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Research Article

Exact Null Controllability of KdV-Burgers Equation with Memory Effect Systems

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This paper is concerned with exact null controllability analysis of nonlinear KdV-Burgers equation with memory. The proposed approach relies upon regression tool to prove controllability property of linearized KdV-Burgers equation via Carleman estimates. The control is distributed along with subdomain $\omega \subset \Omega$ and the external control acts on the key role of observability inequality with memory. This description finally showed the exact null controllability guaranteeing the stability.

1. Introduction

In recent years there has been rapidly increasing interest in mathematical studies of dynamical and statistical property of nonlinear fields described by the Burgers equation (see [1–4]) and it has been motivated by several developments. As Burgers [5] noticed, the Burgers equation is a convenient analytical model for the physical turbulence, which is simultaneously taken into an account of two competing mechanisms: to determine properties of the strong hydrodynamic turbulence: the interior nonlinearity and viscosity.

Moreover, the quantitative description in many physical processes leads to the Burgers equation. One example here is an intense acoustical noise, such as the jet noise [6], where knowledge of dynamical and statistical properties of the Burgers turbulence can be directly applied to an analysis of nonlinear distortions. Another phenomenon is adequately for the nonlinear evolution of gravitational instability and the related characteristic of large-scale cellular structures (see [1, 7]).

Many problems have been modelled on nonlinear Burgers equations. There has been an enormous on-going research and it is directly investigating some nonlinear effects. For example, the simpler model equation with higher dimension, which encapsulates to essential features of the problem. But it is impossible to solve it directly in a higher dimension. After few years, these issues were carried out by Fernández-Cara et al. [8] and it has been discussed via Korteweg-de Vries (KdV). The KdV equation is a prototype of such a model also describing the competition between nonlinear and disperse effects in water waves.

The KdV equations are not directly related to the disperse waves, the density fields in Burgers turbulence, and it is rather difficult to do in higher dimensions. Although the KdV equation of analytical model is not a convenient and also nonefficient model of the strong turbulence, it holds at least one serious drawback from the viewpoint of physical applications (see [6]). Namely, it does not take into account pressure forces that could lead to smoothing of singularity, which appear in density fields driven by Burgerian velocity.

In this context we will consider the KdV-Burgers equations with memory, which is described by the density fields, pressure forces as well as the viscosity and its dispersion, and so forth. In various fields of physics and engineering, problems have been modelled by partial differential equation with memory (see [9]). It is essential to take an account of the effect of past history.

We consider a typical form of KdV-Burgers equation with memory; it is essential and suitable for the above physical situation:

$$\begin{gathered} y_t - y_{xx} + yy_x + y_{xxx} + \int_0^t k(t,\tau) y_{xx} d\tau = \chi_\omega u(t,x) + f(t,x), \quad (t,x) \in (0,T) \times \Omega, \\ y(0,x) = y_0(x), \quad x \in \Omega, \\ y(t,x) = 0, \quad (t,x) \in (0,T) \times \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$. The null controllability of linear parabolic equation without memory kernels has been extensively studied by several authors (see [8, 10–12], and references therein). Barbu and Iannelli [9] discussed the approximate controllability for the similar type of equation with memory. Rosier [13] studied the exact boundary controllability of linear KdV equation with the half-line and Russell and Zhang [14] also discussed the exact controllability and stabilizability of KdV equation. Sakthivel [15] has been proved the asymptotic stability of KdV-Burgers via Lyapunov function technique by using L_2 and L_∞ norms with domain of $[0,1]$. But the domain of attraction always containing $[0,1]$ is impossible; therefore, the problems of KdV-Burgers equations have not been fully investigated, and it is therefore still a challenging problem.

Some nonlinear control systems are modelled by partial differential equations; it will be a strong control. Unfortunately in some cases, even if the linearized control system around the equilibrium is not controllable. But the linear control system around the memory kernel is always controllable. This method has been introduced in Temam [16], Kofman et al. [7], and Coron [17] where they have discussed about the controllability depending on the relationship between the pressure forces and the magnitude of initial velocity in gas models. The distributed control is described by the behavior of water waves in a shallow channel, compressible gas and strong hydrodynamic turbulence, and so forth. It will be indicated as in (1.2) that appears as in control function.

We consider the linearized control KDV-Burgers equation with memory effects of dirichlet boundary conditions, which helps to develop (1.1) (as based on the existing result Fursikov and Imanuvilov [12]):

$$\begin{aligned} &y_t(t,x) - y_{xx}(t,x) + \big(a(t,x)y(t,x)\big)_x + b(t,x)y(t,x) + y_{xxx}(t,x) + \int_0^t k(t,\tau)y_{xx}(\tau,x)d\tau \\ &\quad = \chi_\omega u(t,x) + f(t,x), \quad (t,x) \in (0,T)\times\Omega, \\ &\qquad y(0,x) = y_0(x), \quad x \in \Omega \\ &\qquad y(t,x) = 0, \quad (t,x) \in (0,T)\times\partial\Omega, \end{aligned} \tag{1.2}$$

where $\partial^3 y/\partial x^3 = y_{xxx}$, $y(t,x)$ is a solitary wave of dispersion at the point x and time t, $y_0(x)$ is an initial temperature distribution, the integral kernel $k(t,\tau)$ is called conservation of mass or volume and has support in (t_1,t_2), where $0 < t_1 < t_2 < T$, $f(t,x)$ is a forcing term such as pressure force, χ_ω is a characteristic function of the subset $\omega \subset \Omega$, and $u(t,x)$ is a control over an arbitrary sub domain ω of the domain Ω.

The paper is organized as follows. Section 2 gives some basic assumptions and formation of the problem. Section 3 gives the proof of the Carleman estimate and observability result. Section 4 gives exact null controllability result as based on a unique continuation result of adjoined problems and by using the observability inequality. A conclusion will be given in Section 5.

Notation. We describe some function spaces which will be useful to formulate our results. For each positive integer m and $p > 1$ or $p = \infty$, denote as usual by $W^{m,p}(\Omega)$ the sobolev space of functions in $L^p(\Omega)$ whose weak derivatives are of order less than or equal to m. When $p = 2$ instead of $W^{m,p}(\Omega)$, we will write $H^m(\Omega)$. Besides, we need the space $L^2(0,T : H^1(\Omega))$ of all equivalence classes of square integrable functions from $(0,T)$ to $H^1(\Omega)$. The spaces $L^2(0,T : L^2(\Omega))$ and $L^\infty(0,T : L^2(\Omega))$ are analogously defined. Moreover, we set (see [18])

(i) $\Omega = \{(t,x) : 0 < t < T,\ 0 < x < \infty\}$;

(ii) $\chi_\omega := \begin{cases} 1 & \text{for } x \in \omega, \\ 0 & \text{for } x \in \Omega/\omega; \end{cases}$

(iii) $Q := (0,T)\times\Omega$, $Q_\omega := (0,T)\times\omega$, and $\Sigma := (0,T)\times\partial\Omega$;

(iv) $H^m(\Omega)$:= the Sobolev spaces of functions in $L^2(\Omega)$ whose weak derivatives are of order less than or equal to m, where m is a positive integer;

(v) $L^2(0,T;H^1(\Omega))$:= the space of all equivalence classes of square integrable functions from $(0,T)$ to $H^1(\Omega)$;

(vi) $W^{m,p}(\Omega) = \{y(x) : \|y\|_{W^{m,p}} = \left(\sum_{|\alpha|\le m}\int_\Omega |D^\alpha y|^p dx\right)^{1/p} < \infty\}$.

2. Assumptions and Main Results

A linearized control system (1.2) is a weak solution model of the control problem (1.1) (it is supporting to prove the stabilization), then system (1.2) can be written as

$$\begin{aligned} y_t(t,x) - y_{xx}(t,x) + a(t,x)y_x(t,x) + b(t,x)y(t,x) + y_{xxx}(t,x) + \int_0^t k(t,\tau)y_{xx}(\tau,x)d\tau \\ = \chi_\omega u(t,x) + f(t,x), \quad (t,x) \in (0,T)\times\Omega, \\ y(0,x) = y_0(x), \quad x \in \Omega, \\ y(t,x) = 0, \quad (t,x) \in (0,T)\times\partial\Omega, \end{aligned} \tag{2.1}$$

where $a \in W^{1,\infty}(0,T : L^2(\Omega))$, $b \in L^\infty(Q)$. The kernel $k(t,\cdot)$ is smooth and has support in (t_0,t_1) where $0 < t_0 < t_1 < T$. Moreover, there are plenty of works related to exact and approximate controllability properties of the parabolic system of type (1.2) without the memory effects (see [11]). In this paper we constructed a system with memory; finally by using Hölder's inequality and changing the order of integration the memory term will be observed by $y_{xx} \in \omega \subset \Omega$.

The following lemma is a fundamental tool to proving controllability results.

Lemma 2.1. *Let Ω be an open bounded and connected subset of the boundary $\partial\Omega$ in class C^2, ω an arbitrary subsets of Ω, and $f \in L^2(Q)$ such that*

$$|f(x,t)| \le |f_1(x,t)|e^{s\varphi(x,t)}\varphi_x(x,t)\varphi_{xx}(x,t), \quad a.e.\ (x,t) \in Q. \tag{2.2}$$

For each $y_0 \in L^2(\Omega)$, and $u \in L^2(Q)$ such that

$$\|u\|_{L^2(Q)} \le C\left(|y_0|_2 + \|f_1\|_{L^2(Q)}\right), \tag{2.3}$$

and the space $y^u \in C^1[0,T]; H^1([0,T] : L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ is a solution of (1.2).

Proof. The proof is similar to Theorem 1.3.1 from Barbu [10] and hence it will be omitted. □

Theorem 2.2. *Let Ω,ω be as in Lemma 2.1. System* (1.2) *is exactly null controllable for each $T > 0$, if there exists $u \in L^2(Q)$ and $y^u \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ such that*

$$y^u(T,x) = 0 \quad a.e.\ x \in \Omega, \qquad \|u\|_{L^2(Q)} \le C\left(|y_0|_2 + \|f_1\|^2_{L^2(Q)}\right). \tag{2.4}$$

(The proof of the theorem is given in Section 4).

3. Observability Results and Carleman Estimates

In this section we will derive the observability result via the Carleman estimate of adjoint system (2.1).

To prove the Carleman estimate, the weight functions are necessary. As based on Fernández-Cara et al. [8] and Rosier [13] the weight functions φ and ψ are assumed as

$$\begin{gathered}\varphi(x,t)=\frac{\psi(x)}{\beta(t)},\quad \psi(x)>0,\ \forall x\in\Omega,\ \psi(x)=0,\ \forall x\in\partial\Omega,\ \left|\psi_x(x)\right|>0,\\ \psi_x(x)<0,\ \psi_{xx}(x)<0,\ \left(\psi_{xx}(x)\cdot\psi_x(x)\right)>0,\\ \left(\psi_x(x)\cdot\left(\psi_{xxx}(x)\right)\right)>0,\quad x\in\overline{\Omega\setminus\omega},\end{gathered}\tag{3.1}$$

where $\beta(t)=t(T-t)$.

Suppose that w is an adjoint state variable of system (2.1), then it has some solutions. Therefore,

$$\begin{gathered}w_t(t,x)+w_{xx}(t,x)+c_1w_x(t,x)-c_2w(t,x)+w_{xxx}(t,x)-\int_t^T k(\tau,t)w_{xx}(\tau,x)d\tau\\ =g(t,x),\quad (t,x)\in\Sigma,\\ w(T,x)=w_T(x),\quad x\in\Omega,\\ w(t,x)=0,\quad (t,x)\in(0,T)\times\partial\Omega,\end{gathered}\tag{3.2}$$

where $\|a(t,x)\|\le c_1>0, \|b(t,x)\|\le c_2>0, g\in L^2(Q)$ and $w_T\in L^2(\Omega)$.

Now we will state the theorem to prove system (3.2) of solution w,which will be a solution for (2.1). The question is how the system (3.2) of solution w will be a solution for (2.1). As based on Carleman estimate (assume the weight functions φ,ψ as (3.1)), we can prove that w will be a solution for (2.1) (see [12]).

Theorem 3.1 (Carleman estimate). *The function ψ defined as in* (3.1)*, the kernel has support in* (t_1,t_2) *(where $0<t_1<t_2<T$), there exist positive constants C, and $s\ge s_0$ such that the following inequalities hold when w a solution of* (3.2)*:*

$$\begin{gathered}\int_Q e^{2s\varphi}\Big((s\varphi)^{-1}\left(|w_t|^2+|w_x|^2+|w_{xx}|^2+|w_{xxx}|^2\right)\\ +s^5\varphi_x^4\varphi_{xx}|w|^2+s^3\varphi_x^2\varphi_{xx}|w_x|^2+s\varphi|w_{xx}|^2\Big)dx\,dt\\ \le Ce^{2s\varphi}\left(\int_Q |g|^2dx\,dt+\int_{Q_\omega}s^4\varphi_x^2\varphi_{xx}^2|w|^2dxdt\right).\end{gathered}\tag{3.3}$$

Proof. We set $w = e^{-s\varphi}q$ (where s, φ are positive parameters) and q satisfies (3.2) (for our convenient $c_1 = c_2 = 1$), then

$$\begin{aligned} &q_t + s\varphi_t q + s^2\varphi_x^2 q - s\varphi_{xx}q - 2s\varphi_x q_x + q_{xx} - s^3\varphi_x^3 q + 3s^2\varphi_x\varphi_{xx}q + 3s^2\varphi_x^2 q_x \\ &\quad - 3s\varphi_{xx}q_x - 3s\varphi_x q_{xx} - s\varphi_{xxx}q + q_{xxx} - q - s\varphi_x q + q_x \\ &\quad - e^{s\varphi}\left(\int_t^T k(\tau,t)e^{-s\varphi(\tau)}q(\tau,x)\right)d\tau = ge^{s\varphi}. \end{aligned} \tag{3.4}$$

As based on Barbu [10] we can introduce the operators

$$L_1 q = q_t - s\varphi_t q - s\varphi_{xx}q + s^2\varphi_x^2 q - s^3\varphi_x^3 q + 3s^2\varphi_x\varphi_{xx}q - s\varphi_{xxx}q - q - s\varphi_x q, \tag{3.5}$$

$$L_2 q = q_{xx} + q_{xxx} - 2s\varphi_x q_x + 3s^2\varphi_x^2 q_x - 3s\varphi_{xx}q_x - 3s\varphi_x q_{xx} + q_x. \tag{3.6}$$

It follows from (3.4), (3.5), and (3.6) that

$$L_1 q + L_2 q = h_s \quad \text{in } Q, \tag{3.7}$$

where

$$h_s = ge^{s\varphi} + e^{s\varphi}\left(\int_t^T k(\tau,t)\left(e^{-s\varphi}q\right)_{xx}d\tau\right). \tag{3.8}$$

Taking L_2-norm of both sides of (3.7), we obtain

$$\|h_s\|_{L^2(Q)}^2 = \|L_1 q\|_{L^2(Q)}^2 + \|L_2 q\|_{L^2(Q)}^2 + 2\langle L_1 q, L_2 q\rangle_{L^2(Q)}. \tag{3.9}$$

Let us analyze the scalar product in (3.9) as

$$\langle L_1 q, L_2 q\rangle_{L^2} = E_{ij}, \quad 1 \le i \le 9,\ 1 \le j \le 7, \tag{3.10}$$

where E_{ij} is an integral product of ith term in $L_1 q$ and jth term in $L_2 q$.

Now we will simplify the estimate $\langle L_1 q, L_2 q\rangle_{L^2(Q)}$ by using Green's theorem with integration by parts:

$$
\begin{aligned}
&\langle L_1 q, L_2 q\rangle_{L^2(Q)} \\
&= \int_\Sigma s\varphi_{xxx}\left|\frac{\partial q}{\partial v}\right|^2 d\Sigma + \frac{3}{2}\int_\Sigma s^2\varphi_x\varphi_{xx}\left|\frac{\partial q}{\partial v}\right|^2 d\Sigma + \frac{1}{2}\int_\Sigma s^3\varphi_x^3\left|\frac{\partial q}{\partial v}\right|^2 d\Sigma \\
&\quad - \frac{1}{2}\int_\Sigma s^2\varphi_x^2\left|\frac{\partial q}{\partial v}\right|^2 d\Sigma + \frac{1}{2}\int_\Sigma s\varphi_{xx}\left|\frac{\partial q}{\partial v}\right|^2 d\Sigma + \frac{1}{2}\int_\Sigma s\varphi_t\left|\frac{\partial q}{\partial v}\right|^2 d\Sigma + \int_Q s\varphi_t q_x^2 dxdt \\
&\quad - \frac{1}{2}\int_Q s(\varphi_t)_{xx} q^2 dxdt + \frac{1}{2}\int_Q s(\varphi_t)_{xxx} q^2 dxdt - 2\int_Q s(\varphi_t)_x q_x^2 dxdt + \int_Q s\varphi_t q_x^2 dxdt \\
&\quad + \int_Q s^2(\varphi_t)_x\varphi_x q^2 dxdt + \int_Q s^2\varphi_t\varphi_{xx} q^2 dxdt + \frac{3}{2}\int_Q s^2\varphi_t\varphi_{xx} q^2 dxdt \\
&\quad + \frac{3}{2}\int_Q s^2\varphi_t\varphi_{xx} q^2 dxdt + 3\int_Q s^3\varphi_t\varphi_x\varphi_{xxx} q^2 dxdt + \frac{3}{2}\int_Q s^2(\varphi_t)_x q^2 dxdt \\
&\quad + 3\int_Q s^2\varphi_t\varphi_x q_x^2 dxdt - \frac{3}{2}\int_Q s^2(\varphi_t)_{xx}\varphi_x q^2 dxdt + 2\int_Q s\varphi_{xx} q_x^2 dxdt \\
&\quad - \int_Q s^2\varphi_{xx}^2 q^2 dxdt + \frac{3}{2}\int_Q s^3\varphi_x^2\varphi_{xxx} q^2 dxdt + 3\int_Q s^3(\varphi_x)\varphi_{xx}^2 q^2 dxdt \\
&\quad - 3\int_Q s^2\varphi_x\varphi_{xx} q_x^2 dxdt - 2\int_Q s^2\varphi_x^2 q_x^2 dxdt + \int_Q s^2\varphi_{xx}^2 q^2 dxdt + 2\int_Q s^2\varphi_x\varphi_{xx} q^2 dxdt \\
&\quad - \int_Q s^2\varphi_{xx}\varphi_{xxx} q^2 dxdt - 2\int_Q s^4\varphi_x^3\varphi_{xx} q^2 dxdt + 3\int_Q s^3\varphi_x^3 q^2 dxdt \\
&\quad - 9\int_Q s^3\varphi_x\varphi_{xx}^2 q_x^2 dxdt - \frac{3}{2}\int_Q s^3\varphi_x^2\varphi_{xx} q^2 dxdt + 2\int_Q s^3\varphi_x^3 q_x^2 dxdt \\
&\quad + 7\int_Q s^3\varphi_x\varphi_{xx}\varphi_{xxx} q^2 dxdt + \frac{3}{2}\int_Q s^5\varphi_x^4\varphi_{xx} q^2 dxdt + 9\int_Q s^4\varphi_x^2\varphi_{xx}^2 q^2 dxdt \\
&\quad + \frac{3}{2}\int_Q s^4\varphi_x^3\varphi_{xxx} q^2 dxdt - 3\int_Q s^4\varphi_x^4 q_x^2 dxdt + 6\int_Q s^4\varphi_x^3\varphi_{xxx} q^2 dxdt \\
&\quad + 18\int_Q s^4\varphi_x^2\varphi_{xx}^2 q^2 dxdt - 6\int_Q s^2\varphi_x\varphi_{xx} q_x^2 dxdt + 3\int_Q s^2\varphi_{xx}\varphi_{xxx}^2 q^2 dxdt \\
&\quad + \frac{3}{2}\int_Q s^2\varphi_{xx}\varphi_{xxx} q^2 dxdt + 6\int_Q s^2\varphi_{xx}\varphi_{xxx} q_x^2 dxdt + 3\int_Q s^2\varphi_{xx}\varphi_{xxx} q_x^2 dxdt \\
&\quad + 4\int_Q s^2\varphi_{xx}^2 q^2 dxdt + 6\int_Q s^3\varphi_x\varphi_{xx}^2 q^2 dxdt - \frac{9}{2}\int_Q s^4\varphi_x^3\varphi_{xxx} q^2 dxdt \\
&\quad - \frac{27}{2}\int_Q s^4\varphi_x^2\varphi_{xx} q^2 dxdt + \frac{9}{2}\int_Q s^3\varphi_{xx}^3 q^2 dxdt + 9\int_Q s^3\varphi_x^2\varphi_{xx} q_x^2 dxdt \\
&\quad - \frac{15}{2}\int_Q s^3\varphi_{xx}^3 q^2 dxdt + \int_Q s\varphi_{xxx} q_x^2 dxdt + 2\int_Q s\varphi_{xxx} q_x^2 dxdt \\
&\quad - 6\int_Q s^3\varphi_x^2\varphi_{xx} q_x^2 dxdt - \int_Q s^2\varphi_{xx}\varphi_{xxx} q^2 dxdt - 3\int_Q s^2\varphi_x\varphi_{xxx} q_x^2 dxdt.
\end{aligned}
\tag{3.11}
$$

Since $q = \varphi = 0$ on $\partial\Omega$, some manipulations are dominated by the same parameters and also observed by powers of s. Finally we obtain

$$\sum_{i=1}^{9}\sum_{j=1}^{7} E_{ij} = D_1 + D_2 + \frac{3}{2}\int_Q s^5\varphi_x^4\varphi_{xx}q^2\,dxdt + 3\int_Q s^3\varphi_x^2\varphi_{xx}q_x^2\,dxdt, \tag{3.12}$$

where D_1, D_2 are boundary terms:

$$\begin{aligned}
D_1 = {} & \int_\Sigma s\varphi_{xxx}\left|\frac{\partial q}{\partial v}\right|^2 d\Sigma + \frac{3}{2}\int_\Sigma s^2\varphi_x\varphi_{xx}\left|\frac{\partial q}{\partial v}\right|^2 d\Sigma + \frac{1}{2}\int_\Sigma s^3\varphi_x^3\left|\frac{\partial q}{\partial v}\right|^2 d\Sigma \\
& - \frac{1}{2}\int_\Sigma s^2\varphi_x^2\left|\frac{\partial q}{\partial v}\right|^2 d\Sigma + \frac{1}{2}\int_\Sigma s\varphi_{xx}\left|\frac{\partial q}{\partial v}\right|^2 d\Sigma + \frac{1}{2}\int_\Sigma s\varphi_t\left|\frac{\partial q}{\partial v}\right|^2 d\Sigma, \\
D_2 = {} & \int_Q s\varphi_t q_x^2\,dxdt - \frac{1}{2}\int_Q s(\varphi_t)_{xx}q^2\,dxdt + \frac{1}{2}\int_Q s(\varphi_t)_{xxx}q^2\,dxdt \\
& - 2\int_Q s(\varphi_t)_x q_x^2\,dxdt + \int_Q s\varphi_t q_x^2\,dxdt + \int_Q s^2(\varphi_t)_x\varphi_x q^2\,dxdt + \int_Q s^2\varphi_t\varphi_{xx}q^2\,dxdt \\
& + \frac{3}{2}\int_Q s^2\varphi_t\varphi_{xx}q^2\,dxdt + \frac{3}{2}\int_Q s^2\varphi_t\varphi_{xx}q^2\,dxdt + 3\int_Q s^3\varphi_t\varphi_x\varphi_{xxx}q^2\,dxdt \\
& + \frac{3}{2}\int_Q s^2(\varphi_t)_x q^2\,dxdt + 3\int_Q s^2\varphi_t\varphi_x q_x^2\,dxdt - \frac{3}{2}\int_Q s^2(\varphi_t)_{xx}\varphi_x q^2\,dxdt \\
& + 2\int_Q s^2\varphi_x\varphi_{xx}q^2\,dxdt - \int_Q s^2\varphi_{xx}\varphi_{xxx}q^2\,dxdt - 2\int_Q s^4\varphi_x^3\varphi_{xx}q^2\,dxdt \\
& - \frac{3}{2}\int_Q s^3\varphi_x^2\varphi_{xx}q^2\,dxdt + 7\int_Q s^3\varphi_x\varphi_{xx}\varphi_{xxx}q^2\,dxdt + 27\int_Q s^4\varphi_x^2\varphi_{xx}^2q^2\,dxdt \\
& + \frac{3}{2}\int_Q s^4\varphi_x^3\varphi_{xxx}q^2\,dxdt + 6\int_Q s^4\varphi_x^3\varphi_{xxx}q^2\,dxdt - 6\int_Q s^2\varphi_x\varphi_{xx}q_x^2\,dxdt \\
& + 3\int_Q s^2\varphi_{xx}\varphi_{xxx}^2q^2\,dxdt + \frac{3}{2}\int_Q s^2\varphi_{xx}\varphi_{xxx}q^2\,dxdt + 9\int_Q s^2\varphi_{xx}\varphi_{xxx}q_x^2\,dxdt \\
& + 6\int_Q s^3\varphi_x\varphi_{xx}^2q^2\,dxdt - \frac{9}{2}\int_Q s^4\varphi_x^3\varphi_{xxx}q^2\,dxdt - \frac{27}{2}\int_Q s^4\varphi_x^2\varphi_{xx}q^2\,dxdt \\
& - \int_Q s^2\varphi_{xx}\varphi_{xxx}q^2\,dxdt - 3\int_Q s^2\varphi_x\varphi_{xxx}q_x^2\,dxdt.
\end{aligned} \tag{3.13}$$

By the time derivative definition φ we can write

$$|\varphi_t| \le C\varphi^2, \qquad |\varphi_{tt}| \le C\varphi^3, \tag{3.14}$$

(where C does not depend on s, t, x). Using (3.14) for s is sufficiently large and observes the same powers, then

$$|D_2| \leq C\left[\int_Q s^4\varphi_x^2\varphi_{xx}^2|q|^2dxdt + \int_Q s^2\varphi_x\varphi_{xx}|q_x|^2dxdt\right]. \tag{3.15}$$

Now multiplying (3.7) by $s\varphi q|\nabla\psi|^2$ and integrating over Q, we obtain

$$\int_Q s\varphi q|\nabla\psi|^2L_1q\,dx\,dt + \int_Q s\varphi q|\nabla\psi|^2L_2qdxdt = \int_Q s\varphi q|\nabla\psi|^2h_sdxdt. \tag{3.16}$$

Therefore,

$$\begin{aligned}
\int_Q s\varphi q|\nabla\psi|^2h_sdxdt &= \int_Q s\varphi q|\nabla\psi|^2L_1qdxdt + \int_Q s\varphi|\nabla\psi|^2qq_{xx}dxdt \\
&\quad + \int_Q s\varphi|\nabla\psi|^2qq_{xxx}dxdt - 2\int_Q s^2\varphi\varphi_x|\nabla\psi|^2qq_xdxdt \\
&\quad + 3\int_Q s^3\varphi\left|\nabla_x^2\psi\right|^2qq_xdxdt - 3\int_Q s^2\varphi\varphi_{xx}|\nabla\psi|^2qq_xdxdt \\
&\quad - 3\int_Q s^2\varphi\varphi_x|\nabla\psi|^2qq_{xx}dxdt, \\
\int_Q s\varphi q|\nabla\psi|^2L_1qdxdt &\leq \frac{1}{2}\int_Q s^2\varphi^2|\nabla\psi|^4q^2dxdt + 2\int_Q |L_1q|^2dxdt, \\
\int_Q s\varphi|\nabla\psi|^2qq_{xx}dxdt &= -\int_Q s\varphi|\nabla\psi|^2q_x^2dxdt + \frac{1}{2}\int_Q s\varphi_{xx}|\nabla\psi|^2q^2dxdt \\
&\quad + \int_Q s\varphi_x\Delta\psi(|\nabla\psi|)q^2dxdt, \\
\int_Q s\varphi|\nabla\psi|^2qq_{xxx}dxdt &= \int_Q s\varphi_x|\nabla\psi|^2q_x^2dxdt + \frac{1}{2}\int_Q s\varphi_{xxx}|\nabla\psi|^2q^2dxdt \\
&\quad + \int_Q s\varphi_{xx}(|\nabla\psi|)\Delta\psi q^2dxdt + 2\int_Q s\varphi|\nabla\psi|^2q_x^2dxdt \\
&\quad - \int_Q s\varphi_x|\nabla\psi|^2q^2dxdt - \int_Q s\varphi|\nabla\psi|^2q_xq_{xx}dxdt,
\end{aligned}$$

$$
\begin{aligned}
-2\int_Q s^2\varphi_x\varphi|\nabla\psi|^2 qq_x dxdt &= \int_Q s^2\varphi_x^2|\nabla\psi|^2 q^2 dxdt + \int_Q s^2\varphi_{xx}\varphi|\nabla\psi|^2 q^2 dxdt \\
&\quad + 2\int_Q s^2\varphi_x\varphi\Delta\psi(|\nabla\psi|)q^2 dxdt, \\
3\int_Q s^3\varphi\varphi_x^2|\nabla\psi|^2 qq_x dxdt &\le 3\int_Q s^3\varphi_x^2\varphi|\nabla\psi|^4 q^2 dxdt + \frac{3}{4}\int_Q s^3\varphi\varphi_x^2 q_x^2 dxdt, \\
-3\int_Q s^2\varphi\varphi_{xx}|\nabla\psi|^2 qq_x dxdt &= \frac{3}{2}\int_Q s^2\varphi_x\varphi_{xx}|\nabla\psi|^2 q^2 dxdt + \frac{3}{2}\int_Q s^2\varphi\varphi_{xxx}|\nabla\psi|^2 q^2 dxdt \\
&\quad + 3\int_Q s^2\varphi\varphi_{xx}(|\nabla\psi|)\Delta\psi q^2 dxdt \\
-3\int_Q s^2\varphi_x\varphi|\nabla\psi|^2 qq_{xx} dxdt &= 3\int_Q s^2\varphi\varphi_x|\nabla\psi|^2 q_x^2 dxdt - 3\int_Q s^2\varphi_x\varphi_{xx}|\nabla\psi|^2 q^2 dxdt \\
&\quad - 3\int_Q s^2\varphi_x^2(|\nabla\psi|)\Delta\psi q^2 dxdt - \frac{3}{2}\int_Q s^2\varphi_x\varphi_{xx}|\nabla\psi|^2 q^2 dxdt \\
&\quad - \frac{3}{2}\int_Q s^2\varphi\varphi_{xxx}|\nabla\psi|^2 q^2 dxdt - 3\int_Q s^2\varphi\varphi_{xx}(|\nabla\psi|)\Delta\psi q^2 dxdt \\
&\quad - \int_Q s^2\varphi_x^2(|\nabla\psi|)\Delta\psi q^2 dxdt.
\end{aligned}
\tag{3.17}
$$

By using the above calculation in (3.16), it becomes

$$
\begin{aligned}
&-\frac{3}{4}\int_Q s^3\varphi\varphi_x^2 q_x^2 dxdt \\
&\quad = -\int_Q s\varphi|\nabla\psi|^2 qh_s dxdt + \int_Q s^2\varphi^2|\nabla\psi|^4 q^2 dxdt \\
&\qquad + \int_Q s\varphi|\nabla\psi|^2 q_x^2 dxdt + \frac{1}{4}\int_Q |L_1 q|^2 dxdt \\
&\qquad + 2\int_Q s\varphi\Delta\psi(|\nabla\psi|)q_x^2 dxdt - \int_Q s\varphi|\nabla\psi|^2 q_x q_{xx} dxdt + 3\int_Q s^3\varphi\varphi_x\varphi_{xx}|\nabla\psi|^2 q^2 dxdt \\
&\qquad + \frac{3}{2}\int_Q s^2\varphi_x\varphi_{xx}|\nabla\psi|^2 q^2 dxdt + \int_Q s^2\varphi\varphi_{xx}|\nabla\psi|^2 q^2 dxdt + 2\int_Q s^2\varphi\varphi_x(|\nabla\psi|)\Delta\psi q^2 dxdt \\
&\qquad + \frac{3}{2}\int_Q s^2\varphi_x^2\nabla\left(|\nabla\psi|^2\right)q^2 dxdt + \frac{3}{2}\int_Q s^2\varphi_x\varphi_{xx}|\nabla\psi|^2 q^2 dxdt + \frac{3}{2}\int_Q s^3\varphi\varphi_x^2|\nabla\psi|^4 q^2 dxdt \\
&\qquad + \frac{5}{2}\int_Q s^2\varphi_x\varphi_{xx}|\nabla\psi|^2 q^2 dxdt + \frac{1}{2}\int_Q s\varphi_x|\nabla\psi|^2 q^2 dxdt + 3\int_Q s^2\varphi\varphi_{xxx}|\nabla\psi|^2 q^2 dxdt \\
&\qquad + 3\int_Q s^2\varphi\varphi_x|\nabla\psi|^2 q_x^2 dxdt - 3\int_Q s^2\varphi_x\varphi_{xx}|\nabla\psi|^2 q^2 dxdt - \frac{3}{2}\int_Q s^2\varphi_x\varphi_{xx}|\nabla\psi|^2 q^2 dxdt
\end{aligned}
$$

$$-\frac{3}{2}\int_Q s^2\varphi\varphi_{xxx}|\nabla\psi|^2q^2dxdt-3\int_Q s^2\varphi\varphi_{xx}(|\nabla\psi|)\Delta\psi q^2dxdt$$
$$-3\int_Q s^2\varphi\varphi_x\nabla\left(|\nabla\psi|^2\right)q_x^2dxdt+\frac{1}{4}\int_Q s\varphi|\nabla\psi|^4q_x^2dxdt. \tag{3.18}$$

From the above equation (the right hand side), we can observe the powers of s:

$$\frac{-3}{4}\int_Q s^3\varphi\varphi_x^2q_x^2dxdt=D_3+3\int_Q s^3\varphi\varphi_x^2|\nabla\psi|^4q^2dxdt+\int_Q s\varphi q_{xx}^2dxdt, \tag{3.19}$$

where

$$\begin{aligned}D_3=&\int_Q s\varphi q|\nabla\psi|^2h_sdxdt+\int_Q s\varphi|\nabla\psi|^2qL_1qdxdt+3\int_Q s^2\varphi\varphi_x|\nabla\psi|^2q_x^2dxdt\\&+\frac{3}{2}\int_Q s^2\varphi_x\varphi_{xx}|\nabla\psi|^2q^2dxdt+\int_Q s^2\varphi\varphi_{xx}|\nabla\psi|^2q^2dxdt\\&+2\int_Q s^2\varphi\varphi_x(|\nabla\psi|)\Delta\psi q^2dxdt-3\int_Q s^3\varphi\varphi_x^2(|\nabla\psi|)\Delta\psi q^2dxdt\\&+\frac{3}{2}\int_Q s^2\varphi_x\varphi_{xx}|\nabla\psi|^2q^2dxdt+\frac{3}{2}\int_Q s^2\varphi\varphi_{xxx}|\nabla\psi|^2q^2dxdt\\&+4\int_Q s^2\varphi_x\varphi_{xx}|\nabla\psi|^2q^2dxdt-\frac{3}{2}\int_Q s^2\varphi_x\varphi_{xx}|\nabla\psi|^2q^2dxdt\\&-\frac{3}{2}\int_Q s^2\varphi\varphi_{xxx}|\nabla\psi|^2q^2dxdt-3\int_Q s^2\varphi\varphi_{xx}(|\nabla\psi|)\Delta\psi q^2dxdt.\end{aligned} \tag{3.20}$$

Applying Cauchy's inequality (with $\epsilon=1$) for D_3,

$$\begin{aligned}|D_3|\le&\frac{1}{4}\int_Q|L_1q|^2dxdt+\frac{1}{4}\int_Q|h_s|^2dxdt+\frac{3}{2}\int_Q s^2\varphi_x\varphi_{xx}|\nabla\psi|^2|q|^2dxdt\\&+3\int_Q s^2\varphi\varphi_x^2|\nabla\psi|^2|q_x|^2dxdt.\end{aligned} \tag{3.21}$$

Therefore

$$\begin{aligned}\langle L_1q,L_2q\rangle_{L^2(Q)}=&D_2+D_3+\frac{3}{2}\int_Q s^5\varphi_x^4\varphi_{xx}q^2dxdt+\frac{3}{4}\int_Q s^3\varphi_x^2\varphi_{xx}q_x^2dxdt\\&+3\int_Q s^3\varphi\varphi_x^2|\nabla\psi|^4q^2dxdt+\int_Q s\varphi q_{xx}^2dxdt.\end{aligned} \tag{3.22}$$

By using (3.22) in (3.7), we get

$$\begin{aligned}
&\|L_1 q\|^2 + \|L_2 q\|^2 + 3\int_Q s^5\varphi_x^4\varphi_{xx}q^2\,dxdt + \frac{3}{2}\int_Q s^3\varphi_x^2\varphi_{xx}q_x^2\,dxdt + 2\int_Q s\varphi q_{xx}^2\,dxdt \\
&\quad \le Ce^{2s\varphi}\left[\int_Q |g|^2\,dxdt + \int_Q \left|\int_t^T k(\tau,t)\left(e^{-s\varphi}q\right)_{xx}d\tau\right|^2 dxdt\right. \\
&\qquad \left. + \int_Q s^4\varphi_x^2\varphi_{xx}^2|q|^2\,dxdt + \int_Q s^2\varphi_x\varphi_{xx}|q_x|^2\,dxdt\right] + \frac{1}{2}\int_Q \|L_1 q\|^2\,dxdt \\
&\qquad + 6\int_Q s^3\varphi\varphi_x^2|\nabla\psi|^4|q|^2\,dxdt + 6\int_Q s^2\varphi\varphi_x|\nabla\psi|^2|q_x|^2\,dxdt.
\end{aligned} \tag{3.23}$$

Recalling that s is sufficiently large and also observing that powers of s and $|\nabla\psi| > 0$ in $\Omega\setminus\omega$,

$$\begin{aligned}
&\frac{1}{2}\|L_1 q\|^2 + \|L_2 q\|^2 + 3\int_Q s^5\varphi_x^4\varphi_{xx}q^2\,dxdt + \frac{3}{2}\int_Q s^3\varphi_x^2\varphi_{xx}q_x^2\,dxdt + \int_Q s\varphi q_{xx}^2\,dxdt \\
&\quad \le C\left[\int_Q e^{2s\varphi}|g|^2\,dxdt + \int_Q e^{2s\varphi}\left|\int_t^T k(\tau,t)\left(e^{-s\varphi}q\right)_{xx}d\tau\right|^2 dxdt\right. \\
&\qquad \left. + \int_{Q_\omega} s^4\varphi_x^2\varphi_{xx}^2|q|^2\,dxdt + \int_{Q_\omega} s^2\varphi\varphi_x|q_x|^2\,dxdt\right].
\end{aligned} \tag{3.24}$$

From the above inequality q_x is in $\overline{\omega_0}$,which will be eliminated (because $\overline{\omega_0}$ is subset of ω_0). To prove q_x as q in ω_0 (term $|q_x|^2$ on the right hand side of (3.24)), the truncating function is necessary, so let us define the truncating function $\rho \in C_0^\infty(\Omega)$ with $\rho(x) = 1$ *in* $\overline{\omega_0}$ and $\rho(x) = 0$ in $\Omega\setminus\omega$. Now we will multiply (3.7) by $\rho s\varphi q$ and integrating over Q, then we have

$$\int_Q \rho s\varphi q L_1 q\,dx\,dt + \int_Q \rho s\varphi q L_2 q\,dx\,dt = \int_Q \rho q s\varphi h_s\,dxdt. \tag{3.25}$$

Since

$$\begin{aligned}
\rho\int_Q s\varphi q q_{xx}\,dxdt &= \int_Q \rho s\varphi_{xx}q^2\,dxdt - \int_Q \rho s\varphi_x q_x^2\,dxdt, \\
\int_Q \rho s\varphi q q_{xxx}\,dxdt &= \int_Q \rho s\varphi_x q_x^2\,dxdt - \int_\Sigma s\varphi\left|\frac{\partial q}{\partial v}\right|^2 d\Sigma + \int_Q \rho s\varphi_x q_x^2\,dxdt \\
&\quad - \int_Q \rho s\varphi_{xxx}q^2\,dxdt
\end{aligned}$$

$$
\begin{aligned}
-3\int_Q \rho s^2\varphi\varphi_x q q_{xx}\,dxdt &= 3\int_Q \rho s^2\varphi\varphi_x q_x^2\,dxdt - 3\int_Q \rho s^2\varphi_x\varphi_{xx} q^2\,dxdt \\
&\quad + \frac{3}{2}\int_Q \rho s^2\varphi\varphi_{xxx} q^2\,dxdt \\
-2\int_Q \rho s^2\varphi_x q q_x\,dxdt &= \int_Q \rho s^2\varphi_{xx}\varphi q^2\,dxdt + \int_Q \rho s^2\varphi_x^2 q^2\,dxdt \\
-3\int_Q \rho s^2\varphi\varphi_{xx} q q_x\,dxdt &= \frac{3}{2}\int_Q \rho s^2\varphi_x\varphi_{xx} q^2\,dxdt - \frac{3}{2}\int_Q \rho s^2\varphi\varphi_{xxx} q^2\,dxdt \\
3\int_Q \rho s^3\varphi\varphi_x^2 q q_x\,dxdt &= -\frac{3}{2}\int_Q \rho s^3\varphi_x^3 q^2\,dxdt - 3\int_Q \rho s^3\varphi\varphi_x\varphi_{xx} q^2\,dxdt.
\end{aligned}
\tag{3.26}
$$

The boundary term $-\int_\Sigma s\varphi|\partial q/\partial v|^2 d\Sigma \geq 0$, then

$$
\begin{aligned}
&\int_Q \rho s^2\varphi\varphi_x|q_x|^2\,dxdt \\
&\quad \leq \frac{1}{4}\int_Q e^{2s\varphi}|g|^2\,dxdt - \frac{3}{2}\int_Q \rho s^2\varphi\varphi_{xx}|q|^2\,dxdt + \frac{3}{2}\int_Q \rho s^2\varphi\varphi_{xxx}|q|^2\,dxdt \\
&\qquad + \int_Q \rho s^2\varphi\varphi_t|q|^2\,dxdt + 3\int_Q \rho s^2\varphi_x\varphi_{xx}|q|^2\,dxdt + \int_Q \rho s^2\varphi\varphi_{xxx}|q|^2\,dxdt \\
&\qquad + \int_Q \rho s\varphi_t|q|^2\,dxdt - \int_Q \rho s^3\varphi\varphi_x^2|q|^2\,dxdt + \int_Q \rho s^4\varphi\varphi_x^3|q|^2\,dxdt \\
&\qquad + 2\int_Q \rho s^2\varphi^2|q|^2\,dxdt + \frac{1}{4}\int_Q \rho e^{2s\varphi}\left|\int_t^T k(\tau,t)\left(e^{-s\varphi}q\right)_{xx}d\tau\right|^2 dxdt.
\end{aligned}
\tag{3.27}
$$

Applying as $\rho(x) = 1$ in $\overline{\omega_0} \subset \omega$,

$$
\begin{aligned}
\int_{Q_\omega} s^2\varphi\varphi_x|q_x|^2\,dxdt \leq C\Bigg(&\int_Q e^{2s\varphi}|g|^2\,dxdt + \int_{Q_\omega} e^{2s\varphi}\left|\int_t^T k(\tau,t)\left(e^{-s\varphi}q\right)_{xx}d\tau\right|^2 dxdt \\
&+ \int_{Q_\omega} s^4\varphi\varphi_x^3|q|^2\,dxdt\Bigg).
\end{aligned}
\tag{3.28}
$$

The term s, φ is not in ω, so it will be eliminated from the above inequalities. To eliminate s, φ, we will multiply (3.5) and (3.6) on $(s\varphi)^{-1}$ (see [8, 12]):

$$\begin{aligned}&\int_Q (s\varphi)^{-1}\left(|q_x|^2+|q_{xx}|^2+|q_{xxx}|^2\right)dxdt\\&\quad\le C\left(\int_Q (s\varphi)^{-1}|L_2q|^2dxdt+\int_Q (s\varphi)^{-1}s^2\varphi_x\varphi_{xx}|q|^2dxdt\right)\end{aligned}\tag{3.29}$$

$$\begin{aligned}&\int_Q (s\varphi)^{-1}|q_t|^2dxdt\\&\quad\le C\left(\int_Q (s\varphi)^{-1}|L_1q|^2dxdt+\int_Q s^4\varphi_x^2\varphi_{xx}^2q^2dxdt+\int_Q s^2\varphi_t^2q^2dxdt\right)\end{aligned}\tag{3.30}$$

In view of the estimates (3.22)–(3.30), we obtain

$$\begin{aligned}&\int_Q (s\varphi)^{-1}\left(|q_t|^2+|q_x|^2+|q_{xx}|^2+|q_{xxx}|^2\right)dxdt\\&\quad+\int_Q\left(s^5\varphi_x^4\varphi_{xx}|q|^2+s^3\varphi_x^2\varphi_{xx}|q_x|^2+s\varphi|q_{xx}|^2\right)dxdt\\&\quad\le C\left(\int_Q e^{2s\varphi}|g|^2+\int_{Q_\omega}s^4\left(\varphi_x^2\varphi_{xx}^2+\varphi\varphi_x^3\right)|q|^2dxdt\right.\\&\qquad\left.+\int_{Q_\omega}e^{2s\varphi}\left\{\left|\int_t^T k(\tau,t)\left(e^{-s\varphi}q\right)_{xx}d\tau\right|^2\right\}dxdt\right).\end{aligned}\tag{3.31}$$

From (3.31), we will eliminate the memory kernel term (that appears on the right hand side), because it may not contain in ω. By using Hölder's inequality and changing the order of integration,

$$\begin{aligned}&\int_{Q_\omega}e^{2s\varphi}\left\{\left|\int_t^T k(\tau,t)\left(e^{-s\varphi}q\right)_{xx}d\tau\right|^2\right\}dxdt\\&\quad\le\int_Q e^{2s\varphi}\left|\int_0^T k(\tau,t)\left(e^{-s\varphi}q\right)_{xx}d\tau\right|^2dxdt\\&\quad\le\int_Q e^{2s\varphi}\left(\int_{t_1}^{t_2}|k(\tau,t)|^2s\varphi d\tau\right)\left(\int_{t_1}^{t_2}s^{-1}\varphi^{-1}\left|e^{-s\varphi}q_{xx}\right|^2d\tau\right)dxdt\\&\quad\le C\|k\|_{L^\infty}^2e^{2s\varphi}\int_Q\int_{(t_1,t_2)\times\Omega}e^{2s\varphi}s\varphi\left|\left(e^{-s\varphi}q\right)_{xx}\right|^2dxdt,\end{aligned}\tag{3.32}$$

where C depends on Ω, ω, t_1, t_2, and T (since $e^{2s\varphi}s^{-1}\varphi^{-1} \le C < \infty$, and $e^{-2s\varphi}s\varphi \le C < \infty$ for all $(t,x) \in ((t_1,t_2)\times\Omega)$). Finally reverting to the original variable w to complete the proof, let us choose $\nabla w = e^{-s\varphi}(q_x - s\varphi_x q)$, then

$$\begin{aligned}
&\int_Q (s\varphi)^{-1}e^{2s\varphi}|\nabla w|^2 dxdt \le C\left(\int_Q (s\varphi)^{-1}|q_x|^2 dxdt + \int_Q (s\varphi)^{-1}s\varphi_x|q|^2 dxdt\right),\\
&\int_Q (s\varphi)^{-1}e^{2s\varphi}|\Delta w|^2 dxdt \le C\left(\int_Q (s\varphi)^{-1}|q_{xx}|^2 dxdt + \int_Q (s\varphi)^{-1}s^2\varphi_x^2|q|^2 dxdt\right),\\
&\int_Q (s\varphi)^{-1}e^{2s\varphi}|\nabla\cdot\Delta w|^2 dxdt \le C\left(\int_Q (s\varphi)^{-1}|q_{xxx}|^2 dxdt + \int_Q e^{2s\varphi}s^4\varphi_x^2\varphi_{xx}^2|q|^2 dxdt\right).
\end{aligned} \tag{3.33}$$

Similarly $w_t = e^{-s\varphi}(q_t - s\varphi_t q)$,

$$\int_Q (s\varphi)^{-1}e^{2s\varphi}|w_t|^2 dxdt \le C\left(\int_Q (s\varphi)^{-1}|q_t|^2 dxdt + \int_Q (s\varphi)^{-1}s^2\varphi_t^2|q|^2 dxdt\right). \tag{3.34}$$

From (3.32), we can observe the kernel term as

$$\begin{aligned}
&\int_{Q_\omega} e^{2s\varphi}\left\{\left|\int_t^T k(\tau,t)\left(e^{-s\varphi}q\right)_{xx}d\tau\right|^2\right\}dxdt\\
&\quad\le C\|k\|_{L^\infty}^2\left\{\int_Q e^{2s\varphi}\int_{(t_1,t_2)\times\Omega}|w_{xx}|^2 dxdt\right\}.
\end{aligned} \tag{3.35}$$

If $s \ge s_0$ is sufficiently large, then there exists a constant $C > 0$ such that the above integral terms have been absorbed on $\omega \subset \Omega$ (the right hand side of (3.31)). From the identifier (3.33)–(3.35), we can conform all the terms $|w_{xxx}|^2$, $|w_t|^2$, $|w_{xx}|^2$, $|w|^2$ involving in $s \ge s_0$, $\omega \subset \Omega$ and we obtain (3.3). □

Lemma 3.2. *Suppose that Theorem 3.1 is satisfied. If there exist positive constants C, μ (independent of s) and w will be a solution of* (2.1), *such that the following inequality holds:*

$$\int_\Omega |w(0,x)|^2 dx \le Ce^{\mu s}\left(\int_Q e^{2s\varphi}|g|^2 dxdt + \int_{Q_\omega} e^{2s\varphi}\varphi_x^2\varphi_{xx}^2|w|^2 dxdt\right). \tag{3.36}$$

Proof. We multiply (3.2) by w and integrate on Ω (using Cauchy's inequality, when $\epsilon = 1/2$), we obtain

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\int_\Omega |w|^2 dx + 2\int_\Omega |\nabla w|^2 dx + \frac{1}{2}\int_\Omega |w|^2 dxdt + 2\int_\Omega |\nabla\cdot(\Delta w)|^2 dxdt\\
&\quad\le \int_\Omega\left(2|g|^2 + \frac{3}{2}|w|^2\right)dx + 2\int_\Omega\left|\int_t^T k(\tau,t)w_{xx}d\tau\right|^2 dx.
\end{aligned} \tag{3.37}$$

Define

$$\gamma(t) = \sup\left\{e^{-2s\varphi}\varphi_x^{-2}(x,t)\varphi_{xx}^{-2}(x,t) : x \in \Omega\right\} \le Ce^{\mu s/\beta(t)}, \quad \mu = 2e^{2m\|\varphi\|_\infty}. \tag{3.38}$$

Therefore,

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}\int_\Omega |w|^2 dx \le C\Bigg(&\int_\Omega |g|^2 dx + \int_\Omega \left|\int_t^T k(\tau,t) w_{xx} d\tau\right|^2 dx + \int_\Omega |w|^2 dx \\ &+ \int_\Omega |\nabla w|^2 dx + \int_\Omega |\nabla\cdot(\Delta w)|^2 dx\Bigg). \end{aligned} \tag{3.39}$$

Integrating (3.39) on $(0,t)$,

$$\begin{aligned} &-\int_0^t \frac{d}{dt}\left[\int_\Omega |w|^2 dx\right] ds \\ &\quad \le C\int_0^t \Bigg(\int_\Omega |g|^2 dx + \int_\Omega \left|\int_t^T k(\tau,t) w_{xx} d\tau\right|^2 dx + \int_\Omega |\nabla w|^2 dx \\ &\qquad + \int_\Omega e^{2s\varphi}\gamma(t)\varphi_x^2\varphi_{xx}^2 |w|^2 dx + \int_\Omega |\nabla\cdot(\Delta w)|^2 dx\Bigg) ds, \quad \text{for } t \in (0,T). \end{aligned} \tag{3.40}$$

Therefore,

$$\begin{aligned} \int_\Omega |w(0,x)|^2 dx \le C\Bigg(&\int_\Omega e^{2s\varphi}\varphi_x^2\varphi_{xx}^2 |w|^2 dx + \int_\Omega |g|^2 dx + \int_\Omega \left|\int_t^T k(\tau,t) w_{xx} d\tau\right|^2 dx \\ &+ \int_\Omega |\nabla w|^2 dx + \int_\Omega |\nabla\cdot(\Delta w)|^2 dx\Bigg) \qquad \text{for } t \in (0,T). \end{aligned} \tag{3.41}$$

Now we fix t_1 and t_2 such as $0 < t_1 < t_2 < T$, then integrating the above inequality:

$$\begin{aligned} &\int_{t_1}^{t_2}\int_\Omega |w(0,x)|^2 e^{-\mu s/(t(t-T))} dx dt \\ &\quad \le C\int_{t_1}^{t_2}\Bigg(\int_\Omega e^{2s\varphi}\varphi_x^2\varphi_{xx}^2 |w|^2 dx + \int_\Omega |g|^2 dx + \int_\Omega |\nabla w|^2 dx \\ &\qquad + \int_\Omega \left|\int_t^T k(\tau,t) w_{xx} d\tau\right|^2 dx + \int_\Omega |\nabla\cdot(\Delta w)|^2 dx\Bigg) dt. \end{aligned} \tag{3.42}$$

As based on (3.32), the kernel has been modified as

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\Omega}\left|\int_{t}^{T} k(\tau,t)w_{xx}d\tau\right|^2 dxdt \\
&\quad\le \int_0^T\int_{\Omega}\left(\int_{t_1}^{t_2}|k(\tau,t)|^2 e^{-2s\varphi}s\varphi d\tau\right)\left(\int_{t_1}^{t_2}|w_{xx}|^2 e^{2s\varphi}s^{-1}\varphi^{-1}d\tau\right)dxdt \\
&\quad\le C\|k\|_{L^\infty}^2\int_{t_1}^{t_2}\int_{\Omega} e^{2s\varphi}s^{-1}\varphi^{-1}|w_{xx}|^2 dxdt.
\end{aligned} \tag{3.43}$$

Using (3.43) in (3.42), then

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\Omega}|w(0,x)|^2 e^{-\mu s/(t(t-T))}dxdt \\
&\quad\le C\left(\int_Q |g|^2 dxdt+\int_Q|\nabla w|^2 dxdt+\int_Q|\nabla\cdot(\Delta w)|^2 dxdt\right. \\
&\qquad\left.+\int_{t_1}^{t_2}\int_{\Omega} e^{2s\varphi}\left(\varphi_x^2\varphi_{xx}^2|w|^2+s^{-1}\varphi^{-1}|w_{xx}|^2\right)dxdt\right).
\end{aligned} \tag{3.44}$$

Since

$$\inf_{t\in(t_1,t_2)}\left\{e^{-\mu s/\beta(t)}\right\}\ge C>0 \tag{3.45}$$

and the Carleman estimate (3.3) using in (3.44), we obtain

$$\int_{\Omega}|w(0,x)|^2 dx\le C\left(\int_Q e^{2s\varphi}|g|^2 dxdt+\int_{Q_\omega} e^{2s\varphi}\varphi_x^2\varphi_{xx}^2|w|^2 dxdt\right). \tag{3.46}$$

□

4. Controllability Results

Now we are ready to give the proof of Theorem 2.2 result, which will be a main part of our work.

Proof of Theorem 2.2. Let us fix $T>0$, $a\in W^{1,\infty}(0,T:L^2(\omega))$, $b\in L^\infty(Q)$, and $y_0\in H_0^1(\Omega)$. For every $\epsilon>0$, the penalized formula

$$\text{Minimize}\left\{H_\epsilon(u);u\in L^2(Q)\right\}>0, \tag{4.1}$$

where the functional H_ϵ is (see [9, 11])

$$H_\epsilon(u) = \frac{1}{2}\int_Q |u|^2 dxdt + \frac{1}{2\epsilon}\int_\Omega |y(T,x)|^2 dx. \tag{4.2}$$

Here y is a solution of (1.2), which is associated with the control u. Since H_ϵ is a continuous strictly convex functional in $L^2(Q)$, then H_ϵ has a unique solution (u_ϵ, y_ϵ) (for any $\epsilon > 0$). If (u_ϵ, y_ϵ) (as $\epsilon \to 0$) is a null controllability solution of the system (1.1), then due to penalization property $(1/\epsilon)\int_\Omega |y(T,x)|^2 dx$, the limit exists in an appropriate norm. As based on the pontryagin maximum principle, the maximal condition on the control u_ϵ is

$$u_\epsilon = \chi_\omega w_\epsilon \quad a.e \text{ in } Q, \tag{4.3}$$

and $w^u \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$ is a solution for

$$(w_\epsilon)_t + \Delta w_\epsilon + (w_\epsilon)_x - w_\epsilon + (w_\epsilon)_{xxx} - \int_t^T k(\tau,t)(w_\epsilon)_{xx} d\tau = 0 \quad \text{in } Q, \tag{4.4}$$

$$w_\epsilon = -\frac{1}{\epsilon} y_\epsilon(T,x), \quad \text{in } \Omega, \tag{4.5}$$

$$w_\epsilon = 0, \quad \text{in } \Sigma. \tag{4.6}$$

Now we multiply (4.4) by y_ϵ (where $y = y_\epsilon$), (1.2) by w_ϵ and integrate on Q,

$$\int_{Q_\omega} |y_\epsilon|^2 dxdt + \frac{1}{\epsilon}\int_\Omega |y_\epsilon(T,x)|^2 dx - \int_\Omega y_0(x) w_\epsilon(x,0) dx + \int_Q f w_\epsilon dxdt = 0. \tag{4.7}$$

Applying the observability inequality (3.36), we have

$$\left|\int_\Omega y_0(x) w_\epsilon(x,0) dx\right| \le C\left(\int_{Q_\omega} |w_\epsilon(x,0)|^2 dxdt\right)^{1/2} |y_0|_2. \tag{4.8}$$

Using the Carleman estimate (3.3) and Lemma 2.1 condition,

$$\begin{aligned}\left|\int_Q f w_\epsilon dxdt\right| &\le C\left(\int_Q e^{2s\varphi}\varphi_x^2\varphi_{xx}^2 |w_\epsilon|^2 dxdt\right)^{1/2}\left(\int_Q f_1^2 dxdt\right)^{1/2} \\ &\le C\left(\int_{Q_\omega} e^{2s\varphi}\varphi_x^2\varphi_{xx}^2 |w_\epsilon|^2 dxdt\right)^{1/2} \|f_1\|_{L^2(\Omega)},\end{aligned} \tag{4.9}$$

where C is a positive constant that is independent of ϵ and y_0. By virtue of (4.7)–(4.9), we obtain

$$\int_Q |y_\epsilon|^2 dxdt + \frac{1}{\epsilon}\int_\Omega |y_\epsilon(T,x)|^2 dx \le C\left(|y_0|_2^2 + \|f_1\|_{L^2(Q)}^2\right), \quad \forall \epsilon > 0. \tag{4.10}$$

By the condition (4.10), we can observe

$$\int_Q |u_\epsilon|^2 dxdt + \frac{1}{\epsilon}\int_\Omega \left|y_\epsilon(T,x)\right|^2 dx \le C\left(\left|y_0\right|_2^2 + \left\|f_1\right\|_{L^2(Q)}^2\right). \tag{4.11}$$

Since u_ϵ is bounded in $L^2(Q)$, there exists a subsequence denoted by ϵ such that

$$\begin{gathered} u_\epsilon \longrightarrow u^* \text{ weakly in } L^2(Q), \\ y_\epsilon \longrightarrow y^* \text{ weakly in } L^2\left(0,T;H^1(\Omega)\right) \quad \text{as } \epsilon \longrightarrow 0. \end{gathered} \tag{4.12}$$

Clearly $y^* = y^{u^*}$, letting $\epsilon \to 0$ in (4.11), then we have $|y_\epsilon(T,x)|^2 = 0$ a.e. $x \in \Omega$. This completes the proof of Theorem 2.2. □

5. Conclusions

In this paper, we investigated the null controllability result for linearized KdV-Burgers equation with memory. It is suggested as a model which permits an analytical treatment of the Carleman estimate to prove the stability. The observability inequality has observed the null controllability, and the kernel term expression been observed in y_{xx}. It grants the stability, so our model could be heuristically useful in the studies of nonlinear water waves in a shallow channel, compressible gas, strong hydrodynamic turbulence as well as pressure force, and so forth, [1, 2].

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