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Research Article

Mean Square Consensus for Uncertain Multiagent Systems with Noises and Delays

Yuangong Sun

School of Mathematical Sciences, University of Jinan, Shandong, Jinan 250022, China

Correspondence should be addressed to Yuangong Sun, sunyuangong@yahoo.cn

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This paper investigates the consensus problem in mean square for uncertain multiagent systems with stochastic measurement noises and symmetric or asymmetric time-varying delays. By combining the tools of stochastic analysis, algebraic graph theory, and matrix theory, we analyze the convergence of a class of distributed stochastic approximation type protocols with time-varying consensus gains. Numerical examples are also given to illustrate the theoretical results.

1. Introduction

In recent years, more and more researchers in the control community have focused their attention on distributed coordination of multiagent systems due to its broad applications in many fields such as unmanned aerial vehicles, mobile robots, autonomous underwater vehicles, automated highway systems, and formation control of satellite clusters.

In the cooperative control, a key problem is to design distributed protocols such that group of agents can achieve consensus through local communications. So far, many consensus results have been established for both discrete-time and continuous-time multiagent systems [1–9]. A simple but interesting model of multiple agents moving in the plane was proposed and discussed in [3]. A theoretical framework for consensus problems of continuous-time multi-agent systems was presented in [5]. Some recent progress on consensus of multi-agent systems was given in [6, 7]. When there exist time-varying delays between agents, a reduced-order system approach is used to consensus of multi-agent systems in [8, 9].

For most of consensus results in the literature, it is usually assumed that each agent can obtain its neighbor's information precisely. Since real networks are often in uncertain communication environments, it is necessary to consider consensus problems under

measurement noises. Such consensus problems have been studied by several researchers [10–15]. In [10, 11], the authors studied consensus problems when there exist noisy measurements of the states of neighbors, and a stochastic approximation approach was applied to obtain mean square and almost sure convergence in models with fixed network topologies or with independent communications failures. Necessary and/or sufficient conditions for stochastic consensus of multiagent systems were established for the case of fixed topology and time-varying topologies in [12, 13]. The distributed consensus problem for linear discrete-time multiagent systems with delays and noises was investigated in [14] by introducing a novel technique to overcome the difficulties induced by the delays and noises. In [15], a novel kind of cluster consensus of multiagents systems with several different subgroups was considered based on Markov chains and nonnegative matrix analysis.

Generally speaking, multiagent systems usually can be regarded as a special kind of complex networks. Complex networks have been intensively investigated over the last two decades [16–18]. Note that measurement noises, time delays, and parametric uncertainties may arise naturally in the process of information transmission between agents, for example, because of the congestion of the communication channels, the asymmetry of interactions, and the finite transmission speed due to the physical characteristics of the medium transmitting the information. Then, it is natural to consider the effect of measurement noises, time-varying delays, and parametric uncertainties on consensus problem of multi-agent systems.

To the best of our knowledge, little has been known about the consensus of uncertain multi-agent systems with measurement noises and time-varying delays. In [19], the authors proposed an algorithm which is robust against the bounded time-varying delays and bounded noises. It is natural to conjecture that consensus of multi-agent systems should also be robust to uncertainties. However, it leads to difficulties due to the existence of measurement noises, parametric uncertainties, and symmetric or asymmetric time-varying delays, since most of methods in the literature fail to apply.

In this paper, by taking measurement noises, symmetric or asymmetric time-varying delays, and parametric uncertainties into consideration, we will study the consensus problem for networks of continuous-time integrator agents under dynamically changing and directed topologies. Based on a reduced-order transformation and a new Lyapunov function, we establish two sufficient conditions in terms of linear matrix inequalities such that mean square consensus is achieved asymptotically for all admissible delays and uncertainties. The feasibility of the given linear matrix inequalities is also analyzed.

Throughout this paper, A^T means the transpose of the matrix A. We say that $X > Y$ if $X - Y$ is positive definite, where X and Y are symmetric matrices of same dimensions. $\|\cdot\|$ refers to the Euclidean norm for vectors. $\mathbf{a} = [a \cdots a]^T$ is a column vector of appropriate dimension, where a is a constant. I means an identity matrix of appropriate dimension.

2. Preliminaries

We denote a weighted digraph by $G = (V, E, A)$, where $V = \{1, 2, \ldots, n\}$ is the set of nodes with $n \geq 2$, node i represents the ith agent; $E \subseteq V \times V$ is the set of edges, and an edge of G is denoted by an order pair (i, j); $A = [a_{ij}]$ is an $n \times n$-dimensional weighted adjacency matrix with $a_{ii} = 0$. Say $(i, j) \in E$ if $a_{ji} > 0$. The set of neighbors of the ith agent is denoted by $N_i = \{j \in V : (j, i) \in E\}$. If (i, j) is an edge of G, node i is called the parent of node j. A directed tree is a directed graph, where every node, except one special node without any parent, which is called the root, has exactly one parent, and the root can be connected to any

other nodes through paths. A spanning tree of a digraph is a directed tree formed by graph edges that connect all the nodes of the graph.

The $n \times n$-dimensional Laplacian matrix $L(G) = [l_{ij}]$ of digraph G is defined by $l_{ij} = \sum_{k=1}^{n} a_{ik}$ for $i = j$ and $l_{ij} = -a_{ij}$ for $i \neq j$. It is easy to see that $L(G)$ has at least one zero eigenvalue and $L(G)\mathbf{1} = 0$. Below is an important property of Laplacian matrices shown in [9].

$G = (V, E, A)$ has a spanning tree if and only if the matrix $EL(G)F$ is Hurwitz stable, where

$$E = [\mathbf{1} - I_{n-1}], \qquad F = \begin{bmatrix} \mathbf{0}^T \\ I_{n-1} \end{bmatrix}. \tag{2.1}$$

Consider a network of continuous-time first-order integrator agents with the dynamics

$$\dot{x}_i(t) = u_i(t), \quad i \in V, \tag{2.2}$$

where $x_i \in R$ is the state of the ith agent, $u_i \in R$ is the control input. When only taking measurement noises into consideration, the control input u_i (or protocol) is designed to take the form [12]:

$$u_i(t) = \sum_{j \in N_i} a(t) a_{ij} \left[y_{ij}(t) - x_i(t) \right], \quad i \in V, \tag{2.3}$$

where the consensus-gain function $a(t) : [0, \infty) \to (0, \infty)$ is piecewise continuous; $y_{ij}(t) = x_j(t) + \sigma_{ij}\eta_{ij}(t)$ denotes the measurement of the jth agent's state $x_j(t)$ by the ith agent; $\{\eta_{ij}(t) : i, j \in V\}$ are independent standard white noises; $\sigma_{ij} \geq 0$ is the noise intensity; $\sigma_{ij} = 0$ for $j \notin N_i$. It has been shown that the consensus-gain function plays a key role in the convergence analysis of the designed protocol.

Note that time delays and parametric uncertainties may arise naturally in the process of information transmission between agents. We consider the following protocol of the form:

$$u_i(t) = \sum_{j \in N_i} a(t) \left(a_{ij} + \Delta a_{ij}(t) \right) \left[z_{ij}(t) - x_i\left(t - \tau_{ij}(t)\right) \right], \tag{2.4}$$

where $z_{ij}(t) = x_j(t - \tau_{ij}(t)) + \sigma_{ij}\eta_{ij}(t)$; time delays $\tau_{ij}(t) : [0, \infty) \to [0, \infty)$, $i \in V$, $j \in N_i$, are piecewise continuous and bounded functions; $a(t)$ and σ_{ij} are defined as above; $\Delta a_{ij}(t)$, $i, j \in V$ are parametric uncertainties satisfying $|\Delta a_{ij}(t)| \leq \epsilon_{ij}$ ($\epsilon_{ij} > 0$) and $\Delta a_{ii}(t) = 0$.

For the sake of convenience, let $\tau_{ij}(t) \in \{\tau_k(t) : k = 1, 2, \ldots, m\}$. Denote the ith row of the matrix $A(t) = A + \Delta A(t)$ by $\alpha_i(t)$,

$$\begin{aligned} \Sigma_i &= \operatorname{diag}\{\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{in}\}, \\ \Sigma_t &= \operatorname{diag}\{\alpha_1(t)\Sigma_1, \alpha_2(t)\Sigma_2, \ldots, \alpha_n(t)\Sigma_n\}, \\ \eta_i(t) &= \left[\eta_{i1}(t), \eta_{i2}(t), \ldots, \eta_{in}(t)\right]^T, \\ \eta(t) &= \left[\eta_1^T(t), \eta_2^T(t), \ldots, \eta_n^T(t)\right]^T, \end{aligned} \tag{2.5}$$

and $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$. Substituting the control (2.4) into the system (2.1) leads to

$$\dot{x}(t) = -a(t)\sum_{k=1}^{m} L_k(t)x(t - \tau_k(t)) + a(t)\Sigma_t\eta(t), \tag{2.6}$$

where the $n \times n$-dimensional matrix $L_k(t) = [l_{ij}^{(k)}(t)]$ is defined as followings:

$$l_{ij}^{(k)}(t) = \begin{cases} -a_{ij} - \Delta a_{ij}(t), & j \neq i,\ \tau_k(\cdot) = \tau_{ij}(\cdot) \\ 0, & j \neq i,\ \tau_k(\cdot) \neq \tau_{ij}(\cdot) \\ -\displaystyle\sum_{p=1}^{n} l_{ip}^{(k)}(t), & j = i. \end{cases} \tag{2.7}$$

It is easy to see

$$L_k(t)\mathbf{1} = 0, \qquad \sum_{k=1}^{m} L_k(t) = L(G) + \Delta L(t), \tag{2.8}$$

where $\Delta L(t) = [\Delta l_{ij}(t)]$ with $\Delta l_{ij}(t) = \sum_{k=1}^{n} \Delta a_{ik}(t)$ for $i = j$ and $\Delta l_{ij}(t) = -\Delta a_{ij}(t)$ for $i \neq j$.

Let $y = Ex$, E, and F be defined by (2.1). From (2.6), we have the following reduced-order system:

$$\dot{y}(t) = a(t)\sum_{k=1}^{m} EL_k(t)Fy(t - \tau_k(t)) + a(t)E\Sigma_t\eta(t). \tag{2.9}$$

It is a system driven by an n^2-dimensional standard white noise, which can be written in the form of the Itô stochastic differential equation

$$dy(t) = a(t)\sum_{k=1}^{m} EL_k(t)Fy(t - \tau_k(t))dt + a(t)E\Sigma_t dw(t), \tag{2.10}$$

where $w(t) = [w_{11}(t), \ldots, w_{1n}(t), \ldots, w_{nn}(t)]^T$ is an n^2-dimensional standard Brownian motion. Without loss of generality, we let $\tau_k(t) \equiv \tau(t)$ for $k = 1, 2, \ldots, m$ since the case of multiple delays can be similarly studied. In this case, the protocol (2.4) takes the following simple form:

$$u_i(t) = \sum_{j \in N_i} a(t)\left(a_{ij} + \Delta a_{ij}(t)\right)\left[y_{ij}(t - \tau(t)) - x_i(t - \tau(t))\right], \tag{2.11}$$

and the system (2.10) reduces to

$$dy(t) = a(t)E(L + \Delta L(t))Fy(t - \tau(t))dt + a(t)E\Sigma_t dw(t). \tag{2.12}$$

In the sequel, we assume that the parametric uncertainty $\Delta L(t)$ to be of the form

$$\Delta L(t) = DG(t)H, \tag{2.13}$$

where D and H are constant matrices with appropriate dimensions, and $G(t)$ is an unknown matrix satisfying $G^T(t)G(t) \leq I$. Suppose also that there exists at least one edge $(j, i) \in E$ such that $\sigma_{ij} > 0$, and $\tau(t)$ is a piecewise continuous function on $[0, \infty)$ and $0 \leq \tau(t) \leq h$, where $h > 0$ is a constant. Here, the initial function of the system (2.12) is assumed to satisfy $\phi(t) \equiv x(0) = [x_1(0) \cdots x_n(0)]^T$ on $[-h, 0]$.

We say the system (2.2) under protocol (2.11) asymptotically achieves mean square consensus if $\lim_{t\to\infty} E[(x_i(t) - x_j(t))^2] = 0$ for all $i, j \in V$ and $i \neq j$.

3. Convergence Analysis of Protocol (2.11)

Before establishing the main result of this paper, we first show the relation between a linear matrix inequality and the collectivity of graph G, which can be used to analyze the feasibility of the given consensus condition.

Lemma 3.1. *If G has a spanning tree, then there exist matrices $P > 0$, $Q > 0$, P_1, P_2 of compatible dimensions, constants $h > 0$, and $\gamma > 0$ such that*

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & P_1^T\tilde{L} & P_1^T\tilde{L} \\ * & \Phi_{22} & P_2^T\tilde{L} & P_2^T\tilde{L} \\ * & * & -h^{-1}Q & 0 \\ * & * & * & -\gamma I_n \end{bmatrix} < 0, \tag{3.1}$$

where $\Phi_{11} = \tilde{L}^T P_1 + P_1^T\tilde{L}$, $\Phi_{12} = P - P_1^T + \tilde{L}^T P_2$, $\Phi_{22} = -P_2^T - P_2 + hQ$, and $\tilde{L} = EL(G)F$, E, and F is defined by (2.1).

Proof. By Lemma 3.1, we have that there exists a matrix $P > 0$ such that $\widetilde{L}^T P + P\widetilde{L} < 0$ if G has a spanning tree. Based on the Finsler Lemma [20], there exist matrices P_1 and P_2 such that

$$\begin{bmatrix} \widetilde{L}^T P_1 + P_1^T \widetilde{L} & P - P_1^T + \widetilde{L}^T P_2 \\ * & -P_2^T - P_2 \end{bmatrix} < 0. \tag{3.2}$$

Let $Q = I_n$. Then, (3.2) implies (3.1) by choosing h and γ sufficiently small. □

The following two lemmas will be used in the proof of the main result.

Lemma 3.2 (see [21]). *For any continuous vector $z(t)$ on $[t-\tau(t), t]$ and matrix $W > 0$, where $t \in [0, \infty)$ and $0 \le \tau(t) \le h$, the following inequality holds:*

$$\left(\int_{t-\tau(t)}^{t} z(u)du\right)^T W \int_{t-\tau(t)}^{t} z(u)du \le h \int_{t-\tau(t)}^{t} z^T(u) W z(u) du. \tag{3.3}$$

Lemma 3.3 (see [22]). *Let U, V, and F be real matrices of appropriate dimensions with $F^T F \le I$, then for any scalar $\epsilon > 0$, one has $UFV + V^T F^T U^T \le \epsilon^{-1} U U^T + \epsilon V^T V$.*

Now, let us present the main result of this paper. We assume that the positive consensus-gain function $a(t)$ satisfies one of the following assumptions:

(A1) $\int_0^\infty a(t)dt = +\infty$, $\int_0^\infty a^2(t)dt < \infty$, and $a(t) \le 1$ for sufficiently large t;

(A2) $\int_0^\infty a(t)dt = +\infty$, $\lim_{t\to\infty} a(t) = 0$.

Theorem 3.4. *Assume that (A1) or (A2) holds and $0 \le \tau(t) \le h$. If there exist matrices $P > 0$, $Q > 0$, P_1, P_2 of compatible dimensions, constants $\gamma > 0$ and $\epsilon > 0$ such that*

$$\begin{bmatrix} \widetilde{\Phi}_{11} & \widetilde{\Phi}_{12} & \widetilde{\Phi}_{13} & \widetilde{\Phi}_{14} & P_1^T \widetilde{D} \\ * & \widetilde{\Phi}_{22} & P_2^T \widetilde{L} & P_2^T \widetilde{L} & P_2^T \widetilde{D} \\ * & * & \widetilde{\Phi}_{33} & \widetilde{\Phi}_{34} & 0 \\ * & * & * & \widetilde{\Phi}_{44} & 0 \\ * & * & * & * & -\epsilon I \end{bmatrix} < 0, \tag{3.4}$$

where $\widetilde{\Phi}_{11} = \Phi_{11} + \epsilon F^T H^T H F$, $\widetilde{\Phi}_{12} = \Phi_{12}$, $\widetilde{\Phi}_{13} = P_1^T \widetilde{L} + \epsilon F^T H^T H F$, $\widetilde{\Phi}_{14} = P_1^T \widetilde{L} + \epsilon F^T H^T H F$, $\widetilde{\Phi}_{22} = \Phi_{22}$, $\widetilde{\Phi}_{33} = -h^{-1} Q + \epsilon F^T H^T H F$, $\widetilde{\Phi}_{34} = \epsilon F^T H^T H F$, $\widetilde{\Phi}_{44} = -\gamma I + \epsilon F^T H^T H F$, Φ_{11}, Φ_{12}, and Φ_{22} are defined as in Lemma 3.1, E and F is defined by (2.1), *and $\widetilde{D} = ED$, then the system* (2.2) *under protocol* (2.11) *asymptotically achieves mean square consensus for all admissible uncertainties satisfying* (2.13).

Proof. For the reduced order system (2.12), let

$$\xi(t) = E(L + \Delta L(t)) F y(t - \tau(t)) = \widetilde{L}_t y(t - \tau(t)). \tag{3.5}$$

Then, (2.12) reduces to

$$dy(t) = a(t)\xi(t)dt + a(t)E\Sigma dw(t). \tag{3.6}$$

Note that (3.4) holds. We can choose a constant $\lambda > 0$ sufficiently small such that

$$\begin{bmatrix} \widetilde{\Phi}_{11} + \lambda P & \widetilde{\Phi}_{12} & P_1^T\widetilde{L} & \widetilde{\Phi}_{14} & P_1^T\widetilde{D} \\ * & \widetilde{\Phi}_{22} & P_2^T\widetilde{L} & P_2^T\widetilde{L} & P_2^T\widetilde{D} \\ * & * & -\widetilde{h}^{-1}Q & 0 & 0 \\ * & * & * & \widetilde{\Phi}_{44} & 0 \\ * & * & * & * & -\epsilon I \end{bmatrix} < 0, \tag{3.7}$$

where $\widetilde{h} = e^{-\lambda h}h^{-1}$.

Let

$$\begin{aligned} V(t) = {} & y^T(t)Py(t) + \int_{t-h}^{t} e^{-\lambda\int_s^t a(u)du}(s-t+h)a^2(s)\xi^T(s)Q\xi(s)ds \\ & + \beta\int_{t-h}^{t} e^{-\lambda\int_s^t a(u)du}(s-t+h)a^2(s)ds, \end{aligned} \tag{3.8}$$

where $\beta > 0$ is a constant to be determined. By the Itô formula, we have

$$\begin{aligned} dV(t) = {} & \left[2a(t)y^T(t)P\xi(t) + c(t)a^2(t)\right]dt + 2a(t)y^T(t)PE\Sigma_t dw(t) \\ & + \left[ha^2(t)\xi^T(t)Q\xi(t) - \int_{t-h}^{t} e^{-\lambda\int_s^t a(u)du}a^2(s)\xi^T(s)Q\xi(s)ds \right. \\ & - \lambda a(t)\int_{t-h}^{t} e^{-\lambda\int_s^t a(u)du}(s-t+h)a^2(s)\xi^T(s)Q\xi(s)ds\,dt \\ & \left. -\beta\int_{t-h}^{t} e^{-\lambda\int_s^t a(u)du}a^2(s)ds - \lambda\beta a(t)\int_{t-h}^{t} e^{-\lambda\int_s^t a(u)du}(s-t+h)a^2(s)ds\right]dt, \end{aligned} \tag{3.9}$$

where $c(t) = \mathrm{tr}(\Sigma_t^T E^T PE\Sigma_t) + \beta h$. It is not difficult to see that there exists a scalar $m_1 > 0$ such that $c(t) \le m_1$ due to the fact that $|\Delta a_{ij}(t)| \le \epsilon_{ij}$. Thus,

$$dV(t) + \lambda a(t)V(t)dt = LV(t)dt + 2a(t)y^T(t)PE\Sigma_t dw(t), \tag{3.10}$$

where

$$\begin{aligned} LV(t) &= \lambda a(t) y^T(t) P y(t) + 2a(t) y^T(t) P \xi(t) + c(t) a^2(t) \\ &\quad + h a^2(t) \xi^T(t) Q \xi(t) - \int_{t-h}^{t} e^{-\lambda \int_s^t a(u)du} a^2(s) \xi^T(s) Q \xi(s) ds \\ &\quad - \beta \int_{t-h}^{t} e^{-\lambda \int_s^t a(u)du} a^2(s) ds. \end{aligned} \tag{3.11}$$

Without loss of generality, say $a(t) \leq 1$ for $t \geq 0$. Then, we have

$$\begin{aligned} LV(t) &\leq \lambda a(t) y^T(t) P y(t) + 2a(t) y^T(t) P \xi(t) + m_1 a^2(t) \\ &\quad + h a(t) \xi^T(t) Q \xi(t) - e^{-\lambda h} a(t) \int_{t-\tau(t)}^{t} a^2(s) \xi^T(s) Q \xi(s) ds - \beta e^{-\lambda h} \int_{t-\tau(t)}^{t} a^2(s) ds. \end{aligned} \tag{3.12}$$

On the other hand, integrating (3.7) from $t - \tau(t)$ to t yields

$$y(t) - y(t - \tau(t)) = \int_{t-\tau(t)}^{t} a(s) \xi(s) ds + \int_{t-\tau(t)}^{t} a(s) E \Sigma_s dw(s). \tag{3.13}$$

Therefore, by the definition of $\xi(t)$, we have

$$\begin{aligned} 0 &= \widetilde{L}_t y(t - \tau(t)) - \xi(t) \\ &= \widetilde{L}_t y(t) - \xi(t) - \widetilde{L}_t \int_{t-\tau(t)}^{t} a(s) \xi(s) ds \\ &\quad - \widetilde{L}_t \int_{t-\tau(t)}^{t} a(s) E \Sigma_s dw(s), \end{aligned} \tag{3.14}$$

which implies that

$$\begin{aligned} 0 &= 2a(t) \left[y^T(t) P_1^T + \xi^T(t) P_2^T \right] \\ &\quad \times \left[\widetilde{L}_t \left(y(t) - \int_{t-\tau(t)}^{t} a(s) \xi(s) ds - \int_{t-\tau(t)}^{t} a(s) E \Sigma dw(s) \right) - \xi(t) \right]. \end{aligned} \tag{3.15}$$

By Lemma 3.2, we have

$$\int_{t-\tau(t)}^{t} a^2(s) \xi^T(s) Q \xi(s) ds \geq h^{-1} \left(\int_{t-\tau(t)}^{t} a(s) \xi(s) ds \right)^T Q \int_{t-\tau(t)}^{t} a(s) \xi(s) ds. \tag{3.16}$$

Using the basic inequality $-2u^T v \le \gamma u^T u + \gamma^{-1} v^T v$ for any vector u and v, we have

$$\begin{aligned}
&-2\left[y^T(t)P_1^T + \xi^T(t)P_2^T\right]\widetilde{L}_t \int_{t-\tau(t)}^{t} a(s)E\Sigma_s dw(s) \\
&\quad = -2\left[y^T(t) \quad \xi^T(t)\right] \begin{bmatrix} P_1^T \\ P_2^T \end{bmatrix} \widetilde{L}_t \int_{t-\tau(t)}^{t} a(s)E\Sigma_s dw(s) \\
&\quad \le \gamma^{-1}\left[y^T(t) \quad \xi^T(t)\right] \begin{bmatrix} P_1^T \\ P_2^T \end{bmatrix} \widetilde{L}_t \widetilde{L}_t^T \left[P_1 \quad P_2\right] \begin{bmatrix} y(t) \\ \xi(t) \end{bmatrix} \\
&\qquad + \gamma \left(\int_{t-\tau(t)}^{t} a(s)E\Sigma_s dw(s) \right)^T \int_{t-\tau(t)}^{t} a(s)E\Sigma_s dw(s).
\end{aligned} \tag{3.17}$$

Substituting (3.15)–(3.17) into (3.12) gives

$$\begin{aligned}
LV(t) &\le a(t)\omega^T(t)\Omega(t)\omega(t) + m_1 a^2(t) - \beta e^{-\lambda h} \int_{t-\tau(t)}^{t} a^2(s)ds \\
&\quad + \gamma \left(\int_{t-\tau(t)}^{t} a(s)E\Sigma dw(s) \right)^T \int_{t-\tau(t)}^{t} a(s)E\Sigma dw(s),
\end{aligned} \tag{3.18}$$

where $\omega(t) = [y^T(t) \ \xi^T(t) \ \ -\int_{t-\tau(t)}^{t} a(s)\xi^T(s)ds]^T$, and

$$\Omega(t) = \begin{bmatrix} \Omega_{11} & \Omega_{12} & P_1^T \widetilde{L}_t \\ * & \Omega_{22} & P_2^T \widetilde{L}_t \\ * & * & -\widetilde{h}^{-1} Q \end{bmatrix} + \gamma^{-1} \begin{bmatrix} P_1^T \\ P_2^T \end{bmatrix} \widetilde{L}_t \widetilde{L}_t^T \left[P_1 \quad P_2\right], \tag{3.19}$$

with $\Omega_{11} = \widetilde{L}_t^T P_1 + P_1^T \widetilde{L}_t + \lambda P$, $\Omega_{12} = P - P_1^T + \widetilde{L}_t^T P_2$, $\Omega_{22} = \Phi_{22}$. It is easy to see that $\Omega(t) < 0$ if and only if

$$\Phi + \begin{bmatrix} P_1^T \widetilde{D} \\ P_2^T \widetilde{D} \\ 0 \\ 0 \end{bmatrix} G(t) \left[\widetilde{H} \quad 0 \quad \widetilde{H} \quad \widetilde{H}\right] + \begin{bmatrix} \widetilde{H}^T \\ 0 \\ \widetilde{H}^T \\ \widetilde{H}^T \end{bmatrix} G(t) \left[\widetilde{D}^T \quad P_1 \quad \widetilde{D}^T \quad P_2 \quad 0 \quad 0\right] < 0, \tag{3.20}$$

where Φ is defined by (3.1) and $\widetilde{H} = HF$. By Lemma 3.3, we have that (3.20) is implied by (3.7). Thus, (3.7) yields that $\Omega(t) < 0$ for $t \ge 0$.

Note that $\Omega(t) < 0$ for $t \geq 0$, and

$$E\left[\left(\int_{t-\tau(t)}^{t} a(s)E\Sigma_s dw(s)\right)^T \int_{t-\tau(t)}^{t} a(s)E\Sigma_s dw(s)\right] = \int_{t-\tau(t)}^{t} d(s)a^2(s)ds, \tag{3.21}$$

where $d(t) = \mathrm{tr}(\Sigma_t^T E^T E\Sigma_t)$ satisfying $d(t) \leq m_2$ for some $m_2 > 0$. By choosing $\beta = \gamma m_2 e^{\lambda h}$, we get from (3.10) and (3.18) that

$$E[V(t)] \leq E[V(0)] - \lambda \int_0^t a(s)E[V(s)]ds + m_1 \int_0^t a^2(s)ds. \tag{3.22}$$

By the comparison theorem [23], we have

$$E[V(t)] \leq E[V(0)]e^{-\lambda \int_0^t a(s)ds} + m_1 \int_0^t e^{-\lambda \int_s^t a(u)du} a^2(s)ds. \tag{3.23}$$

If (A1) holds, that is, $\int_0^\infty a(s)ds = \infty$ and $\int_0^\infty a^2(s)ds < \infty$, then, similar to the proof of Theorem 3.2 in [12], we can conclude from (3.23) that $\lim_{t\to\infty} E[V(t)] = 0$. If (A2) holds, by the L'Hopital rule, we have

$$\lim_{t\to\infty} \int_0^t e^{-\lambda \int_s^t a(u)du} a^2(s)ds = \lim_{t\to\infty} \frac{a(t)}{\lambda} = 0, \tag{3.24}$$

which also implies $\lim_{t\to\infty} E[V(t)] = 0$. Based on the construction of $V(t)$ and the transformation $y = Ex$, we have

$$\lim_{t\to\infty} E\left[(x_1(t) - x_{i+1}(t))^2\right] = \lim_{t\to\infty} E\left[y_i^2(t)\right] = 0, \quad i = 1, 2, \ldots, n-1. \tag{3.25}$$

It implies that $\lim_{t\to\infty} E[(x_i(t) - x_j(t))^2] = 0$ for $i, j \in V$ and $i \neq j$. The proof is complete. □

Remark 3.5. By Lemma 3.1, we can easily see that (3.4) holds for appropriate constant $h > 0$ and admissible uncertainties (e.g., $\|H\|$ is sufficiently small) if G has a spanning tree. Therefore, Theorem 3.4 shows that mean square consensus of the system (2.2) under protocol (2.11) is robust to delays and uncertainties if (A1) or (A2) holds. For given matrices D and H, the tolerable upper bound of delay can be derived from (3.4) by using the Matlab's LMI Toolbox.

Remark 3.6. For the case of multiple delays, it is not difficult to conclude that the system (2.2) under protocol (2.4) asymptotically achieves mean square consensus for admissible delays and parametric uncertainties if G has a spanning tree and (A1) or (A2) holds. Since the analysis procedure is similar to the above, we omit it here and leave it to the interested readers.

4. Convergence Analysis for the Case of Asymmetric Delays

The method used in this paper can also be applied to the case when delay only affects the state of neighbors. Assume that there exists at least one agent such that the information exchange between this agent and its neighbors is free of delay, stochastic noises, and parametric uncertainties. For example, among agents there exists a leader x_{i_0} satisfying $\dot{x}_{i_0} = 0$. Without loss of generality, consider the following protocol:

$$\begin{aligned} u_1(t) &= \sum_{j\in N_1} a(t)a_{1j}\left[x_j(t) - x_1(t)\right], \\ u_i(t) &= \sum_{j\in N_i} a(t)\left(a_{ij} + \Delta a_{ij}(t)\right)\left[\tilde{z}_{ij}(t) - x_i(t)\right], \quad i = 2,\ldots,n, \end{aligned} \tag{4.1}$$

where $\tilde{z}_{ij}(t) = x_j(t-\tau(t)) + \sigma_{ij}\eta_{ij}$, τ, a, a_{ij}, Δa_{ij}, σ_{ij} and η_{ij} are defined as above. The case of multiple time-varying delays can be similarly discussed.

Note that

$$\begin{aligned} x_j(t-\tau(t)) - x_i(t) &= \left[x_j(t-\tau(t)) - x_1(t-\tau(t))\right] + \left[x_1(t) - x_i(t)\right] \\ &\quad - \left[x_1(t) - x_1(t-\tau(t))\right], \quad i,j = 2,3,\ldots,n, \\ x_1(t) - x_1(t-\tau(t)) &= \int_{t-\tau(t)}^{t} \dot{x}_1(s)ds. \end{aligned} \tag{4.2}$$

If we set $y = Ex$, where E is defined as above, then we have the following reduced-order system:

$$\dot{y}(t) = a(t)\left[L_1(t)y(t) + L_2(t)y(t-\tau(t)) + L_3(t)\int_{t-\tau(t)}^{t} a(s)y(s)ds\right] + a(t)\hat{\Sigma}_t\hat{\eta}(t), \tag{4.3}$$

where $L_i(t) = L_i + \Delta L_i(t)$ for $i = 1,2,3$,

$$\begin{aligned} L_1 &= -\operatorname{diag}\left\{\sum_{j=1}^{n} a_{2j}, \sum_{j=1}^{n} a_{3j}, \ldots, \sum_{j=1}^{n} a_{nj}\right\} - \mathbf{1}\left[a_{12} \quad a_{13} \quad \cdots \quad a_{1n}\right], \\ \Delta L_1(t) &= -\operatorname{diag}\left\{\sum_{j=1}^{n} \Delta a_{2j}(t), \sum_{j=1}^{n} \Delta a_{3j}(t), \ldots, \sum_{j=1}^{n} \Delta a_{nj}(t)\right\}, \end{aligned}$$

$$L_2 = \begin{bmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix},$$

$$\Delta L_2(t) = \begin{bmatrix} \Delta a_{22}(t) & \Delta a_{23}(t) & \cdots & \Delta a_{2n}(t) \\ \Delta a_{32}(t) & \Delta a_{33}(t) & \cdots & \Delta a_{3n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta a_{n2}(t) & \Delta a_{n3}(t) & \cdots & \Delta a_{nn}(t) \end{bmatrix},$$

$$L_3 = -\operatorname{diag}\left\{\sum_{j=1}^{n} a_{2j}, \sum_{j=1}^{n} a_{3j}, \ldots, \sum_{j=1}^{n} a_{nj}\right\} \mathbf{1}\begin{bmatrix} a_{12} & a_{13} & \cdots & a_{1n} \end{bmatrix},$$

$$\Delta L_3(t) = -\operatorname{diag}\left\{\sum_{j=1}^{n} \Delta a_{2j}(t), \sum_{j=1}^{n} \Delta a_{3j}(t), \ldots, \sum_{j=1}^{n} \Delta a_{nj}(t)\right\} \mathbf{1}\begin{bmatrix} a_{12} & a_{13} & \cdots & a_{1n} \end{bmatrix}, \tag{4.4}$$

$\widehat{\Sigma}_t = -\operatorname{diag}\{\alpha_2(t)\Sigma_2, \ldots, \alpha_n(t)\Sigma_n\}$, $\widehat{\eta}(t) = [\eta_2^T(t)\eta_3^T(t)\cdots\eta_n^T(t)]^T$, $\alpha_i(t)$, Σ_i and η_i are defined as above. Therefore, (4.3) can be written in the form of the Itô stochastic differential equation.

$$dy(t) = a(t)\left[L_1(t)y(t) + L_2(t)y(t-\tau(t)) + L_3(t)\int_{t-\tau(t)}^{t} a(s)y(s)ds\right]dt + a(t)\widehat{\Sigma}_t dw(t). \tag{4.5}$$

It is not difficult to verify that $L_1 + L_2 = EL(G)F$. In the following, we assume that the uncertainties $\Delta L_i(t)$ to be of the form

$$\Delta L_1(t) + \Delta L_2(t) = D_1F(t)H_1, \qquad \Delta L_i(t) = D_iF(t)H_i, \quad i = 2,3, \tag{4.6}$$

where D_i, H_i are constant matrices of appropriate dimensions and $F(t)$ satisfies (2.13).

Theorem 4.1. *Assume that (A1) or (A2) holds and $0 \le \tau(t) \le h$. If there exist matrices $P > 0$, $Q > 0$, $S > 0$, P_1, P_2, and P_3 of compatible dimensions, constants $h > 0$, $\gamma > 0$ and $\epsilon_i > 0$ for $i = 1, 2, 3$ such that*

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & P_1^T L_2 & \Psi_{14} & P_1^T L_2 & P_1^T D_1 & P_1^T D_2 & P_1^T D_3 \\ * & \Psi_{22} & P_2^T L_2 & \Psi_{24} & P_2^T L_2 & P_2^T D_1 & P_2^T D_2 & P_2^T D_3 \\ * & * & \Psi_{33} & L_2^T P_3 & \epsilon_2 E_2^T E_2 & 0 & 0 & 0 \\ * & * & * & \Psi_{44} & P_3^T L_2 & P_3^T D_1 & P_3^T D_2 & 0 \\ * & * & * & * & \Psi_{55} & 0 & 0 & 0 \\ * & * & * & * & * & -\epsilon_1 I & 0 & 0 \\ * & * & * & * & * & * & -\epsilon_2 I & 0 \\ * & * & * & * & * & * & * & -\epsilon_3 I \end{bmatrix} < 0, \tag{4.7}$$

where $\Psi_{11} = (L_1 + L_2)^T P_1 + P_1^T (L_1 + L_2) + hS + \epsilon_1 E_1^T E_1$, $\Psi_{12} = P - P_1^T + (L_1 + L_2)^T P_2$, $\Psi_{14} = P_1^T L_3 + (L_1 + L_2)^T P_3$, $\Psi_{22} = -P_2^T - P_2 + hQ$, and $\Psi_{24} = P_2^T L_3 - P_3$, $\Psi_{33} = -h^{-1}Q + \epsilon_2 E_2^T E_2$, $\Psi_{44} = -h^{-1}S + \epsilon E_3^T E_3$, $\Psi_{55} = -\gamma I + \epsilon_2 E_2^T E_2$, then the system (2.2) *under protocol* (4.1) *asymptotically achieves mean square consensus for all admissible uncertainties satisfying* (2.13).

Proof. Let

$$\xi(t) = L_1(t)y(t) + L_2(t)y(t - \tau(t)) + L_3(t)\int_{t-\tau(t)}^{t} a(s)y(s)ds. \tag{4.8}$$

Then, (4.5) reduces to

$$dy(t) = a(t)\xi(t) + a(t)\widehat{\Sigma}_t dw(t). \tag{4.9}$$

Choose the Lyapunov function as the following:

$$\begin{aligned} V(t) = {} & y^T(t)Py(t) + \int_{t-h}^{t} e^{-\lambda\int_s^t a(u)du}(s - t + h)a^2(s)\xi^T(s)Q\xi(s)ds \\ & + \int_{t-h}^{t} e^{-\lambda\int_s^t a(u)du}(s - t + h)a^2(s)y^T(s)Sy(s)ds \\ & \times \beta\int_{t-h}^{t} e^{-\lambda\int_s^t a(u)du}(s - t + h)a^2(s)ds, \end{aligned} \tag{4.10}$$

where $\beta > 0$ is an appropriate constant to be determined. Without loss of generality, say $a(t) \le 1$ for $t \ge 0$. By (4.9) and the Itô formula, we have

$$dV(t) + \lambda a(t)V(t)dt = LV(t)dt + 2a(t)y^T(t)P\widehat{\Sigma}_t dw(t), \tag{4.11}$$

where

$$\begin{aligned} LV(t) \leq{}& \lambda a(t) y^T(t) P y(t) + 2a(t) y^T(t) P \xi(t) + \widetilde{c}(t) a^2(t) \\ &+ h a(t) \xi^T(t) Q \xi(t) + h a(t) y^T(t) S y(t) \\ &- e^{-\lambda h} a(t) \int_{t-\tau(t)}^{t} a^2(s) \xi^T(s) Q \xi(s) ds \\ &- e^{-\lambda h} a(t) \int_{t-\tau(t)}^{t} a^2(s) y^T(s) S y(s) ds \\ &- \beta e^{-\lambda h} \int_{t-\tau(t)}^{t} a^2(s), \end{aligned} \tag{4.12}$$

and $\widetilde{c}(t) = \operatorname{tr}(\widehat{\Sigma}_t^T P \Sigma_t) + \beta h$. On the other hand, by the definition of $\xi(t)$, we have

$$\begin{aligned} 0 &= L_1(t) y(t) + L_2(t) y(t-\tau(t)) + L_3(t) \int_{t-\tau(t)}^{t} a(s) y(s) ds - \xi(t) \\ &= (L_1(t) + L_2(t)) y(t) + L_3(t) \int_{t-\tau(t)}^{t} a(s) y(s) ds - \xi(t) \\ &\quad - L_2(t) \int_{t-\tau(t)}^{t} a(s) \xi(s) ds - L_2(t) \int_{t-\tau(t)}^{t} a(s) \widehat{\Sigma} dw(s). \end{aligned} \tag{4.13}$$

Then, proceeding as in the proof of Theorem 3.4, we can get a desired result. This completes the proof of Theorem 4.1. □

Remark 4.2. By Lemma 3.1, we can also show that (4.7) holds for appropriate constant $h > 0$ and admissible uncertainties if G has a spanning tree. For given matrices D_i and $H_i (i = 1, 2, 3)$, the tolerable upper bound of delay can be derived from (4.7).

5. Simulation Results

Consider a digraph $G = (V, E, A)$ with six nodes and 0-1 weights, where $a_{21} = a_{32} = a_{43} = a_{54} = a_{65} = a_{16} = 1$. It is evident that G has a spanning tree. Let $a(t) = 1/(t+1)$. Using Matlab to solve (3.4) without uncertainties yields that $h \leq 0.4995$. Thus, by Theorem 3.4, we have that protocol (2.11) asymptotically solves mean square consensus for any time-varying delay $\tau(t)$ satisfying $0 \leq \tau(t) \leq 0.4995$ if (A1) or (A2) holds. Let $a(t) = 1/(t+1)$ and the intensity of the measurement noises $\sigma_{21} = \sigma_{32} = \sigma_{43} = \sigma_{54} = \sigma_{16} = 1$. The state trajectories of the system under protocol (2.11) and a random initial state $x(0)$ are shown in Figure 1 when $a(t) = 1/(t+1)$. Figure 2 shows that the system is divergent when $a(t) = 1$.

Consider again the digraph defined above. Solving (4.7) without uncertainties gives $h \leq 0.5315$. By Theorem 4.1, we have that protocol (4.1) asymptotically solves mean square consensus for any time-varying delay $\tau(t)$ satisfying $0 \leq \tau(t) \leq 0.5315$ if (A1) or (A2) holds.

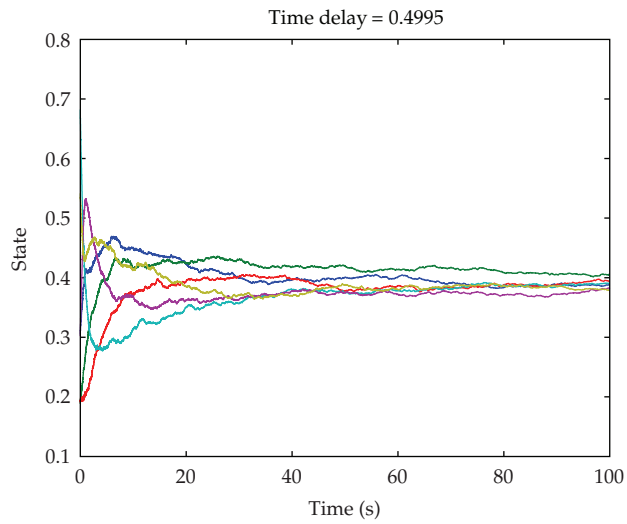

Figure 1: State trajectories under protocol (2.11) and $a(t) = 1/(t+1)$.

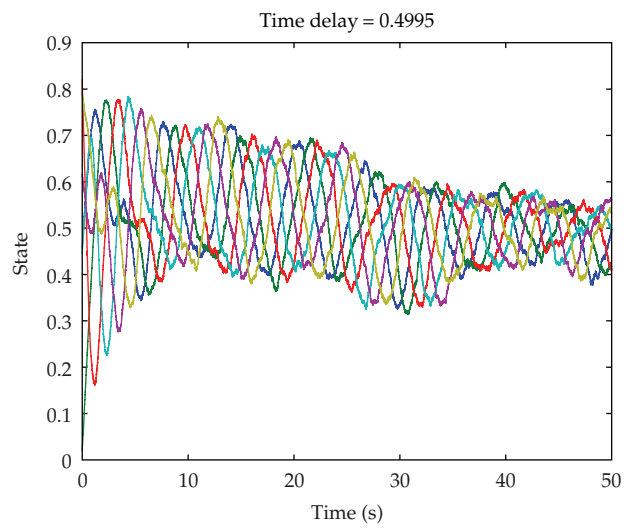

Figure 2: State trajectories under protocol (2.11) and $a(t) = 1$.

Let $a(t) = 1/(t+1)$ and the intensity of the measurement noises $\sigma_{21} = \sigma_{32} = \sigma_{43} = \sigma_{54} = \sigma_{16} = 1$. Under protocol (4.1) and a stochastic initial state $x(0)$, the state trajectories of the system are shown in Figures 3 and 4 for $a(t) = 1/(t+1)$ and $a(t) = 1$, respectively. We see that the system is divergent for the case of $a(t) = 1$.

6. Conclusions

In this paper, we study the mean square consensus problem for continuous-time multi-agent systems with measurement noises, time-varying delays, and parametric uncertainties. By

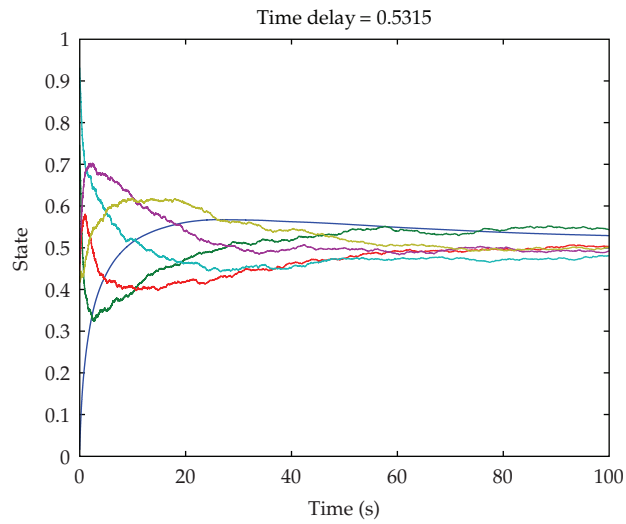

Figure 3: State trajectories under protocol (4.1) and $a(t) = 1/(t+1)$.

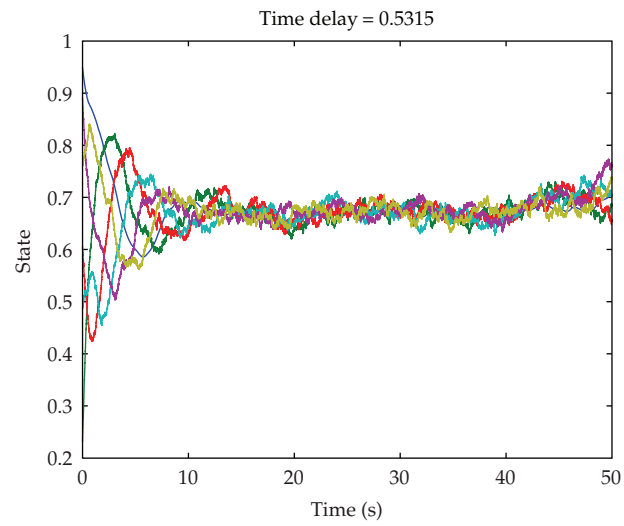

Figure 4: State trajectories under protocol (4.1) and $a(t) = 1$.

introducing a reduced-order transformation and a new Lyapunov function, we combine the tools of stochastic analysis, algebraic graph theory, and matrix theory to analyze the convergence of a class of distributed stochastic approximation type protocols with the time-varying consensus gain. When imposing appropriate conditions on the consensus gain, we show that mean square consensus will be achieved asymptotically for admissible delays and uncertainties if the digraph G has a spanning tree.

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