete partial metric space, m a positive integer, and $A_1, \ldots, A_m$ closed non-empty subsets of X. Let $Y = \cup_{i=1}^{m} A_i$ be a cyclic representation of Y with respect to T. Suppose that T is a cyclic generalized φ-contraction. Then, T has a unique fixed point $z \in \cap_{i=1}^{m} A_i$.*

Proof. It follows from Theorem 2.3 by taking $\phi(t) = t - \varphi(t)$. □

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