Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 603018, 14 pages
doi:10.1155/2012/603018

Research Article

The Difference Problem of Obtaining the Parameter of a Parabolic Equation

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Received 4 April 2012; Accepted 23 April 2012

Academic Editor: Ravshan Ashurov

The boundary value problem of determining the parameter p of a parabolic equation $v'(t)+Av(t) = f(t) + p$ $(0 \leq t \leq 1)$, $v(0) = \varphi$, $v(1) = \psi$ in an arbitrary Banach space E with the strongly positive operator A is considered. The first order of accuracy stable difference scheme for the approximate solution of this problem is investigated. The well-posedness of this difference scheme is established. Applying the abstract result, the stability and almost coercive stability estimates for the solution of difference schemes for the approximate solution of differential equations with parameter are obtained.

1. Introduction

The differential equations with parameters play a very important role in many branches of science and engineering. Some examples were given in temperature overspecification by Dehghan [1], chemistry (chromatography) by Kimura and Suzuki [2], physics (optical tomography) by Gryazin et al. [3].

The differential equations with parameters have been studied extensively by many researchers (see, e.g., [4–20] and the references therein). However, such problems were not well investigated in general.

As a result, considerable efforts have been expanded in formulating numerical solution methods that are both accurate and efficient. Methods of numerical solutions of parabolic problems with parameters have been studied by researchers (see, e.g., [21–29] and the references therein).

It is known that various boundary value problems for parabolic equations with parameter can be reduced to the boundary value problem for the differential equation with parameter p:

$$\begin{gathered}\frac{dv(t)}{dt} + Av(t) = f(t) + p, \quad 0 < t < 1, \\ v(0) = \varphi, \qquad v(1) = \psi\end{gathered} \tag{1.1}$$

in an arbitrary Banach space E with the strongly positive operator A. In the present work, the first order of accuracy difference scheme for the approximate solution of boundary value problem (1.1) is studied. The well-posedness of this difference scheme is established. Applying the abstract result, the stability and almost coercive stability estimates for the solution of difference schemes for the approximate solution of differential equations with parameter are obtained.

2. The Boundary Value Problem for Parabolic Equations

Throughout this work, E is a Banach space, $-A$ is the generator of the analytic semigroup $\exp\{-tA\}(t \geq 0)$ with exponentially decreasing norm, when $t \to +\infty$, that is, the following estimates hold:

$$\|\exp\{-tA\}\|_{E\to E} \leq Me^{-\delta t}, t\|A\exp\{-tA\}\|_{E\to E} \leq M, \quad t > 0,\ M > 0,\ \delta > 0. \tag{2.1}$$

From estimate (2.1), it follows that

$$\|T\|_{E\to E} \leq M(\delta). \tag{2.2}$$

Here, $T = (I - \exp\{-A\})^{-1}$.

Abstract problem (1.1) was investigated in the paper [4] by applying estimates (2.1) and (2.2). The solvability of problem (1.1) in the space $C(E)$ of the continuous E-valued functions $\varphi(t)$ defined on $[0,1]$ equipped with the norm

$$\|\varphi\|_{C(E)} = \max_{0\leq t\leq 1}\|\varphi(t)\|_E \tag{2.3}$$

was studied under the necessary and sufficient conditions for the operator A. The solution depends continuously on the initial and boundary data. More pricisely, we have the following result.

Theorem 2.1. *Assume that $-A$ is the generator of the analytic semigroup $\exp\{-tA\}(t \geq 0)$ and all points $2\pi ik$, $k \in Z$, $k \neq 0$ do not belong to the spectrum $\sigma(A)$. Let $v(0) \in E$, $v(1) \in D(A)$, and $f(t) \in C^{\beta}(E)$ $(0 < \beta \leq 1)$. Then, for the solution $(v(t), p)$ of problem (1.1) in $C(E) \times E$, the estimates*

$$\begin{gathered}\|p\|_E \leq M\left[\|v(0)\|_E + \|v(1)\|_E + \|Av(1)\|_E + \frac{1}{\beta}\|f\|_{C^{\beta}(E)}\right], \\ \|v\|_{C(E)} \leq M\left[\|v(0)\|_E + \|v(1)\|_E + \|f\|_{C(E)}\right]\end{gathered} \tag{2.4}$$

hold, where M does not depend on β, $v(0)$, $v(1)$ and $f(t)$. Here $C^{\beta}(E)$ is the space obtained by completion of the space of all smooth E-valued functions $\varphi(t)$ on $[0,1]$ in the norm

$$\|\varphi\|_{C^{\beta}(E)} = \max_{0\leq t\leq 1}\|\varphi(t)\|_E + \sup_{0\leq t<t+\tau\leq 1}\frac{\|\varphi(t+\tau)-\varphi(t)\|_E}{\tau^{\beta}}. \tag{2.5}$$

With the help of A, we introduce the fractional space $E_{\alpha}(E, A)$, $0 < \alpha < 1$, consisting of all $v \in E$ for which the following norms are finite [6, 30]:

$$\|v\|_{\alpha} = \sup_{\lambda>0}\left\|\lambda^{1-\alpha}A\exp\{-\lambda A\}v\right\|_E + \|v\|_E. \tag{2.6}$$

We say $(v(t), p)$ is the solution of problem (1.1) in $C_0^{\beta,\gamma}(E) \times E_1$ if the following conditions are satisfied:

(i) $v'(t), Av(t) \in C_0^{\beta,\gamma}(E), p \in E_1 \subset E$,

(ii) $(v(t), p)$ satisfies the equation and boundary conditions (1.1).

Here, $C_0^{\beta,\gamma}(E)$, $(0 \leq \gamma \leq \beta, 0 < \beta < 1)$ is the Hölder space with weight obtained by completion of the space of all smooth E-valued functions $\varphi(t)$ on $[0,1]$ in the norm

$$\|\varphi\|_{C_0^{\beta,\gamma}(E)} = \max_{0\leq t\leq 1}\|\varphi(t)\|_E + \sup_{0\leq t<t+\tau\leq 1}\frac{(t+\tau)^{\gamma}\|\varphi(t+\tau)-\varphi(t)\|_E}{\tau^{\beta}}. \tag{2.7}$$

In the paper [23], the exact estimates in $C_0^{\beta,\gamma}(E)$, $(0 \leq \gamma \leq \beta, 0 < \beta < 1)$ and $C_0^{\beta,\gamma}(E_{\alpha-\beta})$ $(0 \leq \gamma \leq \beta \leq \alpha, 0 < \alpha < 1)$ Hölder spaces for the solution of problem (1.1) were proved. In applications, exact estimates for the solution of the boundary value problems for parabolic equations were obtained.

Now, we consider the application of Theorem 2.1. First, the boundary-value problem on the range $\{0 \leq t \leq 1, x \in \mathrm{R}^n\}$ for the $2m$-order multidimensional parabolic equation is considered:

$$\begin{gathered}\frac{\partial v(t,x)}{\partial t} + \sum_{|r|=2m} a_r(x)\frac{\partial^{|r|}v(t,x)}{\partial x_1^{r_1}\cdots\partial x_n^{r_n}} + \sigma v(t,x) = f(t,x) + p(x), \quad 0 < t < 1, \\ v(0,x) = \varphi(x), \qquad v(1,x) = \psi(x), \quad x \in \mathrm{R}^n, \quad |r| = r_1 + \cdots + r_n,\end{gathered} \tag{2.8}$$

where $a_r(x)$ and $f(t,x)$ are given as sufficiently smooth functions. Here, σ is a sufficiently large positive constant.

It is assumed that the symbol

$$B^x(\xi) = \sum_{|r|=2m} a_r(x)(i\xi_1)^{r_1}\cdots(i\xi_n)^{r_n}, \qquad \xi = (\xi_1,\ldots,\xi_n) \in \mathrm{R}^n \tag{2.9}$$

of the differential operator of the form

$$B^x = \sum_{|r|=2m} a_r(x)\frac{\partial^{|r|}}{\partial x_1^{r_1}\cdots\partial x_n^{r_n}} \tag{2.10}$$

acting on functions defined on the space R^n satisfies the inequalities

$$0 < M_1|\xi|^{2m} \le (-1)^m B^x(\xi) \le M_2|\xi|^{2m} < \infty \tag{2.11}$$

for $\xi \neq 0$.

Problem (2.8) has a unique smooth solution. This allows us to reduce problem (2.8) to problem (1.1) in a Banach space $E = C^\mu(R^n)$ of all continuous bounded functions defined on $\mathbb{R}^n$ satisfying a Hölder condition with the indicator $\mu \in (0,1)$.

Theorem 2.2. *For the solution of boundary problem* (2.8)*, the following estimates are satisfied:*

$$\begin{gathered}\|p\|_{C^\mu(R^n)} \le M\left[\|\varphi\|_{C^\mu(R^n)} + \|\psi\|_{C^{2m+\mu}(R^n)} + \frac{1}{\beta}\|f\|_{C^\beta(C^\mu(R^n))}\right],\\ \|v\|_{C(C^\mu(R^n))} \le M\left[\|\varphi\|_{C^\mu(R^n)} + \|\psi\|_{C^\mu(R^n)} + \|f\|_{C(C^\mu(R^n))}\right],\end{gathered} \tag{2.12}$$

where M is independent of $\varphi(x)$, $\psi(x)$, and $f(t,x)$.

The proof of Theorem 2.2 is based on the abstract Theorem 2.1 and on the strongly positivity of the operator $A = B^x + \sigma I$ defined by formula (2.10) (see, [31–33]).

Second, let Ω be the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1, 1 \le k \le n)$ with boundary $S, \overline{\Omega} = \Omega \cup S$. In $[0,1]\times\Omega$, we consider the mixed boundary value problem for the multidimensional parabolic equation

$$\begin{gathered}\frac{\partial v(t,x)}{\partial t} - \sum_{r=1}^{n}\alpha_r(x)\frac{\partial^2 v(t,x)}{\partial x_r^2} + \sigma v(t,x) = f(t,x) + p(x),\\ x = (x_1,\ldots,x_n) \in \Omega, \quad 0 < t < 1,\\ v(0,x) = \varphi(x), v(1,x) = \psi(x), \quad x \in \overline{\Omega},\\ v(t,x) = 0, \quad x \in S,\end{gathered} \tag{2.13}$$

where $\alpha_r(x)$ $(x \in \Omega)$, $\varphi(x)$, $\psi(x)(\overline{\Omega})$, and $f(t,x)(t \in (0,1), x \in \Omega)$ are given smooth functions and $\alpha_r(x) \ge a > 0$.Here, σ is a sufficiently large positive constant.

We introduce the Banach spaces $C_{01}^{\beta}(\overline{\Omega})(\beta = (\beta_1, \ldots, \beta_n),\ 0 < x_k < 1,\ k = 1, \ldots, n)$ of all continuous functions satisfying a Hölder condition with the indicator $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_k \in (0,1), 1 \le k \le n$, and with weight $x_k^{\beta_k}(1 - x_k - h_k)^{\beta_k}$, $0 \le x_k < x_k + h_k \le 1$, $1 \le k \le n$ which is equipped with the norm

$$\begin{aligned} \|f\|_{C_{01}^{\beta}(\overline{\Omega})} = \|f\|_{C(\overline{\Omega})} + \sup_{\substack{0 \le x_k < x_k + h_k \le 1, \\ 1 \le k \le n}} & \left| f(x_1, \ldots, x_n) - f(x_1 + h_1, \ldots, x_n + h_n) \right| \\ & \times \prod_{k=1}^{n} h_k^{-\beta_k} x_k^{\beta_k} (1 - x_k - h_k)^{\beta_k}, \end{aligned} \tag{2.14}$$

where $C(\overline{\Omega})$ is the space of the all continuous functions defined on $\overline{\Omega}$, equipped with the norm

$$\|f\|_{C(\overline{\Omega})} = \max_{x \in \overline{\Omega}} |f(x)|. \tag{2.15}$$

It is known that the differential expression [34]

$$Av(x) = -\sum_{r=1}^{n} \alpha_r(x) \frac{\partial^2 v(x)}{\partial x_r^2} + \sigma v(x) \tag{2.16}$$

defines a positive operator A acting on $C_{01}^{\beta}(\overline{\Omega})$ with the domain $D(A) = \{v(x), \partial^2 v(x)/\partial x_r^2 \in C_{01}^{\beta}(\overline{\Omega}), v(x) = 0 \text{ on } S\}$.

Therefore, we can replace mixed problem (2.13) by the abstract boundary problem (1.1). Using the results of Theorem 2.1, we can obtain the following theorem on stability.

Theorem 2.3. *For the solution of mixed boundary value problem* (2.18), *the following estimates are valid:*

$$\begin{aligned} \|p\|_{C^{\mu}(R^n)} &\le M\left[\|\varphi\|_{C_{01}^{\mu}(\overline{\Omega})} + \|\psi\|_{C_{01}^{24\mu}(\overline{\Omega})} + \frac{1}{\beta}\|f\|_{C^{\beta}(C_{01}^{\mu}(\overline{\Omega}))}\right], \\ \|v\|_{C(C_{01}^{\mu}(\overline{\Omega}))} &\le M\left[\|\varphi\|_{C_{01}^{\mu}(\overline{\Omega})} + \|\psi\|_{C_{01}^{\mu}(\overline{\Omega}))} + \|f\|_{C(C_{01}^{\mu}(\overline{\Omega})))}\right], \end{aligned} \tag{2.17}$$

where M does not depend on $\varphi(x)$, $\psi(x)$, and $f(t,x)$.

Third, we consider the mixed boundary value problem for parabolic equation

$$\begin{gathered} \frac{\partial v(t,x)}{\partial t} - a(x)\frac{\partial^2 v(t,x)}{\partial x^2} + \sigma v(t,x) = f(t,x) + p(x), \quad 0 < t < 1,\ 0 < x < 1, \\ v(0,x) = \varphi(x), \qquad v(1,x) = \psi(x), \quad 0 \le x \le 1, \\ v(t,0) = v(t,1), \qquad v_x(t,0) = v_x(t,1), \quad 0 \le t \le 1, \end{gathered} \tag{2.18}$$

where $a(x)$, $\varphi(x)$, $\psi(x)$, and $f(t,x)$ are given sufficiently smooth functions and $a(x) \geq a > 0$. Here, σ is a sufficiently large positive constant.

We introduce the Banach spaces $C^{\beta}[0,1]$ $(0 < \beta < 1)$ of all continuous functions $\varphi(x)$ satisfying a Hölder condition for which the following norms are finite,

$$\|\varphi\|_{C^{\beta}[0,1]} = \|\varphi\|_{C[0,1]} + \sup_{0\leq x<x+\tau\leq 1} \frac{|\varphi(x+\tau)-\varphi(x)|}{\tau^{\beta}}, \tag{2.19}$$

where $C[0,1]$ is the space of the all continuous functions $\varphi(x)$ defined on $[0,1]$ with the usual norm

$$\|\varphi\|_{C[0,1]} = \max_{0\leq x\leq 1}|\varphi(x)|. \tag{2.20}$$

It is known that the differential expression [30]

$$Av = -a(x)v''(x) + \sigma v(x) \tag{2.21}$$

defines a positive operator A acting in $C^{\beta}[0,1]$ with the domain

$$D(A) = \left\{v(x), v''(x) \in C^{\beta}[0,1], v(0) = v(1), v_x(0) = v_x(1)\right\}. \tag{2.22}$$

Therefore, we can replace the mixed problem (2.18) by the abstract boundary value problem (1.1). Using the result of Theorem 2.1, we can obtain the following theorem on stability.

Theorem 2.4. *For the solution of mixed problem* (2.18), *the following estimates are valid:*

$$\begin{gathered}\|p\|_{C^{\mu}[0,1]} \leq M\left[\|\varphi\|_{C^{\mu}[0,1]} + \|\psi\|_{C^{2+\mu}[0,1]} + \frac{1}{\beta}\|f\|_{C^{\beta}(C^{\mu}[0,1])}\right], \\ \|v\|_{C(C^{\mu}[0,1])} \leq M\left[\|\varphi\|_{C^{\mu}[0,1]} + \|\psi\|_{C^{\mu}[0,1]} + \|f\|_{C(C^{\mu}[0,1])}\right],\end{gathered} \tag{2.23}$$

where M is independent of $\varphi(x)$, $\psi(x)$, and $f(t,x)$.

3. Rothe Difference Scheme for Parabolic Equations with an Unknown Parameter

In this section, our focus is the well-posedness of the Rothe difference scheme

$$\begin{gathered}\tau^{-1}(u_k - u_{k-1}) + Au_k = \varphi_k + p, \qquad \varphi_k = f(t_k), \\ t_k = k\tau, \quad 1 \leq k \leq N, \qquad N\tau = 1, \\ u_0 = \varphi, \qquad u_N = \psi\end{gathered} \tag{3.1}$$

for approximately solving problem (1.1).

Let $[0,1]_\tau = \{t_k = k\tau, k = 0,1,\ldots,N, N\tau = 1\}$ be the *uniform grid space* with step size $\tau > 0$, where N is a fixed positive integer.

Throughout the section, $C([0,1]_\tau, E)$ denotes the linear space of grid functions $\varphi^\tau = \{\varphi_k\}_1^N$ with values in the Banach space E.

Let $C_\tau(E) = C([0,1]_\tau, E)$ be the Banach space of bounded grid functions with the norm

$$\|\varphi^\tau\|_{C_\tau(E)} = \max_{1\le k\le N} \|\varphi_k\|_E. \tag{3.2}$$

For $\alpha \in [0,1]$, let $C^\alpha(E) = C^\alpha([0,1]_\tau, E)$ be the Hölder space with the following norm:

$$\|\varphi^\tau\|_{C^\alpha(E)} = \|\varphi^\tau\|_{C_\tau(E)} + \max_{1\le k<k+r\le N} \frac{\|\varphi_{k+r} - \varphi_k\|_E}{(r\tau)^\alpha}. \tag{3.3}$$

Let us start with some lemmas we need in the following.

Lemma 3.1 (see [31]). *The following estimates hold:*

$$\begin{gathered} \left\|R^k\right\|_{E\to E} \le \frac{1}{(1+\delta\tau)^k}, \quad k \ge 1, \\ \left\|\tau A R^k\right\|_{E\to E} \le \frac{1}{k}, \quad k \ge 1, \end{gathered} \tag{3.4}$$

for some $M, \delta > 0$, which are independent of τ, where τ is a positive small number and $R = (I + \tau A)^{-1}$ is the resolvent of A.

Lemma 3.2. *The operator $I - R^N$ has an inverse $T_\tau = (I - R^N)^{-1}$ and the following estimate is satisfied:*

$$\|T_\tau\|_{E\to E} \le M(\delta). \tag{3.5}$$

Let us now obtain the formula for the solution of problem (3.1). It is clear that the first order of accuracy difference scheme

$$\begin{gathered} \tau^{-1}(u_k - u_{k-1}) + Au_k = p + \varphi_k, \qquad \varphi_k = f(t_k), \\ t_k = k\tau, \quad 1 \le k \le N, \qquad N\tau = 1, \\ u_0 = \varphi, \end{gathered} \tag{3.6}$$

has a solution and the following formula holds:

$$u_k = R^k\varphi + \sum_{j=1}^{k} R^{k-j+1}(p + \varphi_j)\tau, \quad 1 \le k \le N. \tag{3.7}$$

Applying formula (3.7) and the boundary condition

$$u_N = \psi, \tag{3.8}$$

we can write

$$\psi = R^N \varphi + \sum_{j=1}^{N} R^{N-j+1} \varphi_j \tau + \sum_{j=1}^{N} R^{N-j+1} \tau p. \tag{3.9}$$

Since

$$\sum_{j=1}^{N} R^{N-j+1} \tau = A^{-1}(I - R) \sum_{j=1}^{N} R^{N-j} = A^{-1}\left(I - R^N\right), \tag{3.10}$$

we have that

$$\psi = R^N \varphi + \sum_{j=1}^{N} R^{N-j+1} \varphi_j \tau + A^{-1}\left(I - R^{\mathrm{N}}\right) p. \tag{3.11}$$

Using Lemma 3.2, we get

$$p = T_\tau \left(A\psi - AR^N \varphi - \sum_{j=1}^{N} AR^{N-j+1} \varphi_j \tau \right). \tag{3.12}$$

Using formulas (3.7) and (3.12), we get

$$u_k = R^k \varphi + \sum_{j=1}^{k} R^{k-j+1} \varphi_j \tau + \sum_{j=1}^{k} R^{k-j+1} \tau T_\tau \left(A\psi - AR^N \varphi - \sum_{j=1}^{N} AR^{N-j+1} \varphi_j \tau \right), \quad 1 \le k \le N. \tag{3.13}$$

Since

$$\sum_{j=1}^{k} R^{k-j+1} \tau = A^{-1}(I - R) \sum_{j=1}^{k} R^{k-j} = A^{-1}\left(I - R^k\right), \tag{3.14}$$

we have that

$$u_k = R^k \varphi + \sum_{j=1}^{k} R^{k-j+1} \varphi_j \tau + \left(I - R^k\right) T_\tau \left(\psi - R^N \varphi - \sum_{j=1}^{N} R^{N-j+1} \varphi_j \tau \right), \quad 1 \le k \le N. \tag{3.15}$$

Hence, difference equation (3.1) is uniquely solvable, and, for the solution, formulas (3.12) and (3.15) are valid.

Theorem 3.3. *For the solution* $(\{u_k\}_{k=1}^N, p)$ *of problem* (3.1) *in* $C_\tau(E) \times E$, *the stability estimates*

$$\|p\|_E \le M\left[\|\varphi\|_E + \|A\psi\|_E + \frac{1}{\beta}\left\|\{\varphi_k\}_{k=1}^N\right\|_{C_\tau^\beta(E)}\right], \tag{3.16}$$

$$\left\|\{u_k\}_{k=1}^N\right\|_{C_\tau(E)} \le M\left[\|\varphi\|_E + \|\psi\|_E + \left\|\{\varphi_k\}_{k=1}^N\right\|_{C_\tau(E)}\right] \tag{3.17}$$

hold, where M *is independent of* τ, φ, ψ, *and* $\{\varphi_k\}_{k=1}^N$.

Proof. From formulas (3.7) and (3.12), it follows that

$$p = T_\tau\left(A\psi - AR^N\varphi - \sum_{j=1}^{N-1} AR^{N-j+1}(\varphi_j - \varphi_N)\tau - \left(I - R^N\right)\varphi_N\right). \tag{3.18}$$

Using this formula, the triangle inequality, and estimates (3.4), we obtain

$$\begin{aligned}\|p\|_E &\le \|T_\tau\|_{E\to E}\Bigg(\|A\psi\|_H + \left\|AR^N\right\|_{E\to E}\|\varphi\|_E \\ &\qquad + \sum_{j=1}^{N-1}\left\|AR^{N-j+1}\right\|_{E\to E}\|\varphi_j - \varphi_N\|_E\tau + \left(1 + \left\|R^N\right\|_{E\to E}\right)\|\varphi_N\|_E\Bigg) \\ &\le M\left[\|\varphi\|_E + \|A\psi\|_E + \frac{1}{\beta}\left\|\{\varphi_k\}_{k=1}^N\right\|_{C_\tau^\beta(E)}\right].\end{aligned} \tag{3.19}$$

□

The estimate (3.16) is proved. Using formula (3.15), the triangle inequality, and estimates (3.4), we obtain

$$\begin{aligned}\|u_k\|_E &\le \Bigg(\left\|R^k\right\|_{E\to E}\|\varphi\|_E + \sum_{j=1}^{k}\left\|R^{k-j+1}\right\|_{E\to E}\|\varphi_j\|_E\tau + \left(1 + \left\|R^k\right\|_{E\to E}\right)\|T_\tau\|_{E\to E} \\ &\qquad \times\left(\|\psi\|_E + \left\|R^N\right\|_{E\to E}\|\varphi\|_E + \sum_{j=1}^{N}\left\|R^{N-j+1}\right\|_{E\to E}\|\varphi_j\|_E\tau\right)\Bigg) \\ &\le M\left[\|\varphi\|_E + \|\psi\|_E + \left\|\{\varphi_k\}_{k=1}^N\right\|_{C_\tau(E)}\right]\end{aligned} \tag{3.20}$$

for any k. From that it follows estimate (3.17). Theorem 3.3 is proved.

Theorem 3.4. *For the solution* $(\{u_k\}_{k=1}^N, p)$ *of problem* (3.1) *in* $C_\tau(E)\times E$, *the almost coercive stability estimates*

$$\|p\|_E \le M\left[\|\varphi\|_E + \|A\psi\|_E + \min\left\{\ln\frac{1}{\tau}, |\ln\|A\|_{E\to E}|\right\}\left\|\{\varphi_k\}_{k=1}^N\right\|_{C_\tau(E)}\right], \tag{3.21}$$

$$\begin{aligned}&\left\|\{\tau^{-1}(u_k-u_{k-1})\}_{k=1}^N\right\|_{C_\tau(E)} + \left\|\{Au_k\}_{k=1}^N\right\|_{C_\tau(E)}\\ &\quad\le M\left[\|A\varphi\|_E + \|A\psi\|_E + \min\left\{\ln\frac{1}{\tau}, |\ln\|A\|_{E\to E}|\right\}\left\|\{\varphi_k\}_{k=1}^N\right\|_{C_\tau(E)}\right]\end{aligned} \tag{3.22}$$

hold, where M *does not depend on* τ, φ, ψ, *and* $\{\varphi_k\}_{k=1}^N$.

Proof. Using formula (3.12), the triangle inequality and estimates (3.4), we obtain

$$\begin{aligned}\|p\|_E &\le \|T_\tau\|_{E\to E}\left(\|A\psi\|_H + \left\|R^N\right\|_{E\to E}\|A\varphi\|_E + \sum_{j=1}^{N-1}\left\|AR^{N-j+1}\right\|_{E\to E}\|\varphi_j\|_E\tau\right)\\ &\le M\left[\|A\varphi\|_E + \|A\psi\|_E + \sum_{j=1}^{N-1}\left\|AR^{N-j+1}\right\|_{E\to E}\tau\left\|\{\varphi_k\}_{k=1}^N\right\|_{C_\tau(E)}\right].\end{aligned} \tag{3.23}$$

Since [31]

$$\sum_{j=1}^{N-1}\left\|AR^{N-j+1}\right\|_{E\to E}\tau \le M\min\left\{\ln\frac{1}{\tau}, |\ln\|A\|_{E\to E}|\right\}, \tag{3.24}$$

we have estimate (3.21). Using formula (3.15), the triangle inequality, and estimates (3.4), (3.24), we obtain

$$\begin{aligned}\|Au_k\|_E &\le \left(\left\|R^k\right\|_{E\to E}\|A\varphi\|_E + \sum_{j=1}^{k}\left\|AR^{k-j+1}\right\|_{E\to E}\|\varphi_j\|_E\tau + \left(1+\left\|R^k\right\|_{E\to E}\right)\|T_\tau\|_{E\to E}\right.\\ &\qquad\left.\times\left(\|A\psi\|_E + \left\|R^N\right\|_{E\to E}\|A\varphi\|_E + \sum_{j=1}^{N}\left\|AR^{N-j+1}\right\|_{E\to E}\|\varphi_j\|_E\tau\right)\right)\\ &\le M\left[\|A\varphi\|_E + \|A\psi\|_E + \min\left\{\ln\frac{1}{\tau}, |\ln\|A\|_{E\to E}|\right\}\left\|\{\varphi_k\}_{k=1}^N\right\|_{C_\tau(E)}\right]\end{aligned} \tag{3.25}$$

for any k. Therefore,

$$\left\|\{Au_k\}_{k=1}^N\right\|_{C_\tau(E)} \le M\left[\|A\varphi\|_E + \|A\psi\|_E + \min\left\{\ln\frac{1}{\tau}, |\ln\|A\|_{E\to E}|\right\}\left\|\{\varphi_k\}_{k=1}^N\right\|_{C_\tau(E)}\right]. \tag{3.26}$$

This estimate, triangle inequality, and (1.1) yield estimate (3.22). Theorem 3.4 is proved. □

4. Applications

Now, we consider the applications of Theorems 3.3 and 3.4. The boundary value problem (2.18) for the parabolic differential equation is considered. The discretization of problem (2.18) is carried out in two steps. In the first step, we define the grid space

$$[0,1]_h = \{x = x_n : x_n = nh, \quad 0 \le n \le M, Mh = 1\}. \tag{4.1}$$

Let us introduce the Banach space $C_h = C([0,1]_h)$ of the grid functions $\varphi^h(x) = \{\varphi_n\}_1^{M-1}$ defined on $[0,1]_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{C_h} = \max_{x\in[0,1]_h} \left|\varphi^h(x)\right|. \tag{4.2}$$

To the differential operator A generated by problem (2.18), we assign the difference operator A_h^x by the formula

$$A_h^x \varphi^h(x) = \left\{-\left(a(x)\varphi_{\bar{x}}\right)_{x,n} + \sigma\varphi_n\right\}_1^{M-1} \tag{4.3}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_n\}_1^{M-1}$ satisfying the conditions $\varphi_0 = \varphi_M$, $\varphi_1 - \varphi_0 = \varphi_M - \varphi_{M-1}$. It is wellknown that A_h^x is a strongly positive operator in C_h. With the help of A_h^x, we arrive at the boundary value problem

$$\begin{gathered} \frac{du^h(t,x)}{dt} + A_h^x u^h(t,x) = p^h(x) + f^h(t,x), \quad 0 < t < 1, \ x \in [0,1]_h, \\ u^h(0,x) = \varphi^h(x), \qquad u^h(1,x) = \psi^h(x), \ x \in [0,1]_h. \end{gathered} \tag{4.4}$$

In the second step, we replace (4.4) with the difference scheme (3.1)

$$\begin{gathered} \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + A_h^x u_k^h(x) = p^h(x) + f_k^h(x), \\ f_k^h(x) = f^h(t_k,x), \qquad t_k = k\tau, \quad 1 \le k \le N, \ x \in [0,1]_h, \\ u^h(0,x) = \varphi^h(x), \qquad u^h(1,x) = \psi^h(x), \quad x \in [0,1]_h. \end{gathered} \tag{4.5}$$

Theorem 4.1. *The solution pairs* $(\{u_k^h(x)\}_0^N, p^h(x))$ *of problem* (4.5) *satisfy the stability estimates*

$$\begin{gathered} \left\|p^h\right\|_{C_h} \le M_1 \left[\left\|\varphi^h\right\|_{C_h} + \left\|\psi^h\right\|_{C_h} + \left\|A_h^x\psi^h\right\|_{C_h} + \frac{1}{\beta}\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau^\beta(C_h)}\right], \\ \left\|\left\{u_k^h\right\}_1^N\right\|_{C_\tau(C_h)} \le M_2 \left[\left\|\varphi^h\right\|_{C_h} + \left\|\psi^h\right\|_{C_h} + \left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau(C_h)}\right], \end{gathered} \tag{4.6}$$

where M_1 *and* M_2 *do not depend on* β,φ^h, ψ^h, *and* f_k^h, $1 \le k \le N$.

Here, $C_\tau^\beta(C_h)$ is the grid space of grid functions $\{f_k^h\}_1^N$ defined on $[0,1]_\tau \times [0,1]_h$ with norm

$$\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau^\beta(C_h)} = \left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau(C_h)} + \sup_{1\le k<k+r\le N} \frac{\left\|f_{k+r}^h - f_k^h\right\|_{L_{2h}}}{(r\tau)^\beta},$$
$$\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau(C_h)} = \max_{1\le k\le N}\left\|f_k^h\right\|_{C_h}. \tag{4.7}$$

The proof of Theorem 4.1 is based on Theorem 3.3 and the positivity property of the operator A_h^x defined by formula (4.3).

Theorem 4.2. *The solution pairs $(\{u_k^h(x)\}_0^N, p^h(x))$ of problem (4.5) satisfy the almost coercive stability estimates*

$$\left\|p^h\right\|_{C_h} \le M_1\left[\left\|A_h^x\varphi^h\right\|_{C_h} + \left\|A_h^x\psi^h\right\|_{C_h} + \ln\frac{1}{\tau+h}\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau(C_h)}\right],$$
$$\left\|\left\{\frac{u_k^h - u_{k-1}^h}{\tau}\right\}_1^N\right\|_{C_\tau(C_h)} + \left\|\left\{A_h^x u_k^h\right\}_1^N\right\|_{C_\tau(C_h)}$$
$$\le M_2\left[\left\|A_h^x\varphi^h\right\|_{C_h} + \left\|A_h^x\psi^h\right\|_{C_h} + \ln\frac{1}{\tau+h}\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau(C_h)}\right], \tag{4.8}$$

where M_1 and M_2 are independent of φ^h, ψ^h, and f_k^h, $1 \le k \le N$.

The proof of Theorem 4.2 is based on Theorem 3.4 and the positivity property of the operator A_h^x defined by formula (4.3) and on the estimate

$$\min\left\{\ln\frac{1}{\tau}, \left|\ln\left\|A_h^x\right\|_{C_h\to C_h}\right|\right\} \le M\ln\frac{1}{\tau+h}. \tag{4.9}$$

Note that, in a similar manner, we can construct the difference schemes of the first order of accuracy with respect to one variable for approximate solutions of boundary value problems (2.8) and (2.13). Abstract theorems given from above permit us to obtain the stability, the almost stability estimates for the solutions of these difference schemes.

5. Conclusion

In this work, the first order of accuracy Rothe difference scheme for the approximate solution of the boundary value problem of determining the parameter p of a parabolic equation

$$v'(t) + Av(t) = f(t) + p(0 \le t \le 1), \qquad v(0) = \varphi, \qquad v(1) = \psi \tag{5.1}$$

in arbitrary Banach space E with the strongly positive operator A is studied. The well-posedness of the difference scheme is established. Some results in this paper in Hilbert

space H with self adjoint positive definite operator A were obtained in the paper [25]. The investigation of this paper in arbitrary Banach space E with the strongly positive operator A permits us to obtain the stability and almost stability estimates for the solution of difference schemes for the approximate solution of differential equations with parameter are obtained. Of course, such type results for the solution of difference scheme for the following boundary value problems

$$\begin{aligned} &v'(t) + Av(t) = f(t) + p(0 \le t \le 1), \qquad v(0) = \varphi, \qquad v(\lambda) = \psi, \quad 0 < \lambda \le 1, \\ &v'(t) - Av(t) = f(t) + p(0 \le t \le 1), \qquad v(1) = \varphi, \qquad v(\lambda) = \psi, \quad 0 \le \lambda < 1 \end{aligned} \tag{5.2}$$

in an arbitrary Banach space with positive operator A and an unknown parameter p hold.

Acknowledgment

The authors are grateful to Professor Allaberen Ashyralyev (Fatih University,Turkey) for his comments and suggestions to improve the quality of the paper.

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