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Research Article

Existence and Uniqueness of Solutions for Initial Value Problem of Nonlinear Fractional Differential Equations

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We discuss the initial value problem for the nonlinear fractional differential equation $L(D)u = f(t,u)$, $t \in (0,1]$, $u(0) = 0$, where $L(D) = D^{s_n} - a_{n-1}D^{s_{n-1}} - \cdots - a_1 D^{s_1}$, $0 < s_1 < s_2 < \cdots < s_n < 1$, and $a_j < 0$, $j = 1, 2, \ldots, n-1$, D^{s_j} is the standard Riemann-Liouville fractional derivative and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. We extend the basic theory of differential equation, the method of upper and lower solutions, and monotone iterative technique to the initial value problem. Some existence and uniqueness results are established.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary noninteger order, so fractional differential equations have wider application. Fractional differential equations have gained considerable importance; it can describe many phenomena in various fields of science and engineering such as control, porous media, electrochemistry, viscoelasticity, and electromagnetic.

In the recent years, there has been a significant development in fractional calculus and fractional differential equations; see Kilbas et al. [1], Miller and Ross [2], Podlubny [3], Baleanu et al. [4], and so forth. Research on the solutions of fractional differential equations is very extensive, such as numerical solutions, see El-Mesiry et al. [5] and Hashim et al. [6], mild solutions, see Chang et al. [7] and Chen et al. [8], the existence and uniqueness of solutions for initial and boundary value problem, see [9–30], and so on.

With the deep study, many papers that studied the fractional equations contained more than one fractional differential operator; see [16–20].

Babakhani and Daftardar-Gejji in [16] considered the initial value problem of nonlinear fractional differential equation

$$L(D)u = f(t,u), \quad u(0) = 0, \quad 0 < t < 1. \tag{1.1}$$

By using Banach fixed point theorem and fixed point theorem on a cone some results of existence and uniqueness of solutions are established.

Zhang in [17] studied the singular initial value problem for fractional differential equation by nonlinear alternative of Leray-Schauder theorem:

$$L(D)u = f(t,u), \quad t^{1-s_n}u(t)|_{t=0} = 0, \quad 0 < t \leq 1. \tag{1.2}$$

In above two equations, $L(D)$ is defined $L(D) := D^{s_n} - a_{n-1}D^{s_{n-1}} - \cdots - a_1 D^{s_1}$, where $0 < s_1 < s_2 < \cdots < s_n < 1$, and $a_j > 0$, $j = 1, 2, \ldots, n-1$, D^{s_j} is the standard Riemann-Liouville fractional derivative.

McRae in [14] studied the initial value problem by the method of upper and lower solutions and monotone iterative technique:

$$\begin{gathered} D^q u = f(t,u), \quad t \in (t_0, T], \ 0 < q < 1, \\ u(t_0) = u^0 = u(t)(t - t_0)^{1-q}|_{t=t_0}. \end{gathered} \tag{1.3}$$

In this paper, we use similar method as in [16] to consider the initial value problem:

$$\begin{gathered} L(D)u = f(t,u), \quad t \in (0,1], \\ u(0) = 0, \end{gathered} \tag{1.4}$$

where $L(D) = D^{s_n} - a_{n-1}D^{s_{n-1}} - \cdots - a_1 D^{s_1}$, $0 < s_1 < s_2 < \cdots < s_n < 1$, and $a_j < 0$, $j = 1, 2, \ldots, n-1$, D^{s_j} is the standard Riemann-Liouville fractional derivative and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

Since f is assumed continuous, the IVP (1.4) is equivalent to the following Volterra fractional integral equation:

$$u(t) = \sum_{j=1}^{n-1} \frac{a_j}{\Gamma(s_n - s_j)} \int_0^t (t-s)^{s_n - s_{n-1} - 1} u(s)ds + \frac{1}{\Gamma(s_n)} \int_0^t (t-s)^{s_n - 1} f(s, u(s))ds. \tag{1.5}$$

In Section 2, we give some definitions and lemmas that will be useful to our main results. In Section 3, we will use the basic theory of differential equation, the method of upper and lower solutions, and monotone iterative technique to investigate the initial value problem (1.4), and some existence and uniqueness results are established. In Section 4, an example is presented to illustrate the main results.

2. Preliminaries

In this section, we need the following definitions and lemmas that will be useful to our main results. These materials can be found in the recent literatures; see [1, 11, 16].

Definition 2.1 (see [1]). Let $\Omega = [a,b](-\infty < a < b < +\infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by

$$\begin{aligned} I_{a+}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a, \\ I_{b-}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b (x-t)^{\alpha-1} f(t)dt, \quad x < b, \end{aligned} \tag{2.1}$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the left-sided and the right-sided fractional integrals. We denote $I_{0+}^{\alpha} f(x)$ by $I^{\alpha} f(x)$ in the following paper.

Definition 2.2 (see [1]). Let $\Omega = [a,b]$ $(-\infty < a < b < +\infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional derivatives $D_{a+}^{\alpha} f$ and $D_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by

$$\begin{aligned} D_{a+}^{\alpha} f(x) &= \left(\frac{d}{dx}\right)^n (I_{a+}^{n-\alpha}) f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} f(t)dt, \quad x > a, \\ D_{b-}^{\alpha} f(x) &= \left(-\frac{d}{dx}\right)^n (I_{b-}^{n-\alpha}) f(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^n \int_x^b (x-t)^{n-\alpha-1} f(t)dt, \quad x < b, \end{aligned} \tag{2.2}$$

respectively, where $n = [\alpha] + 1$, $[\alpha]$ means the integral part of α. These derivatives are called the left-sided and the right-sided fractional derivatives. We denote $D_{0+}^{\alpha} f(x)$ by $D^{\alpha} f(x)$ in the following paper.

Definition 2.3. Letting $v, w \in C([0,1],\mathbb{R})$ be locally Hölder continuous with exponent $s_n < \lambda < 1$, we say that w is an upper solution of (1.4) if

$$\begin{aligned} L(D)w &\geq f(t,w), \\ w(0) &\geq 0, \end{aligned} \tag{2.3}$$

and v is a lower solution of (1.4) if

$$\begin{aligned} L(D)v &\leq f(t,v), \\ v(0) &\leq 0. \end{aligned} \tag{2.4}$$

Next, we will list the following lemma from [11] that is useful for our main results.

Lemma 2.4 (see [11, Lemma 2.1]). *Let $m \in C([0,1],\mathbb{R})$ be locally Hölder continuous with exponent $q < \lambda < 1$ such that for any $t_1 \in (0,1]$, we have*

$$m(t_1) = 0, \quad m(t) \leq 0 \quad \text{for } 0 \leq \mathrm{t} \leq \mathrm{t}_1. \tag{2.5}$$

Then it follows that $D^q m(t_1) \geq 0$.

Corollary 2.5. *Let $m \in C([0,1],\mathbb{R})$ be locally Hölder continuous with exponent $s_n < \lambda < 1$ such that for any $t_1 \in (0,1]$, we have*

$$m(t_1) = 0, \quad m(t) \leq 0 \quad \text{for } 0 \leq \mathrm{t} \leq \mathrm{t}_1. \tag{2.6}$$

Then it follows that $L(D)m(t_1) \geq 0$ provided $a_j < 0$, $j = 1, 2, \dots, n-1$.

Lemma 2.6. *Let $\{u_\epsilon(t)\}$ be a family of continuous functions on $[0,1]$, for each $\epsilon > 0$, where $L(D)u_\epsilon(t) = f(t, u_\epsilon(t))$, $u_\epsilon(0) = 0$ and $|f(t,u_\epsilon(t))| \leq M$ for $0 \leq t \leq 1$. Then the family $\{u_\epsilon(t)\}$ is equicontinuous on $[0,1]$.*

Proof. Since $\{u_\epsilon(t)\}$ is a family of continuous functions on $[0,1]$, there exists $l > 0$ such that $|u_\epsilon(t)| \leq l$ for $0 \leq t \leq 1$.

Let $\delta < \min\{(\sum_{j=1}^{n-1} \epsilon\Gamma(s_n - s_j + 1)/(4l|a_j|))^{1/(s_n - s_{n-1})}, (\epsilon\Gamma(s_n+1)/(4M))^{1/s_n}\}$. For $0 \leq t_1 < t_2 \leq 1$, $t_2 - t_1 < \delta$, we get

$$\begin{aligned} |u_\epsilon(t_2) - u_\epsilon(t_1)| &= \left| \sum_{j=1}^{n-1} I^{s_n - s_j} a_j u(t_2) - \sum_{j=1}^{n-1} I^{s_n - s_j} a_j u(t_1) + I^{s_n} f(t_2, u(t_2)) - I^{s_n} f(t_1, u(t_1)) \right| \\ &= \left| \sum_{j=1}^{n-1} \frac{a_j}{\Gamma(s_n - s_j)} \int_0^{t_1} \left[(t_2 - s)^{s_n - s_j - 1} - (t_1 - s)^{s_n - s_j - 1} \right] u(s)ds \right. \\ &\quad \left. + \sum_{j=1}^{n-1} \frac{a_j}{\Gamma(s_n - s_j)} \int_{t_1}^{t_2} (t_2 - s)^{s_n - s_j - 1} u(s)ds + \frac{1}{\Gamma(s_n)} \int_{t_1}^{t_2} (t_2 - s)^{s_n - 1} f(s, u(s))ds \right. \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(s_n)}\int_0^{t_1}\left[(t_2-s)^{s_n-1}-(t_1-s)^{s_n-1}\right]f(s,u(s))ds\Bigg| \\
&\leq \sum_{j=1}^{n-1}\frac{l|a_j|}{\Gamma(s_n-s_j)}\int_0^{t_1}\left[(t_1-s)^{s_n-s_j-1}-(t_2-s)^{s_n-s_j-1}\right]ds \\
&\quad+\sum_{j=1}^{n-1}\frac{l|a_j|}{\Gamma(s_n-s_j)}\int_{t_1}^{t_2}(t_2-s)^{s_n-s_j-1}ds+\frac{M}{\Gamma(s_n)}\int_{t_1}^{t_2}(t_2-s)^{s_n-1}ds \\
&\quad+\frac{M}{\Gamma(s_n)}\int_0^{t_1}\left[(t_1-s)^{s_n-1}-(t_2-s)^{s_n-1}\right]ds \\
&=\sum_{j=1}^{n-1}\frac{l|a_j|}{\Gamma(s_n-s_j+1)}\left(t_1^{s_n-s_j}-t_2^{s_n-s_j}\right)+\sum_{j=1}^{n-1}\frac{2l|a_j|}{\Gamma(s_n-s_j+1)}(t_2-t_1)^{s_n-s_j} \\
&\quad+\frac{M}{\Gamma(s_n+1)}\left(t_1^{s_n}-t_2^{s_n}\right)+\frac{2M}{\Gamma(s_n+1)}(t_2-t_1)^{s_n} \\
&\leq \sum_{j=1}^{n-1}\frac{2l|a_j|}{\Gamma(s_n-s_j+1)}(t_2-t_1)^{s_n-s_{n-1}}+\frac{2M}{\Gamma(s_n+1)}(t_2-t_1)^{s_n} \\
&\leq \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.
\end{aligned}
\tag{2.7}
$$

Thus, $\{u_\epsilon(t)\}$ is equicontinuous on $[0,1]$. □

Lemma 2.7 (see [16, Theorem 4.2]). *Let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be continuous and Lipschitz with respect to second variable with Lipschitz constant L. Let a_j satisfy*

$$
0<\frac{L}{\Gamma(s_n+1)}+\sum_{j=1}^{n-1}\frac{|a_j|}{\Gamma(s_n-s_j+1)}<1. \tag{2.8}
$$

Then IVP (1.4) *has a unique solution.*

Lemma 2.8. *Let $v,w \in C([0,1],\mathbb{R})$ be locally Hölder continuous with exponent $q<\lambda<1$, $f \in C([0,1]\times\mathbb{R},\mathbb{R})$ and*

$$
L(D)w \geq f(t,w), \quad L(D)v \leq f(t,v), \quad 0<t\leq 1 \tag{2.9}
$$

one of the nonstrict inequalities being strict. Then $v(0)<w(0)$ implies $v(t)<w(t)$, $0\leq t\leq 1$.

Proof. Suppose that $v(t)<w(t)$, $0\leq t\leq 1$ is not true. We suppose the inequality $L(D)>f(t,w(t))$. Letting $m(t)=v(t)-w(t)$, there exists $0<t_1\leq 1$ such that $m(t)\leq 0$, $0\leq t\leq t_1$, and

$m(t_1) = 0$. Then by Corollary 2.5, we can obtain $L(D)m(t_1) \geq 0$. From the conditions and the definition of $m(t)$, we have

$$f(t_1, v(t_1)) \geq L(D)v(t_1) \geq L(D)w(t_1) > f(t_1, w(t_1)). \tag{2.10}$$

This is a contradiction to $v(t_1) = w(t_1)$. The proof is complete. □

Lemma 2.9. *Assume that the conditions of Lemma 2.8 hold with nonstrict inequalities* (2.3) *and* (2.4). *Furthermore, suppose that*

$$f(t, x) - f(t, y) \leq N(x - y), \quad \text{where } x \geq y,\ N > 0. \tag{2.11}$$

Then $v(0) \leq w(0)$ *implies* $v(t) \leq w(t)$, $0 \leq t \leq 1$ *provided* $N < 1/\Gamma(1 - s_n) - \sum_{j=1}^{n-1} a_j/\Gamma(1 - s_j)$.

Proof. Let $w_\epsilon(t) = w(t) + \epsilon$. For small $\epsilon > 0$, we have

$$w_\epsilon(0) > w(0), \qquad w_\epsilon(t) > w(t), \quad 0 \leq t \leq 1. \tag{2.12}$$

Then, from (2.11) and (2.12) we get

$$\begin{aligned} L(D)w_\epsilon(t) &= L(D)w(t) + L(D)\epsilon \\ &= f(t, w(t)) + \epsilon\left[\frac{t^{-s_n}}{\Gamma(1 - s_n)} - \sum_{j=1}^{n-1}\frac{a_j t^{-s_j}}{\Gamma(1 - s_j)}\right] \\ &\geq f(t, w_\epsilon(t)) - N\epsilon + \epsilon\left[\frac{t^{-s_n}}{\Gamma(1 - s_n)} - \sum_{j=1}^{n-1}\frac{a_j t^{-s_j}}{\Gamma(1 - s_j)}\right] \\ &\geq f(t, w_\epsilon(t)) - N\epsilon + \epsilon\left[\frac{1}{\Gamma(1 - s_n)} - \sum_{j=1}^{n-1}\frac{a_j}{\Gamma(1 - s_j)}\right] \\ &> f(t, w_\epsilon(t)), \quad 0 < t \leq 1. \end{aligned} \tag{2.13}$$

Applying Lemma 2.8, we obtain $v(t) < w_\epsilon(t)$, $0 \leq t \leq 1$. By the arbitrariness of $\epsilon > 0$, we can conclude that $v(t) \leq w(t)$. The proof is complete. □

Corollary 2.10. *The function* $f(t, u) = \sigma(t)u$, *where* $\sigma(t) \leq N$, *is admissible in Lemma 2.9 to yield* $v(t) \leq 0$ *on* $0 \leq t \leq 1$.

3. Main Results

In this section, we establish the existence and uniqueness criteria of solutions for initial value problem (1.4).

Theorem 3.1. *Assume that $f \in C(R_0, \mathbb{R})$, where $R_0 = \{(t,u) : 0 \le t \le 1, |u(t)| \le b\}$ and $|f(t,u)| \le M$. Then IVP* (1.4) *possesses at least one solution $u(t)$ on $0 \le t \le \alpha$, where $\alpha = \min\{1, (b\Gamma(1+s_n)/(2M))^{1/s_n}, (\sum_{j=1}^{n-1} \Gamma(s_n - s_j + 1)/(2|a_j|))^{1/(s_n - s_{n-1})}\}$.*

Proof. Let $u_0(t)$ be a continuous function on $[-\delta, 0]$, $\delta > 0$, such that $u_0(0) = 0$, $|u_0(t)| \le b$ and $|L(D)u_0(t)| \le M$, where $D_{0-}^{s_j} u_0(t)$, $j = 1, 2, \ldots, n-1$ are the continuous fractional derivatives. For $0 < \epsilon \le \delta$, we define the function $u_\epsilon(t) = u_0(t)$ on $[-\delta, 0]$ and

$$u_\epsilon(t) = \frac{1}{\Gamma(s_n)} \int_0^t (t-s)^{s_n - 1} f(s, u_\epsilon(s-\epsilon)) ds + \sum_{j=1}^{n-1} \frac{a_j}{\Gamma(s_n - s_j)} \int_0^t (t-s)^{s_n - s_j - 1} u_\epsilon(s-\epsilon) ds \tag{3.1}$$

on $[0, \alpha_1]$, where $\alpha_1 = \min\{\epsilon, \alpha\}$. We observe that $D^{s_j} u_\epsilon(t)$, $j = 1, 2, \ldots, n$ exist for $t \in [0, \alpha_1]$ and

$$\begin{aligned}
|u_\epsilon(t)| &\le \frac{1}{\Gamma(s_n)} \int_0^t (t-s)^{s_n-1} \left| f(s, u_\epsilon(s-\epsilon)) \right| ds + \sum_{j=1}^{n-1} \frac{\left|a_j\right|}{\Gamma(s_n - s_j)} \int_0^t (t-s)^{s_n - s_j - 1} |u_\epsilon(s-\epsilon)| ds \\
&\le \frac{M}{\Gamma(s_n)} \int_0^t (t-s)^{s_n-1} ds + \sum_{j=1}^{n-1} \frac{b\left|a_j\right|}{\Gamma(s_n - s_j)} \int_0^t (t-s)^{s_n - s_j - 1} ds \\
&= \frac{M}{\Gamma(s_n+1)} t^{s_n} + \sum_{j=1}^{n-1} \frac{b\left|a_j\right|}{\Gamma(s_n - s_j + 1)} t^{s_n - s_j} \\
&\le \frac{M}{\Gamma(s_n+1)} \alpha^{s_n} + \sum_{j=1}^{n-1} \frac{b\left|a_j\right|}{\Gamma(s_n - s_j + 1)} \alpha^{s_n - s_{n-1}} \\
&\le \frac{b}{2} + \frac{b}{2} = b.
\end{aligned} \tag{3.2}$$

If $\alpha_1 < \alpha$, we can employ (3.1) to extend $u_\epsilon(t)$ as a continuously fractional differentiable function on $[-\delta, \alpha_2]$, $\alpha_2 = \min\{\alpha, 2\epsilon\}$ such that $u_\epsilon(t) \le b$ holds. Continuing this process, we can define $u_\epsilon(t)$ over $[-\delta, \alpha]$ so that $u_\epsilon(t) \le b$; it has a continuous fractional derivative and satisfies (3.1) on the same interval $[-\delta, \alpha]$. Furthermore, $|L(D)u_\epsilon(t)| \le M$, since $|f(t, u_\epsilon(t-\epsilon))| \le M$ on R_0. Therefore, from Lemma 2.6, the family $\{u_\epsilon(t)\}$ is an equicontinuous and uniformly bounded function. An application of Ascoli-Arzela Theorem shows the existence of a sequence $\{\epsilon_n\}$ such that $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n \to 0$ as $n \to \infty$, and $u(t) = \lim_{n\to\infty} u_{\epsilon_n}(t)$ exists uniformly on $[-\delta, \alpha]$. Due to f being uniformly continuous, we can obtain $f(t, u_{\epsilon_n}(t-\epsilon_n))$ which uniformly tends to $f(t, u(t))$, and $u_{\epsilon_n}(t - \epsilon_n)$ uniformly tends to $u(t)$ as $n \to \infty$. Therefore, term by term, integration of (3.1) with $\epsilon = \epsilon_n$, $\alpha_1 = \alpha$ yields

$$u(t) = \frac{1}{\Gamma(s_n)} \int_0^t (t-s)^{s_n-1} f(s, u(s)) ds + \sum_{j=1}^{n-1} \frac{a_j}{\Gamma(s_n - s_j)} \int_0^t (t-s)^{s_n - s_j - 1} u(s) ds. \tag{3.3}$$

This proves that $u(t)$ is a solution of IVP (1.4) and the proof is complete. □

Theorem 3.2. *Let $v, w \in C([0,1],\mathbb{R})$ be lower and upper solutions of the IVP* (1.4) *which are locally Hölder continuous with exponent $s_n < \lambda < 1$ such that $v(t) \le w(t)$, $t \in [0,1]$ and $f \in C(\Omega,\mathbb{R})$, where $\Omega = \{(t,u) : v(t) \le u(t) \le w(t), t \in [0,1]\}$. Furthermore, suppose that*

$$\left(\sum_{j=1}^{n-1}\frac{\Gamma(s_n - s_j + 1)}{2|a_j|}\right)^{1/(s_n - s_{n-1})} \ge 1. \tag{3.4}$$

Then there exists a solution $u(t)$ of IVP (1.4) *satisfying $v(t) \le u(t) \le w(t)$ on* $[0,1]$.

Proof. For the need of proof, we define function $p(t,u) : [0,1]\times\mathbb{R} \to \mathbb{R}$ as

$$p(t,u) = \max\{v(t), \min\{u, w(t)\}\}. \tag{3.5}$$

Therefore, $f(t,p(t,u))$ defines a continuous extension of f to $[0,1]\times R$ which is also bounded because f is bounded on Ω. Then by Theorems 3.1 and 3.2, we can obtain that the initial value problem

$$\begin{gathered} L(D)u = f(t,p(t,u)), \quad t \in (0,1], \\ u(0) = 0, \end{gathered} \tag{3.6}$$

has a solution on $[0,1]$.

Clearly, from the definition of function $p(t,u)$, we know that if IVP (3.6) exits a solution $u(t)$ satisfying $v(t) \le u(t) \le w(t)$ on $[0,1]$, then $u(t)$ is also a solution of IVP (1.4). In the following, we will prove that the solution $u(t)$ of IVP (3.6) satisfies $v(t) \le u(t) \le w(t)$ on $[0,1]$.

For any $\epsilon > 0$, we consider

$$w_\epsilon(t) = w(t) + \epsilon, \qquad v_\epsilon(t) = v(t) - \epsilon. \tag{3.7}$$

Then, we get

$$w_\epsilon(0) = w(0) + \epsilon, \qquad v_\epsilon(0) = v(0) - \epsilon. \tag{3.8}$$

Therefore, it follows that $v_\epsilon(0) < u(0) < w_\epsilon(0)$. Next, we will show that $v_\epsilon(t) < u(t) < w_\epsilon(t)$, $t \in [0,1]$. Suppose that it is not true. Then there exists $t_1 \in (0,1]$ such that

$$u(t_1) = w_\epsilon(t_1), \quad v_\epsilon(t) < u(t) < w_\epsilon(t), \quad 0 \le t < t_1. \tag{3.9}$$

Therefore, $u(t_1) > w(t_1)$, $p(t_1, u(t_1)) = w(t_1)$ and $v(t_1) \leq p(t_1, u(t_1)) \leq w(t_1)$. Letting $m(t) = u(t) - w_\epsilon(t)$, we have $m(t_1) = 0$ and $m(t) \leq 0$, $0 \leq t \leq t_1$. Then from Corollary 2.5, we can obtain $L(D)m(t_1) \geq 0$ and

$$\begin{aligned} f(t_1, w(t_1)) &= f(t_1, p(t_1, w(t_1))) \\ &= L(D)u(t_1) \geq L(D)w_\epsilon(t_1) \\ &= L(D)w(t_1) + L(D)\epsilon(t_1) \\ &= L(D)w(t_1) + \epsilon\left[\frac{t_1^{-s_n}}{\Gamma(1-s_n)} - \sum_{j=1}^{n-1}\frac{a_j t_1^{-s_j}}{\Gamma(1-s_j)}\right] \\ &> L(D)w(t_1) = f(t_1, w(t_1)), \end{aligned} \tag{3.10}$$

which is a contradiction. The other case $v_\epsilon(t) < u(t)$ can be proved similarly.

Hence, we get $v_\epsilon(t) < u(t) < w_\epsilon(t)$ on $[0,1]$. Letting $\epsilon \to 0$, we obtain $v(t) \leq u(t) \leq w(t)$ on $[0,1]$. The proof is complete. □

Now, we will give the existence of maximal and minimal solutions of initial value problem (1.4).

Theorem 3.3. *Let $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$, v_0, w_0 be lower and upper solutions of* (1.4) *such that $v_0 \leq w_0$ on* $[0,1]$. *Furthermore, suppose that*

$$f(t,x) - f(t,y) \geq -N(x-y), \quad \text{for } \mathrm{v}_0 \leq \mathrm{y} \leq \mathrm{x} \leq \mathrm{w}_0,\ \mathrm{N} \geq 0, \tag{3.11}$$

and a_j satisfy

$$0 < \frac{N}{\Gamma(s_n+1)} + \sum_{j=1}^{n-1}\frac{|a_j|}{\Gamma(s_n - s_j + 1)} < 1. \tag{3.12}$$

Then there exist monotone sequences $\{v_n\}$ and $\{w_n\}$ such that $v_n \to \rho$, $w_n \to r$ as $n \to \infty$ uniformly on $[0,1]$, *where ρ and r are minimal and maximal solutions of IVP* (1.4), *respectively.*

Proof. For any $\eta \in C([0,1], \mathbb{R})$ satisfying $v_0 \leq \eta \leq w_0$, we consider the following linear fractional differential equation:

$$\begin{gathered} L(D)u = f(t,\eta) - N(u-\eta), \quad t \in (0,1], \\ u(0) = 0. \end{gathered} \tag{3.13}$$

Obviously, the right hand side of (3.13) satisfies the Lipschitz condition. From (3.11) and Lemma 2.7, it is clear that for every η, there exists a unique solution u of (3.13) on $[0,1]$.

Define the operator T by $T\eta = u$ and use it to construct the sequences $\{v_n\}$, $\{w_n\}$. We need to prove the following propositions hold:

(i) $v_0 \leq Tv_0$, $w_0 \geq Tw_0$;

(ii) T is a monotone operator on the segment

$$\langle v_0, w_0 \rangle = \{u \in C([0,1], \mathbb{R}) : v_0 \leq u \leq w_0\}. \tag{3.14}$$

To prove (i), let $Tv_0 = v_1$, where v_1 is the unique solution of (3.13) with $\eta = v_0$. Letting $p = v_0 - v_1$, we have

$$\begin{gathered} L(D)p = L(D)v_0 - L(D)v_1 \leq f(t, v_0) - \left[f(t, v_0) - N(v_1 - v_0)\right] = -Np, \\ p(0) = v_0(0) - v_1(0) \leq 0. \end{gathered} \tag{3.15}$$

By Corollary 2.10, we can obtain that $p(t) \leq 0$ on $[0,1]$, that is, $v_0 \leq Tv_0$.

Similarly, we can get $w_0 \geq Tw_0$.

To prove (ii), let $\eta_1, \eta_2 \in \langle v_0, w_0 \rangle$ such that $\eta_1 \leq \eta_2$. Assume that $u_1 = T\eta_1$ and $u_2 = T\eta_2$. Setting $p = u_1 - u_2$, then using the condition (3.11), we have

$$\begin{aligned} L(D)p &= L(D)u_1 - L(D)u_2 = f(t, \eta_1) - N(u_1 - \eta_1) - \left[f(t, \eta_2) - N(u_2 - \eta_2)\right] \\ &\leq -N(\eta_1 - \eta_2) - N(u_1 - \eta_1) + N(u_2 - \eta_2) = -Np, \\ &\qquad p(0) = u_1(0) - u_2(0) = 0. \end{aligned} \tag{3.16}$$

From Corollary 2.10, we can obtain that $p(t) \leq 0$ on $[0,1]$, which implies $T\eta_1 \leq T\eta_2$. And (ii) is proved.

Therefore, we can define the sequences $v_n = Tv_{n-1}$, $w_n = Tw_{n-1}$. From the previous discussion, we can get

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq w_n \leq \cdots \leq w_1 \leq w_0 \quad \text{on } [0,1]. \tag{3.17}$$

Clearly, the sequences $\{v_n\}$, $\{w_n\}$ are uniformly bounded on $[0,1]$. From (3.13), we have $|L(D)v_n|$, $|L(D)w_n|$ which are also uniformly bounded. By Lemma 2.6, we know that $\{v_n\}$, $\{w_n\}$ are equicontinuous on $[0,1]$. Then applying Ascoli-Arzela Theorem, there exist subsequences $\{v_{n_k}\}$, $\{w_{n_k}\}$ that converge uniformly on $[0,1]$. From (3.17), we can see that the entire sequences $\{v_n\}$, $\{w_n\}$ converge uniformly and monotonically to ρ, r, respectively, as $n \to \infty$. It is now easy to show that ρ, r are solutions of IVP (1.4) by the corresponding Volterra fractional integral equation for (3.13).

In the following, we will prove that ρ and r are the minimal and maximal solutions of IVP (1.4), respectively. We need to show that if u is any solution of IVP (1.4) satisfying $v_0 \leq u \leq w_0$ on $[0,1]$, then we have $v_0 \leq \rho \leq u \leq r \leq w_0$ on $[0,1]$.

We assume that for some n, $v_n \leq u \leq w_n$ on $[0,1]$ and letting $p = v_{n+1} - u$, we have

$$\begin{aligned} L(D)p &= L(D)v_{n+1} - L(D)u = f(t, v_n) - N(v_{n+1} - v_n) - f(t,u) \\ &\leq -N(v_n - u) - N(v_{n+1} - v_n) = -Np, \\ p(0) &= v_{n+1}(0) - u(0) = 0, \end{aligned} \tag{3.18}$$

which implies $v_{n+1} \leq u$. Similarly, we have $u \leq w_{n+1}$ on $[0,1]$. Since $v_0 \leq u \leq w_0$ on $[0,1]$, this proves $v_n \leq u \leq w_n$ for all n by induction. Letting $n \to \infty$, we conclude that $\rho \leq u \leq r$ on $[0,1]$ and the proof is complete. □

Theorem 3.4. *Suppose that the conditions of Theorem 3.3 hold. In addition, we assume*

$$f(t,x) - f(t,y) \leq N(x-y), \quad v_0 \leq y \leq x \leq w_0, \quad N > 0. \tag{3.19}$$

Then $\rho = r = u$ is the unique solution of IVP (1.4) *provided* $N < 1/\Gamma(1-s_n) - \sum_{j=1}^{n-1} a_j/\Gamma(1-s_j)$.

Proof. We have proved $\rho \leq r$ in Theorem 3.3, so we just need to prove $\rho \geq r$. Letting $p = r - \rho$, we get

$$\begin{aligned} L(D)p &= f(t,r) - f(t,\rho) \leq Np, \\ p(0) &= r(0) - \rho(0) = 0. \end{aligned} \tag{3.20}$$

From Corollary 2.10, we obtain $p \leq 0$ on $[0,1]$, which implies $\rho \geq r$. Hence, $\rho = r = u$ is the unique solution of IVP (1.4). □

4. Examples

In this paper, we will present an example to illustrate the main results.

Example 4.1. Consider the initial value problem of fractional differential equation

$$\begin{aligned} D^{0.8}u + 0.4D^{0.6}u &= \frac{u^2 t^{0.2}}{10\Gamma(0.2)} - \frac{u t^{0.4}}{2\Gamma(0.4)}, \quad t \in (0,1], \\ u(0) &= 0. \end{aligned} \tag{4.1}$$

Choose $w = 5$, $v = -5$; then we can obtain

$$\begin{aligned} D^{0.8}w + 0.4D^{0.6}w &\geq \frac{w^2 t^{0.2}}{10\Gamma(0.2)} - \frac{w t^{0.4}}{2\Gamma(0.4)}, \\ D^{0.8}v + 0.4D^{0.6}v &\leq \frac{v^2 t^{0.2}}{10\Gamma(0.2)} - \frac{v t^{0.4}}{2\Gamma(0.4)}. \end{aligned} \tag{4.2}$$

That is, v and w are the lower and upper solutions of initial value problem (4.1). Furthermore, v and w are locally continuous with exponent $1 > \lambda > 0.8$.

Since

$$\left(\frac{\Gamma(0.8-0.6+1)}{2|-0.4|}\right)^{1/(0.8-0.6)} = 1.9914 > 1, \tag{4.3}$$

then by Theorem 3.2, there exists a solution $u(t)$ of initial value problem (4.1) satisfying $-5 \le u(t) \le 5$.

Next, we will prove the existence of maximal and minimal solutions for initial value problem (4.1) by using Theorem 3.3.

Let $v_0 = -5$ and $w_0 = 5$ be lower and upper solutions of (4.1). Furthermore, for any $-5 \le y \le x \le 5$, we have

$$\begin{aligned} f(t,x) - f(t,y) &= \frac{x^2 t^{0.2}}{10\Gamma(0.2)} - \frac{x t^{0.4}}{2\Gamma(0.4)} - \frac{y^2 t^{0.2}}{10\Gamma(0.2)} + \frac{y t^{0.4}}{2\Gamma(0.4)} \\ &= \frac{t^{0.2}}{10\Gamma(0.2)}(x-y)(x+y) - \frac{t^{0.4}}{2\Gamma(0.4)}(x-y) \\ &\ge -\frac{1}{2\Gamma(0.4)}(x-y). \end{aligned} \tag{4.4}$$

Then let $N = 1/2\Gamma(0.4) \approx 0.2254$. We get

$$0 < \frac{N}{\Gamma(0.8+1)} + \frac{|-0.4|}{\Gamma(0.8-0.6+1)} \approx 0.6777 < 1. \tag{4.5}$$

Thus, from Theorem 3.3, there exist monotone sequences $\{v_n\}$ and $\{w_n\}$ such that $v_n \to \rho$, $w_n \to r$ as $n \to \infty$ uniformly on $[0,1]$, where ρ and r are minimal and maximal solutions of initial value problem (4.1), respectively.

In addition,

$$\begin{aligned} f(t,x) - f(t,y) &= \frac{x^2 t^{0.2}}{10\Gamma(0.2)} - \frac{x t^{0.4}}{2\Gamma(0.4)} - \frac{y^2 t^{0.2}}{10\Gamma(0.2)} + \frac{y t^{0.4}}{2\Gamma(0.4)} \\ &= \frac{t^{0.2}}{10\Gamma(0.2)}(x-y)(x+y) - \frac{t^{0.4}}{2\Gamma(0.4)}(x-y) \\ &\le \frac{10}{10\Gamma(0.2)}(x-y) \\ &\le N(x-y), \\ 0 < N &< \frac{1}{\Gamma(1-0.8)} - \frac{-0.4}{\Gamma(1-0.6)} \approx 0.3982. \end{aligned} \tag{4.6}$$

Hence, by Theorem 3.4, initial value problem (4.1) has a unique solution.

5. Conclusion

In this paper, we considered the initial value problem of nonlinear fractional differential equation

$$\begin{aligned} L(D)u &= f(t,u), \quad t \in (0,1], \\ u(0) &= 0. \end{aligned} \tag{5.1}$$

The basic theory of differential equation, the method of upper and lower solutions, and monotone iterative technique have been applied for the existence and uniqueness of solutions of the initial value problem. And several results were obtained. Besides, we studied the existence of minimal and maximal solutions. In Section 4, we also give an example to illustrate our results.

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