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Research Article

On Semi-(B, G)-Preinvex Functions

Xiaoling Liu and D. H. Yuan

Department of Mathematics, Hanshan Normal University, Chaozhou, Guangdong 521041, China

Correspondence should be addressed to D. H. Yuan, dhyuan@hstc.edu.cn

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We firstly construct a concrete semi-invex set which is not invex. Basing on concept of semi-invex set, we introduce some kinds of generalized convex functions, which include semi-(B, G)-preinvex functions, strictly semi-(B, G)-preinvex functions and explicitly semi-(B, G)-preinvex functions. Moreover, we establish relationships between our new generalized convexity and generalized convexity introduced in the literature. With these relationships and the well-known results pertaining to common generalized convexity, we obtain results for our new generalized convexities. We extend the existing results in the literature.

1. Introduction

It is well known that convexity has been playing a key role in mathematical programming, engineering, and optimization theory. The generalization of convexity is one of the most important aspects in mathematical programming and optimization theory. There have been many attempts to weaken the convexity assumptions in the literature [1–17].

One of generalization of convexity, invexity, was introduced by Hanson in [5]. Further, he proved that invexity has a common property that Karush-Kuhn-Tucker conditions are sufficient for global optimality of nonlinear programming under the invexity assumptions. Ben-Israel and Mond [6] introduced the concept of preinvex functions, which is a special case of invexity. On the other hand, Avriel [1] introduced the definition of r-convex functions which is another generalization of convex functions. He also discussed some characterizations and the relations between r-convexity and other generalization of convexity. In [18], Antczak introduced the concept of a class of r-preinvex functions which is a generalization of r-convex functions and preinvex functions, obtained some optimality results under r-preinvexity assumption for constrained optimization problems.

Recently, Antczak [19] extended invexity concept to G-invexity for scalar differentiable functions. In the natural way, Antczak's definition of G-invexity was also extended to the differentiable vector-valued case in [20]. With vector G-invexity, Antczak [21] proved new

duality results for nonlinear differentiable multiobjective programming problems. To deal with programming which is not necessarily differential, Antczak [22] introduced the concept of G-preinvexity, which unifies the concepts of nondifferentiable convexity, preinvexity, and r-preinvexity. Antczak [23], Luo and Wu [24] also discussed relations between concepts of different preinvexity. Further, various concepts of D-η-properly prequasi-invex functions were introduced in [25].

Note that characterizing the generalized convex functions are important in mathematical programming and optimization theory. Many researchers have extensively studied the properties of different generalized convex functions. Yang et al. [26] presented characterizations for prequasiinvex functions, semistrictly prequasi-invex functions, and strictly prequasi-invex functions. In [16, 17], Yang and Li presented characterizations for preinvex functions and semistrictly preinvex functions. Next, Luo and Wu [27], Luo and Xu [28], Luo et al. [29] obtained the same results or even more general ones under weaker assumptions. Luo and Wu [27] also gave characterization for strictly preinvex functions under mild conditions. Yang et al. [30] proved that the explicit B-preinvexity, together with the intermediate-point B-preinvexity, implies B-preinvexity, while the explicit B-preinvexity, together with a lower semicontinuity, implies the B-preinvexity. Characterizations of D-η-properly prequasi-invex functions were presented in [25, 31, 32].

Motivated by [10, 11, 14, 16, 17, 22–24, 31], we present some new kinds of generalized convex functions, which include semi-(B,G)-preinvex functions, strictly semi-(B,G)-preinvex functions and explicitly semi-(B,G)-preinvex functions. We have managed to characterize these new kinds of generalized convex functions. The rest of the paper is organized as follows. In Section 2, we firstly construct a concrete set which is not invex but semi-invex; basing on the semi-invex set, we define some new classes of generalized convex functions and discuss the relations with each other; we also establish relation theorems with common generalized convex functions introduced in the literature; moreover, we present the optimality properties for semi-(B,G)-preinvex functions and explicitly semi-(B,G)-preinvex functions. Section 3 obtains properties for these new kinds of generalized convexity. In Section 4, we discuss relations between (B,G)-preinvexity and explicitly (B,G)-preinvexity; we also obtain the characterizations of (B,G)-preinvexity and explicitly (B,G)-preinvexity. Section 5 gives some conclusions.

2. Definitions and Preliminaries

In this section, we provide some definitions and some results which we will use throughout the paper.

Definition 2.1. Let $X \subset \mathbb{R}^n$, $\eta : X \times X \times [0,1] \to \mathbb{R}^n$. The set X is said to be semi-invex at $u \in X$ with respect to η if for all $x \in X$, $\lambda \in [0,1]$ such that

$$u + \lambda\eta(x,u,\lambda) \in X. \tag{2.1}$$

X is said to be semi-invex set with respect to η if X is semi-invex at each $u \in X$. If $\eta(x,u,\lambda)$ is independent with respect to the third argument λ, then semi-invex set is called invex with respect to η.

Remark 2.2. If X is an invex set with respect to η, then X is a semi-invex set with respect to η. But the converse is not true. See the following example.

Example 2.3. Let X be a subset in $\mathbb{R}^n$ defined as follows:

$$X =: \left\{ (x_1, x_2) \mid 0 < x_2 < x_1^2,\ 0 < x_1 < 2 \right\} \cup \{(0,0)\}. \tag{2.2}$$

Consider the point $u = (0,0)$. Since the tangent line of the curve $x_2 = x_1^2$ at point u is the line $x_2 = 0$. Then, for any $x \in X \setminus \{u\}$, there exists $0 < \lambda_0 < 1$ such that

$$u + \lambda\eta(x,u) \notin X, \quad \forall \lambda \in (0, \lambda_0). \tag{2.3}$$

Therefore, there exists no vector-valued function $\eta(x,u) \neq 0$ such that

$$u + \lambda\eta(x,u) \in X, \quad \forall \lambda \in (0,1). \tag{2.4}$$

However, define $\eta(x,u;\lambda) := (x_1, (1/2)\lambda x_2)$ for $x = (x_1, x_2)$, then

$$u + \lambda\eta(x,u,\lambda) \in X, \quad \forall \lambda \in (0,1). \tag{2.5}$$

Hence, X is semi-invex at u with respect to η.

Definition 2.4 (see [33]). Let X be a nonempty semi-invex subset of $\mathbb{R}^n$. A real-valued function $f : X \to \mathbb{R}$ is said to be semi-B-preinvex at $u \in X$ with respect to η if there exist vector-valued function $\eta : X \times X \times [0,1] \to \mathbb{R}^n$ and real functions $b^1, b^2 : X \times X \times [0,1] \to \mathbb{R}_+$ such that for all $x \in X$

$$\begin{gathered} f\big(u + \lambda\eta(x,u,\lambda)\big) \leq b^1(x,u;\lambda) f(x) + b^2(x,u;\lambda) f(u), \\ b^1(x,u;1) = b^2(x,u;0) = 1, \quad b^1(x,u;\lambda) + b^2(x,u;\lambda) = 1, \quad \lambda \in (0,1), \end{gathered} \tag{2.6}$$

where $\lim_{\lambda \to 0} \lambda\eta(x,u,\lambda) = 0$. The real-valued function f is said to be semi-B-preinvex on X with respect to η if f is semi-B-preinvex at each $u \in X$ with respect to η; f is said to be strictly semi-B-preinvex on X with respect to η if strict inequality (2.6) holds for all $x, u \in X$ such that $x \neq u$; f is said to be explicitly semi-B-preinvex on X with respect to η if strict inequality (2.6) holds for all $x, u \in X$ such that $f(x) \neq f(u)$.

Remark 2.5. Note that semi-B-preinvexity is a special kind of (ϕ_1, ϕ_2) convexity defined in [11, 12]. Furthermore, assume that X is an invex subset. Then semi-B-preinvexity is B-preinvexity [14]; explicitly semi-B-preinvexity is explicitly B-preinvexity [30]; strictly semi-B-preinvexity is strictly B-preinvexity [34]. Moreover, if X be a convex set, then semi-B-preinvexity is B-vexity defined in [8, 9].

Definition 2.6. Let X be a nonempty semi-invex subset of $\mathbb{R}^n$. A real-valued function $f : X \to \mathbb{R}$ is said to be semi-(B,G)-preinvex at u on X with respect to η if there exists a continuous real-valued function $G : I_f(X) \to \mathbb{R}$ such that G is a strictly increasing function on its domain,

a vector-valued function $\eta : X \times X \times [0,1] \to \mathbb{R}^n$, and real functions $b^1, b^2 : X \times X \times [0,1] \to \mathbb{R}_+$ such that for all $x \in X$

$$\begin{gathered} f(u + \lambda\eta(x,u,\lambda)) \leq G^{-1}\Big(b^1(x,u;\lambda)G(f(x)) + b^2(x,u;\lambda)G(f(u))\Big), \\ b^1(x,u;1) = b^2(x,u;0) = 1, \quad b^1(x,u;\lambda) + b^2(x,u;\lambda) = 1, \quad \lambda \in (0,1). \end{gathered} \tag{2.7}$$

If inequality (2.7) holds for any $u \in X$, then f is semi-(B,G)-preinvex on X with respect to η; f is said to be strictly semi-(B,G)-preinvex on X with respect to η if strict inequality (2.7) holds for all $x, u \in X$ such that $x \neq u$; f is said to be explicitly semi-(B,G)-preinvex on X with respect to η if strict inequality (2.7) holds for all $x, u \in X$ such that $f(x) \neq f(u)$.

Remark 2.7. Let X be an invex subset. Then semi-(B,G)-preinvexity, strictly semi-(B,G)-preinvexity, and explicitly semi-(B,G)-preinvexity are called (B,G)-preinvexity, strictly (B,G)-preinvexity, and explicitly (B,G)-preinvexity, respectively.

Remark 2.8. Every G-preinvex function with respect to η introduced in [19, 22] is semi-(B,G)-preinvex function with respect to η, where $b^1(x,u;\lambda) = \lambda$, $b^2(x,u;\lambda) = 1 - \lambda$, $\lambda \in (0,1)$; every semi-B-preinvex function with respect to η introduced in [14] is semi-(B,G)-preinvex function with respect to η, where $G(a) = a$, $a \in \mathbb{R}$. The converse results are, in general, not true, see Example 2.10.

Remark 2.9. Every semistrictly G-preinvex function with respect to η introduced in [24] is explicitly (B,G)-preinvex function with respect to η, where $b^1(x,u;\lambda) = \lambda$, $b^2(x,u;\lambda) = 1 - \lambda$, $\lambda \in (0,1)$; every explicitly semi-B-preinvex function with respect to η introduced in [30] is explicitly semi-(B,G)-preinvex function with respect to η, where $G(a) = a$, $a \in \mathbb{R}$. The converse results are, in general, not true. See Example 2.10 too.

Example 2.10. Let X be the subset defined in Example 2.3, $x = (x_1, x_2)$, $u = (u_1, u_2) \in X$. Define

$$\eta(x,u,\lambda) = \begin{cases} \left(x_1, \dfrac{1}{2}\lambda x_2\right), & u = (0,0), \\ x^0 - u, & u \neq (0,0), \end{cases} \tag{2.8}$$

where $x^0 \in X$ is a point on the line between u and x, which is different from u, such that $\bigcup(u, \|u - x^0\|) \subset X$. Define

$$\begin{gathered} f(x) = \ln(x_1 + x_2 + 2), \quad x = (x_1, x_2) \in X, \\ b^1(x,u;\lambda) = \lambda, \quad b^2(x,u;\lambda) = 1 - \lambda, \quad \lambda \in (0,1), \\ G(a) = e^a, \quad a \in \mathbb{R}. \end{gathered} \tag{2.9}$$

Then, it is easy to check that f is both an explicitly semi-(B,G)-preinvex function and a semi-(B,G)-preinvex function on X with respect to η. However, f is not a G-preinvex function on

X with respect to η and f is also not a semistrictly G-preinvex function on X with respect to η, because X is not an invex set. Moreover, by letting $u = (0,0)$, $x = (1,1/2)$, $\lambda = 1/2$, we have

$$\begin{aligned} f(u+\lambda\eta(x,u,\lambda)) &= f\left(\frac{1}{2},\frac{1}{16}\right) = \ln\left(\frac{41}{16}\right) \\ &> \frac{1}{2}\ln 2 + \frac{1}{2}\ln\left(\frac{7}{2}\right) = \lambda f(x) + (1-\lambda) f(u). \end{aligned} \tag{2.10}$$

Hence, f is not an explicitly semi-B-preinvex function and f is also not a semi-B-preinvex function on X with respect to η.

From Definition 2.6, the inverse of function G must exist. Hence function G must be a strictly increasing one. Thus, we can assume that function G is a strictly increasing function on its domain. Now we give the following useful lemma.

Lemma 2.11. *Let $f : X \to \mathbb{R}$. Then:*

(i) *f is semi-(B,G)-preinvex on X with respect to η if and only if $G(f)$ is semi-B-preinvex on X with respect to η;*

(ii) *f is strictly semi-(B,G)-preinvex on X with respect to η if and only if $G(f)$ is strictly semi-B-preinvex on X with respect to η;*

(iii) *f is explicitly semi-(B,G)-preinvex on X with respect to η if and only if $G(f)$ is explicitly semi-B-preinvex on X with respect to η.*

Proof. (i) By the monotonicity of G, we know that the inequality (2.7) is equivalent with

$$\begin{gathered} G(f(u+\lambda\eta(x,u,\lambda))) \le b^1(x,u;\lambda)G(f(x)) + b^2(x,u;\lambda)G(f(u)), \\ b^1(x,u;1) = b^2(x,u;0) = 1, \quad b^1(x,u;\lambda) + b^2(x,u;\lambda) = 1, \quad \lambda \in (0,1). \end{gathered} \tag{2.11}$$

Therefore, by Definitions 2.6 and 2.4, f is semi-(B,G)-preinvex on X with respect to η if and only if $G(f)$ is semi-B-preinvex on X with respect to η.

Similar to part (i), we can prove (ii) and (iii). This completes the proof. □

Theorems 2.12 and 2.13, present the optimality properties for semi-(B,G)-preinvex functions and explicitly semi-(B,G)-preinvex functions, respectively.

Theorem 2.12. *Let X be a nonempty semi-invex set in $\mathbb{R}^n$ with respect to $\eta : X \times X \times [0,1] \to \mathbb{R}^n$, and $f : X \to \mathbb{R}$ be a semi-(B,G)-preinvex function on X with respect to η. If $\overline{x} \in X$ is a local minimum to the problem of minimizing $f(x)$ subject to $x \in X$, then $\overline{x}$ is a global one.*

Proof. Let f be a semi-(B,G)-preinvex function on X with respect to η. Then, by Lemma 2.11(i), $G(f)$ is a semi-B-preinvex function on X with respect to η. Since G is increasing on its domain $I_f(x)$, then $\overline{x} \in X$ is a local minimum to the problem of minimizing $f(x)$ subject to $x \in X$ if and only if $\overline{x} \in X$ is a local minimum to the problem of minimizing $G(f)(x)$ subject to $x \in X$. Therefore, by Theorem 3.1 in [33], $\overline{x} \in X$ is a global one to the problem of minimizing $G(f)(x)$ subject to $x \in X$. Hence $\overline{x} \in X$ is a global one for the problem of minimizing $f(x)$ subject to $x \in X$. This completes the proof. □

Theorem 2.13. *Let X be a nonempty semi-invex set in $\mathbb{R}^n$ with respect to $\eta : X \times X \times [0,1] \to \mathbb{R}^n$, and $f : X \to \mathbb{R}$ be an explicitly semi-(B,G)-preinvex function on X with respect to η. If $\overline{x} \in X$ is a local minimum to the problem of minimizing $f(x)$ subject to $x \in X$, then $\overline{x}$ is a global one.*

Proof. Similar to the proof of Theorem 2.12, from Theorem 3.1 in [17], we can establish the result. □

From Example 2.10, Theorems 2.12 and 2.13, we can conclude that these new generalized convex functions constitutes an important class of generalized convex functions in mathematical programming.

3. Properties of Semi-(B,G)-Preinvex Functions

In this section, we first discuss the relations between our new kinds of generalized convex functions. By definitions of strictly semi-(B,G)-preinvexity, explicitly semi-(B,G)-preinvexity, and semi-(B,G)-preinvexity, the following result is obviously true.

Theorem 3.1. *If f is strictly semi-(B,G)-preinvex function on X with respect to η, then f is both an explicitly semi-(B,G)-preinvex function and a semi-(B,G)-preinvex function on X with respect to η.*

The following example illustrates that semi-(B,G)-preinvexity does not imply strictly semi-(B,G)-preinvexity; also explicitly semi-(B,G)-preinvexity does not imply strictly semi-(B,G)-preinvexity.

Example 3.2. Let X be the set defined in Example 2.3; let $\eta(x,u,\lambda)$, $b^1(x,u;\lambda)$, and $b^2(x,u;\lambda)$ be functions defined in Example 2.10. define

$$f(x) = \begin{cases} 1, & x = (0,0), \\ 0, & x \neq (0,0). \end{cases} \tag{3.1}$$

Then f is both an explicitly semi-(B,G)-preinvex function and a semi-(B,G)-preinvex function on X with respect to η, but f is not a strictly semi-(B,G)-preinvex function on X with respect to η, where $G(a) = a$, $a \in \mathbb{R}$.

Note that B-preinvex function is semi-(B,G)-preinvex, and explicitly B-preinvex function is explicitly semi-(B,G)-preinvex, where $G(a) = a$, $a \in R$. Examples 2.1 and 2.2 in [30] can illustrate that semi-(B,G)-preinvexity does not imply explicitly semi-(B,G)-preinvexity, and also explicitly semi-(B,G)-preinvexity does not imply semi-(B,G)-preinvexity.

Next, we present properties of semi-(B,G)-preinvex functions and explicitly semi-(B,G)-preinvex functions.

Theorem 3.3. *Let X be a nonempty semi-invex set in $\mathbb{R}^n$ with respect to $\eta : X \times X \times [0,1] \to \mathbb{R}^n$, $f : X \to \mathbb{R}$ be an explicitly semi-(B,G_1)-preinvex function on X with respect to η, and $G_2 : I_{G_1(f)}(X) \to \mathbb{R}$ be both a convex function and an increasing function. Then f is an explicitly semi-$(B,G_2(G_1))$-preinvex function on X with respect to the same η.*

Proof. If f is an explicitly semi-(B,G_1)-preinvex function on X with respect to η. Then, by Lemma 2.11(i), $G_1(f)$ is an explicitly semi-B-preinvex function on X with respect to η.

Therefore, there exist $b^1, b^2 : X \times X \times [0,1] \to \mathbb{R}_+$ such that, for any $x, u \in X$, $f(x) \neq f(u)$, the inequality

$$\begin{aligned} &G_1(f(u+\lambda\eta(x,u,\lambda))) < b^1(x,u;\lambda)G_1(f(x)) + b^2(x,u;\lambda)G_1(f(u)), \\ &b^1(x,u;1) = b^2(x,u;0) = 1, \quad b^1(x,u;\lambda) + b^2(x,u;\lambda) = 1, \quad \lambda \in (0,1) \end{aligned} \tag{3.2}$$

holds. Note the convexity and monotonicity of G_2, we have

$$\begin{aligned} G_2(G_1(f(u+\lambda\eta(x,u,\lambda)))) &< G_2\Big(b^1(x,u;\lambda)G_1(f(x)) + b^2(x,u;\lambda)G_1(f(u))\Big) \\ &\leq b^1(x,u;\lambda)G_2(G_1(f(x))) + b^2(x,u;\lambda)G_2(G_1(f(u))). \end{aligned} \tag{3.3}$$

Hence, $G_2(G_1(f))$ is an explicitly semi-B-preinvex function on X with respect to η. Again, by Lemma 2.11(i), f is an explicitly semi-$(B, G_2(G_1))$-preinvex function on X with respect to η. This completes the proof. □

Theorem 3.4. *Let X be a nonempty semi-invex set in $\mathbb{R}^n$ with respect to $\eta : X \times X \times [0,1] \to \mathbb{R}^n$, $f_i : X \to \mathbb{R}$ $(i \in K = \{1,\ldots,k\})$ be semi-(B,G)-preinvex function on X with respect to the same η, G, b^1, and b^2. Moreover, G is both a convex function and a concave function on $\mathbb{R}$. Then, for any $\lambda_i > 0$, $\sum_{i=1}^k \lambda_i = 1$, the function $h(x) := \sum_{i=1}^k \lambda_i f_i(x)$ is semi-(B,G)-preinvex on X with respect to the same η, G, b^1, and b^2. Further, if there exists $i_0 \in K$ such that f_{i_0} is explicitly semi-(B,G)-preinvex on X with respect to the same η, G, b^1, and b^2, then h is explicitly semi-(B,G)-preinvex on X with respect to the same η, G, b^1, and b^2.*

Proof. If f_i is semi-(B,G)-preinvex on X with respect to the same η, G, b^1, and b^2, $i \in K$. Then, by Lemma 2.11(i), $G(f_i)$ is a semi-B-preinvex function on X with respect to the same η, b^1, and b^2, $i \in K$. Therefore, for any $x, u \in X$, the inequality

$$\begin{aligned} &G(f_i(u+\lambda\eta(x,u,\lambda))) \leq b^1(x,u;\lambda)G(f_i(x)) + b^2(x,u;\lambda)G(f_i(u)), \\ &b^1(x,u;1) = b^2(x,u;0) = 1, \quad b^1(x,u;\lambda) + b^2(x,u;\lambda) = 1, \quad \lambda \in (0,1) \end{aligned} \tag{3.4}$$

holds for $i \in K$. Since G is both a convex function and a concave function on $\mathbb{R}$, then

$$G\left(\sum_{i=1}^k \lambda_i (f_i(y))\right) = \sum_{i=1}^k \lambda_i G(f_i(y)). \tag{3.5}$$

Multiplying (3.4) by λ_i, we have

$$G(h(u+\lambda\eta(x,u,\lambda))) \leq b^1(x,u;\lambda)G(h(x)) + b^2(x,u;\lambda)G(h(u)). \tag{3.6}$$

Hence, $G(h)$ is a semi-B-preinvex function on X with respect to η, b^1, and b^2. Again, by Lemma 2.11(i), h is a semi-(B,G)-preinvex function on X with respect to η, G, b^1, and b^2.

Furthermore, if there exists $i_0 \in K$ such that f_{i_0} is an explicitly semi-(B,G)-preinvex function on X with respect to the same η, G, b^1, and b^2, then, the inequality

$$G\big(f_{i_0}\big(u+\lambda\eta(x,u,\lambda)\big)\big) < b^1(x,u;\lambda)G\big(f_{i_0}(x)\big) + b^2(x,u;\lambda)G\big(f_{i_0}(u)\big) \tag{3.7}$$

holds for any $x,u \in X$ and $f_{i_0}(x) \neq f_{i_0}(u)$. Hence, for any $x,u \in X$ and $f_{i_0}(x) \neq f_{i_0}(u)$,

$$G\big(h\big(u+\lambda\eta(x,u,\lambda)\big)\big) < b^1(x,u;\lambda)G(h(x)) + b^2(x,u;\lambda)G(h(u)). \tag{3.8}$$

Therefore, $G(h)$ is an explicitly semi-B-preinvex function on X with respect to η, b^1, and b^2. Again, by Lemma 2.11(i), h is an explicitly semi-(B,G)-preinvex function on X with respect to η, G, b^1 and b^2. This completes the proof. □

Theorem 3.5. *Let X be a nonempty semi-invex set in $\mathbb{R}^n$ with respect to $\eta : X \times X \times [0,1] \to \mathbb{R}^n$, $f_i : X \to \mathbb{R}$ be semi-(B,G)-preinvex on X with respect to the same η, G, b^1 and b^2, $i \in \mathbb{N}$, where $\mathbb{N}$ is a finite or infinite index set. Then function $h(x) := \sup_{i\in\mathbb{N}} f_i(x)$ is a semi-(B,G)-preinvex function on X with respect to the same η, G, b^1, and b^2.*

Proof. If f_i is a semi-(B,G)-preinvex function on X with respect to the same η, G, b^1 and b^2, $i \in \mathbb{N}$. Then, by Lemma 2.11(i), $G(f_i)$ is a semi-B-preinvex function on X with respect to the same η, b^1 and b^2, $i \in \mathbb{N}$. Therefore, for any $x,u \in X$, the inequality

$$\begin{gathered} G\big(f_i\big(u+\lambda\eta(x,u,\lambda)\big)\big) \le b^1(x,u;\lambda)G\big(f_i(x)\big) + b^2(x,u;\lambda)G\big(f_i(u)\big), \\ b^1(x,u;1) = b^2(x,u;0) = 1, \quad b^1(x,u;\lambda) + b^2(x,u;\lambda) = 1, \quad \lambda \in (0,1) \end{gathered} \tag{3.9}$$

holds for $i \in \mathbb{N}$. Define $h^*(x) = \sup_{i\in\mathbb{N}} G(f_i(x))$. Then,

$$h^*(x) = \sup_{i\in\mathbb{N}} G\big(f_i(x)\big) = G\left(\sup_{i\in\mathbb{N}} f_i(x)\right) = G(h(x)). \tag{3.10}$$

Therefore, we have

$$G\big(h\big(u+\lambda\eta(x,u,\lambda)\big)\big) \le b^1(x,u;\lambda)G(h(x)) + b^2(x,u;\lambda)G(h(u)). \tag{3.11}$$

Hence, $G(h)$ is a semi-B-preinvex function on X with respect to η. Again, by Lemma 2.11(i), h is a semi-(B,G)-preinvex function on X with respect to η, G, b^1, and b^2. This completes the proof. □

We remark that explicitly semi-(B,G)-preinvexity does not possess an analogous property, see the following example.

Example 3.6. Let $X_1 = [-6,-2] \subset \mathbb{R}$, $X_1 = [-1,6] \subset \mathbb{R}$ and $X = X_1 \cup X_2$. Define

$$f_1(x) = \begin{cases} 1, & x = 0, \\ 0, & x \in X \setminus \{0\}, \end{cases} \qquad f_2(x) = \begin{cases} 1, & x = 1, \\ 0, & x \in X \setminus \{1\}, \end{cases} \tag{3.12}$$

and define

$$G(a) = a, \quad a \in \mathbb{R},$$
$$b^1(x,u,\lambda) = \lambda, \qquad b^2(x,u,\lambda) = 1-\lambda,$$
$$\eta(x,y,\lambda) = \begin{cases} x-y, & x,y \in X_2, \\ x-y, & x,y \in X_1, \\ 7-y, & x \in X_2,\ y \in X_1, \\ -y, & x \in X_1,\ y \in X_2 \setminus \{0\}, \\ \dfrac{1}{6}x, & x \in X_1,\ y = 0. \end{cases} \tag{3.13}$$

It is obvious that f_1 and f_2 are explicitly semi-(B,G)-preinvex functions on X. Further, it can be verified that

$$h(x) = \sup\{f_i(x), i = 1,2\} = \begin{cases} 1, & x = 0 \text{ or } x = 1, \\ 0, & x \in X \setminus \{0,1\}. \end{cases} \tag{3.14}$$

Taking $x = -1$, $y = 1$, $\lambda = 1/2$, we have

$$G(h(x)) = G(h(-1)) = 0 < 1 = G(h(1)) = G(h(y)). \tag{3.15}$$

On the other hand,

$$\begin{aligned} G(h(y+\lambda\eta(x,y,\lambda))) = h(0) = 1 > \frac{1}{2} &= \frac{1}{2}G(h(-1)) + \frac{1}{2}G(h(1)) \\ &= b^1(x,u,\lambda)G(h(x)) + b^2(x,u,\lambda)G(h(y)). \end{aligned} \tag{3.16}$$

Hence, h is not an explicitly semi-(B,G)-preinvex functions on X.

But we have the following result.

Theorem 3.7. *Let X be a nonempty semi-invex set in $\mathbb{R}^n$ with respect to $\eta : X \times X \times [0,1] \to \mathbb{R}^n$, $f_i : X \to \mathbb{R}$ be both a semi-(B,G)-preinvex function and an explicitly semi-(B,G)-preinvex function on X with respect to the same η, G, b^1, and b^2, $i \in \mathbb{N}$, where $\mathbb{N}$ is a finite or infinite index set. Define function $h(x) := \sup_{i\in\mathbb{N}} f_i(x)$, for every $x \in X$. Assume that for every $x \in X$, there exists an $i_0 := i(x) \in \mathbb{N}$, such that $h(x) = f_{i_0}(x)$. Then function $h(x)$ is both a semi-(B,G)-preinvex function and an explicitly semi-(B,G)-preinvex function on X with respect to the same η, G, b^1 and b^2.*

Proof. By Theorem 3.5, we know that h is a semi-(B,G)-preinvex function on X with respect to η. It suffices to show that h is an explicitly semi-(B,G)-preinvex function on X with respect

to η. Assume that h is not an explicitly semi-(B,G)-preinvex function on X. Then, there exist $x, y \in X$, $h(x) \neq h(y)$ such that

$$G(h(y+\lambda\eta(x,y,\lambda))) \geq b^1(x,y,\lambda)G(h(x)) + b^2(x,y,\lambda)G(h(y)), \quad \forall\lambda \in (0,1). \tag{3.17}$$

By the semi-(B,G)-preinvexity of h, we have

$$G(h(y+\lambda\eta(x,y,\lambda))) \leq b^1(x,y,\lambda)G(h(x)) + b^2(x,y,\lambda)G(h(y)). \tag{3.18}$$

Hence

$$G(h(y+\lambda\eta(x,y,\lambda))) = b^1(x,y,\lambda)G(h(x)) + b^2(x,y,\lambda)G(h(y)). \tag{3.19}$$

Denote $z = y+\lambda\eta(x,y,\lambda)$. From the assumptions of the theorem, there exist $i(z) := i_0$, $i(x) := i_1$ and $i(y) := i_2$, satisfying

$$h(z) = h_{i_0}(z), \qquad h(x) = h_{i_1}(x), \qquad h(y) = h_{i_2}(y). \tag{3.20}$$

Then, (3.19) implies that

$$G(f_{i_0}(z)) = b^1(x,y,\lambda)G(f_{i_1}(x)) + b^2(x,y,\lambda)G(f_{i_2}(y)). \tag{3.21}$$

(i) If $f_{i_0}(x) \neq f_{i_0}(y)$, by the explicitly semi-(B,G)-preinvexity of f_{i_0}, we have

$$G(f_{i_0}(z)) < b^1(x,y,\lambda)G(f_{i_0}(x)) + b^2(x,y,\lambda)G(f_{i_0}(y)). \tag{3.22}$$

From $f_{i_0}(x) \leq f_{i_1}(x)$, $f_{i_0}(y) \leq f_{i_2}(y)$ and (3.22), we have

$$G(f_{i_0}(z)) < b^1(x,y,\lambda)G(f_{i_1}(x)) + b^2(x,y,\lambda)G(f_{i_2}(y)), \tag{3.23}$$

which contradicts (3.19).

(ii) If $f_{i_0}(x) = f_{i_0}(y)$, by the semi-(B,G)-preinvexity of f_{i_0}, we have

$$G(f_{i_0}(z)) \leq b^1(x,y,\lambda)G(f_{i_0}(x)) + b^2(x,y,\lambda)G(f_{i_0}(y)). \tag{3.24}$$

Since $h(x) \neq h(y)$, at least one of the inequalities $f_{i_0}(x) \leq f_{i_1}(x) = h(x)$ and $f_{i_0}(y) \leq f_{i_2}(y) = h(y)$ has to be a strict inequality. From (3.24), we obtain

$$G(h(z)) = G(f_{i_0}(z)) < b^1(x,y,\lambda)G(h(x)) + b^2(x,y,\lambda)G(h(y)), \tag{3.25}$$

which contradicts (3.19). This completes the proof. □

4. Characterizations of (B, G)-Preinvexity

In this section, we consider (B, G)-preinvexity and explicitly (B, G)-preinvexity, which are special cases of semi-(B, G)-preinvexity and explicitly semi-(B, G)-preinvexity, respectively. We obtain two sufficient conditions or characterizations for (B, G)-preinvexity under the Condition C, which was introduced by Mohan and Neogy in [13]. We say that the function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the Condition C if the following identities

$$\begin{aligned} \eta(y, y + \lambda\eta(x, y)) &= -\lambda\eta(x, y), \\ \eta(x, y + \lambda\eta(x, y)) &= (1 - \lambda)\eta(x, y) \end{aligned} \tag{4.1}$$

hold for any $x, y \in X$ and for any $\lambda \in [0, 1]$.

The upper and lower semicontinuity of a real function f is defined as follows.

Definition 4.1. Let X be a nonempty subset of $\mathbb{R}^n$. The function $f : X \to \mathbb{R}$ is said to be upper semicontinuous at $\overline{x} \in X$, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in X$, if $\|x - \overline{x}\| < \delta$, then

$$f(x) < f(\overline{x}) + \epsilon. \tag{4.2}$$

If $-f$ is upper semicontinuous at $\overline{x} \in X$, then f is said to be lower semicontinuous at $\overline{x} \in X$.

Theorem 4.2. *Let X be a nonempty invex set in $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, where η satisfies the Condition C, and $f : X \to \mathbb{R}$ be an explicitly (B, G)-preinvex function on X with respect to η. If there exists a $\lambda \in (0, 1)$ such that for every $x, y \in X$ the following inequality holds*

$$G(f(y + \lambda\eta(x, y))) \leq b^1(x, y, \lambda)G(f(x)) + b^2(x, y, \lambda)G(f(y)). \tag{4.3}$$

Then f is (B, G)-preinvex on X with respect to the same η, G, b^1, and b^2.

Proof. Since f is an explicitly (B, G)-preinvex function on X with respect to η, G, b^1, and b^2. Then, by Lemma 2.11(iii), $G(f)$ is an explicitly B-preinvex function on X with respect to η, b^1, and b^2. Therefore, from Theorem 4.1 in [30], we deduce that $G(f)$ is a B-preinvex function on X with respect to η, b^1, and b^2. Again, from Lemma 2.11(iii), f is a (B, G)-preinvex function on X with respect to the same η, G, b^1, and b^2. □

By Theorem 4.2, we get the following corollary, which is Theorem 2 in [24].

Corollary 4.3. *Let X be a nonempty invex set in $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, where η satisfies the Condition C, and $f : X \to \mathbb{R}$ be a semistrictly G-preinvex function on X with respect to η and G. If there exists a $\lambda \in (0, 1)$ such that for every $x, y \in X$ the following inequality holds*

$$G(f(y + \lambda\eta(x, y))) \leq \lambda G(f(x)) + (1 - \lambda)G(f(y)). \tag{4.4}$$

Then f is a G-preinvex function on X with respect to the same η and G.

Theorem 4.4. *Let X be a nonempty invex set in $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, where η satisfies the Condition C and $f : X \to \mathbb{R}$ be an explicitly (B, G)-preinvex function on X with respect*

to η, G, b^1, and b^2. Assume that f is a lower semicontinuous function and G is a continuous one on $I_f(X)$. Then f is a (B,G)-preinvex function on X with respect to the same η, G, b^1, and b^2.

Proof. Since f is a lower semicontinuous function, and G is a continuous one, then $G(f)$ is a lower semicontinuous one. By the assumption of theorem, $G(f)$ is an explicitly B-preinvex function on X with respect to η, b^1, and b^2. Therefore, from Theorem 4.2 in [30], we deduce that $G(f)$ is a B-preinvex function on X with respect to η, b^1, and b^2. From Lemma 2.11(iii), f is a (B,G)-preinvex function on X with respect to η, G, b^1, and b^2. □

As an anonymous reviewer pointed out, an interesting question is to investigate under what conditions, the (B,G)-preinvex function is also a explicitly (B,G)-preinvex function. Until now, we have no definite answer to this question. However, we have Theorem 4.5 which is Theorem 1 in [24] for a special case $b^1(x,u;\lambda) = \lambda$ and $b^2(x,u;\lambda) = 1-\lambda$.

Theorem 4.5. *Let X be a nonempty invex set in $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, where η satisfies the Condition C, and $f : X \to \mathbb{R}$ be a G-preinvex function on X with respect to η. If there exists a $\lambda \in (0,1)$ such that for every $x, y \in X$, $f(x) \neq f(y)$, the inequality*

$$G\big(f\big(y+\lambda\eta(x,y)\big)\big) < \lambda G\big(f(x)\big) + (1-\lambda)G\big(f(y)\big) \tag{4.5}$$

holds, then f is explicitly G-preinvex on X with respect to the same η.

5. Conclusions

In this paper, we firstly construct a concrete set which is not invex but semi-invex; basing on the semi-invex set, we have introduced some new kinds of generalized convex functions, which include semi-(B,G)-preinvex functions, strictly semi-(B,G)-preinvex functions and explicitly semi-(B,G)-preinvex functions. From Example 2.10, Theorems 2.12 and 2.13, we can conclude that these new generalized convex functions constitutes an important class of generalized convex functions in mathematical programming. Moreover, we have established the relationships between the new kinds of generalized convex functions defined in this paper and the corresponding common kinds of generalized convex one introduced in the literature. Basing on these relationships and using the well-known results pertaining to common generalized convex functions, we have obtained results for these new kinds of generalized convex functions.

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