

## Research Article

# Perturbation Bound of the Group Inverse and the Generalized Schur Complement in Banach Algebra

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We investigate the relative perturbation bound of the group inverse and also consider the perturbation bound of the generalized Schur complement in a Banach algebra.

## 1. Introduction

Let  $\mathcal{A}$  denote a Banach algebra with unit 1. The symbols  $\mathcal{A}^{-1}$ ,  $\mathcal{A}^D$ ,  $\mathcal{A}^d$ ,  $\mathcal{A}_g$ ,  $\mathcal{A}^{\text{nil}}$ ,  $\mathcal{A}^{\text{qnil}}$ , and  $\mathcal{A}^\bullet$  stand for the sets of all invertible, Drazin invertible, generalized Drazin invertible, group invertible, nilpotent, quasinilpotent, and idempotent elements of a Banach algebra  $\mathcal{A}$ , respectively.

Some definitions will be given in the following.

Letting  $a \in \mathcal{A}^D$ , there is an unique element  $x \in \mathcal{A}$  such that

$$a^{k+1}x = a^k, \quad xax = x, \quad ax = xa. \quad (1.1)$$

Then  $x$  is called the Drazin inverse of  $a$ , denoted by  $a^D$ . The smallest nonnegative integer  $k$  which satisfies (1.1) is called the index of  $a$ , denoted by  $\text{Ind}(a) = k$ . If  $\text{Ind}(a) \leq 1$ , then  $a^D = a^\#$  (or  $a^g$ ).

Let  $a \in \mathcal{A}$ , if the conditions (1.1) are replaced by

$$axa = a, \quad xax = x, \quad ax = xa. \quad (1.2)$$

Then  $x$  is called the group inverse of  $a$ , denoted by  $x = a^\#$ . If the conditions (1.1) are replaced by

$$xax = x, \quad ax = xa, \quad a(1 - ax) \text{ is quasinilpotent.} \quad (1.3)$$

Then  $x$  is called the generalized Drazin inverse of  $a$ , denoted by  $x = a^d$ .

Some notations of the Schur complement are given in the following.

For a  $2 \times 2$  block complex matrix  $M$  is defined as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad (1.4)$$

where  $A \in C^{m \times m}$ ,  $D \in C^{p \times p}$ ,  $B \in C^{m \times p}$ , and  $C \in C^{p \times m}$ . If  $A$  is nonsingular, then the classical Schur complement of  $A$  in  $M$  is given as follows (see [1]):

$$S = D - CA^{-1}B. \quad (1.5)$$

In [2], Benítez and Thome considered the expression

$$N = \begin{bmatrix} A^- + A^-BS^-CA^- & -A^-BS^- \\ -S^-CA^- & S^- \end{bmatrix}, \quad (1.6)$$

and  $N$  is called the generalized Schur form of the matrix  $M$  given in (1.4) being  $S = D - CA^-B$  for some fixed generalized inverses  $A^- \in A\{1\}$ ,  $S^- \in S\{1\}$ , where  $S$  is called generalized Schur complement of  $A$  in  $M$ . In [2, Theorem 2], Benítez and Thome investigated the expression of the group inverse of  $M$  in (1.4) by the generalized Schur complement, where (1.5) is replaced by

$$S = D - CA^\#B. \quad (1.7)$$

Similar results also were given by Sheng and Chen in [3, Theorem 3.2]. The Drazin inverse of a  $2 \times 2$  block complex square matrix in (1.4) with a singular generalized Schur complement was considered in [4–6], where

$$S = A - CA^D D. \quad (1.8)$$

For the expression of a  $2 \times 2$  block operator matrix was investigated by Deng and Wei in [7].

Some notations for the block matrix form of a given element  $a \in \mathcal{A}$  are introduced in [8]. Let  $a \in \mathcal{A}$  and  $s \in \mathcal{A}^\bullet$  (see [8, Chapter VII]) which denotes the set of all idempotent elements in  $\mathcal{A}$ . Then we write

$$a = sas + sa(1-s) + (1-s)as + (1-s)a(1-s) \quad (1.9)$$

and use the notations

$$a_{11} = sas, \quad a_{12} = sa(1-s), \quad a_{21} = (1-s)as, \quad a_{22} = (1-s)a(1-s). \quad (1.10)$$

For a representation of arbitrary element  $a \in \mathcal{A}$  is given as the following matrix form:

$$a = \begin{bmatrix} sas & sa(1-s) \\ (1-s)as & (1-s)a(1-s) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_s. \quad (1.11)$$

In this paper, we will consider some results on the relative perturbation bounds of group inverse and also give the perturbation bounds of the generalized Schur complement of an element  $a \in \mathcal{A}$  under some certain conditions in a Banach algebra.

## 2. Perturbation Bound of $\|(a+b)^\# - a^d\|$ in Banach Algebra

In recent years, perturbation theory for the Drazin inverse of a given matrix  $A \in \mathbb{C}^{n \times n}$  and its applications have been considered in [9–20]. In [12], Yimin and Guorong gave a perturbation result for the Drazin inverse under condition  $(\mathcal{W})$  (see [12] for details). In [8, Ch 5], Djordjević and Rakočević extended the perturbation bound of Yimin and Guorong [12] to Banach algebra. In [13], Wei had discussed the upper perturbation bound of  $\|B^\# - A^\#\|/\|A^\#\|$  with  $B = A + E$  and had answered the question of Campbell and Meyer [21] when  $\text{Ind}(A) = 1$ . In [14], Wei and Wu presented the perturbation upper bounds of  $\|B^D - A^D\|/\|A^D\|$  under the weaker condition  $\text{core} - \text{rank } B = \text{core} - \text{rank } A$  and completely answered the question of Campbell and Meyer in [21]. In [16], Wei derived a relative perturbation upper bound of  $\|B^\# - A^D\|/\|A^D\|$  by Jordan canonical of  $A$ . In [5], Li gave sharper upper bounds for  $\|B^\# - A^D\|$  under weaker conditions:  $\text{rank}(B) = \text{rank}(A^k)$  and  $\|A^D\|\|E\| < 1/(1 + \|A^D\|\|A\|)$ . In [17], Wei et al. derived constructive perturbation bound of the Drazin inverse of a square matrix by using a technique proposed by Stewart and based on perturbation theory for invariant subspaces. In [18], Xu et al. gave some upper bounds for  $\|B^D - A^D\|/\|A^D\|$  only under the condition that  $B$  is a stable perturbation of  $A$ . In [22], González and Koliha investigated the perturbation of the Drazin inverse of a closed linear operator and derived explicit bounds for the perturbations under certain restrictions on the perturbing operators. In [23], González and Vélez-Cerrada analyzed the perturbation of the Drazin inverse and also gave explicit upper bounds of  $\|B^\# - A^D\|$  and  $\|BB^\# - AA^D\|$  and obtained a result on the continuity of the group inverse for operators on Banach space.

In this section, we will investigate the relative perturbation bound of the group inverse in Banach algebra.

At first, we will give some concepts and lemmas as follows.

For  $a \in \mathcal{A}^d$ , let  $p = aa^d$  and  $p \in \mathcal{A}^\bullet$  (see [24]):

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p, \quad a^d = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p, \quad a^\pi = 1 - p = \begin{bmatrix} 0 & 0 \\ 0 & 1 - p \end{bmatrix}_p, \quad (2.1)$$

where  $a_1 \in p\mathcal{A}p$  is invertible and  $a_2 \in (1-p)\mathcal{A}(1-p)$  is quasinilpotent.

For any  $a \in \mathcal{A}^d$ , we write  $\sigma(a)$ ,  $\rho(a)$ , and  $r(a)$  for the spectrum, the resolvent set, and the spectral radius of  $a$ , respectively. For  $\lambda \in \rho(a)$  and let  $R(\lambda, a) = (\lambda - a)^{-1}$ . If 0 is an isolated point of  $\sigma(a)$ , then the spectral idempotent corresponding to the set  $\{0\}$  is defined by

$$a^\pi = \frac{1}{2\pi i} \int_\gamma R(\lambda, a) d\lambda, \quad (2.2)$$

where  $\gamma$  is a small circle surrounding 0 and separating 0 from  $\sigma(a) \setminus \{0\}$ .

Some lemmas will be useful for the following proof in this paper.

**Lemma 2.1** (see [24, Theorem 2.3]). *Let  $x, y \in \mathcal{A}$ , and let  $p \in \mathcal{A}^\bullet$ . Assume that*

$$x = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}_p, \quad y = \begin{bmatrix} b & 0 \\ c & a \end{bmatrix}_{1-p}. \quad (2.3)$$

(i) *If  $a \in (p\mathcal{A}p)^d$  and  $b \in ((1-p)\mathcal{A}(1-p))^d$ , then  $x, y \in \mathcal{A}^d$  and*

$$x^d = \begin{bmatrix} a^d & u \\ 0 & b^d \end{bmatrix}_p, \quad y^d = \begin{bmatrix} b^d & 0 \\ u & a^d \end{bmatrix}_{1-p}, \quad (2.4)$$

where  $u = \sum_{n=0}^{\infty} (a^d)^{n+2} cb^n b^\pi + \sum_{n=0}^{\infty} a^\pi a^n c (b^d)^{n+2} - a^d cb^d$ .

(ii) *If  $x \in \mathcal{A}^d$  and  $a \in (p\mathcal{A}p)^d$ , then  $b \in [(1-p)\mathcal{A}(1-p)]^d$  and  $x^d$  is given by (2.4).*

**Lemma 2.2** (see [24, Corollary 3.4]). *If  $a, b \in \mathcal{A}$  are generalized Drazin invertible,  $b$  is quasinilpotent, and  $ab = 0$ , then  $a + b$  is generalized Drazin invertible and*

$$(a + b)^d = \sum_{n=0}^{\infty} b^n (a^d)^{n+1}. \quad (2.5)$$

The following lemma is a generalization of [25, Theorem 1].

**Lemma 2.3.** *Let  $a, b \in \mathcal{A}^d$  such that  $ab = ba$ . Then  $a + b \in \mathcal{A}^d$  if and only if  $1 + a^d b \in \mathcal{A}^d$ . In this case*

$$(a + b)^d = a^d (1 + a^d b)^d b b^d + \sum_{n=0}^{\infty} b^n (-b)^n (a^d)^{n+1} + \sum_{n=0}^{\infty} (b^d)^{n+1} (-a)^n a^\pi. \quad (2.6)$$

Now we will state a lemma for the representation of the group inverse of an element  $a \in \mathcal{A}^d$  with  $2 \times 2$  block form in Banach algebra (see [26, Theorem 7.7.7] and [23, Theorem 2.2.] which were established for a finite dimensional case and partitioned operators matrix, resp.).

**Lemma 2.4.** *Let  $z \in \mathcal{A}$ , and it has the block matrix form as  $z = \begin{bmatrix} z_1 & z_{12} \\ z_{21} & z_2 \end{bmatrix}_p$ , where  $p \in \mathcal{A}^\bullet$  is an idempotent element,  $z_1$  is invertible in  $p\mathcal{A}p$ , and  $z_2 = z_{21}z_1^{-1}z_{12}$ . Let  $\delta = p + z_1^{-1}z_{12}z_{21}z_1^{-1}$ . Then  $z$  is group invertible if and only if  $\delta$  is an invertible element in  $p\mathcal{A}p$ . In this case*

$$\begin{aligned} z^\# &= \begin{bmatrix} (\delta z_1 \delta)^{-1} & (\delta z_1 \delta)^{-1} z_1^{-1} z_{12} \\ z_{21} z_1^{-1} (\delta z_1 \delta)^{-1} & z_{21} z_1^{-1} (\delta z_1 \delta)^{-1} z_1^{-1} z_{12} \end{bmatrix}_p, \\ z^\pi &= \begin{bmatrix} p - \delta^{-1} & -\delta^{-1} z_1^{-1} z_{12} \\ -z_{21} z_1^{-1} \delta^{-1} & 1 - p - z_{21} z_1^{-1} \delta^{-1} z_1^{-1} z_{12} \end{bmatrix}_p. \end{aligned} \quad (2.7)$$

Let  $b \in \mathcal{A}$  be a perturbation element of  $a$ . According to (2.1), we obtain

$$b = \begin{bmatrix} b_1 & b_{12} \\ b_{21} & b_2 \end{bmatrix}_p, \quad a + b = \begin{bmatrix} a_1 + b_1 & b_{12} \\ b_{21} & a_2 + b_2 \end{bmatrix}_p, \quad (2.8)$$

where  $p = aa^d$ .

**Theorem 2.5.** *Let  $a \in \mathcal{A}^d$  and  $b \in \mathcal{A}$  be a perturbation element of  $a$ ,  $a$  and  $a + b$  which are defined as (2.1) and (2.8), respectively. If  $\|a^d b a a^d\| < 1$ , then  $a_1 + b_1$  is invertible in subalgebra  $p\mathcal{A}p$ . Furthermore let  $a_2 + b_2 = b_{21}(a_1 + b_1)^{-1}b_{12}$  and  $\delta = p + [p(a + b)p]^d b(1 - p)b[p(a + b)p]^d \in p\mathcal{A}p$ . Then  $a + b$  is group invertible if and only if  $\delta \in p\mathcal{A}p$  is invertible and  $\delta$  is invertible if and only if  $[p(a + b)p]^2 + pb(1 - p)bp \in p\mathcal{A}p$  is invertible. In this case,*

$$\begin{aligned} \frac{\|(a + b)^\# - a^d\|}{\|a^d\|} &\leq \frac{T_1^2}{1 - \|a_1^{-1}b_1\|} (\|b_{12}\| + \|b_{21}\|) + T_1^2 \frac{\|a_1^{-1}\|^3}{(1 - \|a_1^{-1}b_1\|)^4} \|b_{12}\| \|b_{21}\| \\ &+ \left( T_1 T_2 + T_2 + \frac{\|a_1^{-1}b_1\|}{1 - \|a_1^{-1}b_1\|} T_1 \right) T_1, \end{aligned} \quad (2.9)$$

where

$$\begin{aligned} a_1 &= pap, & b_1 &= pbp, & b_{12} &= pb(1 - p), & b_{21} &= (1 - p)bp, \\ T_1 &= \|a_1 + b_1\|^2 \left\| \left( (a_1 + b_1)^2 + b_{12}b_{21} \right)^{-1} \right\|, & T_2 &= \frac{\|a_1^{-1}\| \|b_{12}b_{21}\|}{1 - \|a_1^{-1}b_1\|}. \end{aligned} \quad (2.10)$$

*Proof.* Let  $p = aa^d$ . Then  $a$ ,  $a^d$ ,  $b$  and  $a + b$  have the matrix form as (2.1) and (2.8), respectively, where  $a_1$  is invertible in  $p\mathcal{A}p$  and  $a_2$  is quasiniptent in  $(1 - p)\mathcal{A}(1 - p)$ .

It follows from the hypothesis  $\|a^d b a a^d\| < 1$  that  $\|a_1^{-1} b_1\| < 1$ . Thus, it implies that  $p + a_1^{-1} b_1 \in p\mathcal{A}p$  is invertible. It is easy to see that  $(a_1 + b_1)^{-1} = (p + a_1^{-1} b_1)^{-1} a_1^{-1}$ . Let  $\delta = p + [p(a+b)p]^d b(1-p)b[p(a+b)p]^d \in p\mathcal{A}p$ ; that is, we have  $\delta = p + (a_1 + b_1)^{-1} b_{12} b_{21} (a_1 + b_1)^{-1}$ . Therefore, we have

$$\begin{aligned} \delta &= (a_1 + b_1)^{-1} [(a_1 + b_1)p(a_1 + b_1) + b_{12} b_{21}] (a_1 + b_1)^{-1} \\ &= (a_1 + b_1)^{-1} [(a_1 + b_1)^2 + b_{12} b_{21}] (a_1 + b_1)^{-1} \\ &= [p(a+b)p]^d \left[ [p(a+b)p]^2 + pb(1-p)bp \right] [p(a+b)p]^d. \end{aligned} \quad (2.11)$$

From the previous equations, we get that  $\delta$  is invertible if and only if  $[p(a+b)p]^2 + pb(1-p)bp$  is invertible. Since  $a_2 + b_2 = b_{21}(a_1 + b_1)^{-1} b_{12}$  and by Lemma 2.4, we obtain that  $a + b$  is group invertible if and only if  $\delta \in p\mathcal{A}p$  is invertible.

In the following, we consider the upper bound of  $\|(a+b)^\# - a^d\|/\|a^d\|$ .

Applying Lemma 2.4, we obtain

$$(a+b)^\# = \begin{bmatrix} \eta & \eta(a_1 + b_1)^{-1} b_{12} \\ b_{21}(a_1 + b_1)^{-1} \eta & b_{21}(a_1 + b_1)^{-1} \eta(a_1 + b_1)^{-1} b_{12} \end{bmatrix}_p, \quad (2.12)$$

where  $\eta = (\delta(a_1 + b_1)\delta)^{-1}$ .

Note that

$$\begin{aligned} \eta - a_1^{-1} &= (\delta(a_1 + b_1)\delta)^{-1} - a_1^{-1} = \delta^{-1}(a_1 + b_1)^{-1} \delta^{-1} - a_1^{-1} \\ &= \delta^{-1} a_1^{-1} \delta^{-1} + \delta^{-1} \sum_{n=1}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} \delta^{-1} - a_1^{-1} \\ &= \delta^{-1} (a_1^{-1} - \delta a_1^{-1} \delta) \delta^{-1} + \delta^{-1} \sum_{n=1}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} \delta^{-1} \\ &= \delta^{-1} (a_1^{-1} - (p + \theta) a_1^{-1} (p + \theta)) \delta^{-1} + \delta^{-1} \sum_{n=1}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} \delta^{-1} \\ &= \delta^{-1} \left[ a_1^{-1} - (p a_1^{-1} p + \theta a_1^{-1} p + p a_1^{-1} \theta + \theta a_1^{-1} \theta) \right] \delta^{-1} + \delta^{-1} \sum_{n=1}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} \delta^{-1} \\ &= -\delta^{-1} (\theta a_1^{-1} p + p a_1^{-1} \theta + \theta a_1^{-1} \theta) \delta^{-1} + \delta^{-1} \sum_{n=1}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} \delta^{-1} \\ &= -\delta^{-1} (\theta a_1^{-1} p + \delta a_1^{-1} \theta) \delta^{-1} + \delta^{-1} \sum_{n=1}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} \delta^{-1} \\ &= -\delta^{-1} \theta a_1^{-1} \delta^{-1} - a_1^{-1} \theta \delta^{-1} + \delta^{-1} \sum_{n=1}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} \delta^{-1}, \end{aligned} \quad (2.13)$$

$$\begin{aligned}\|\delta^{-1}\| &= \left\| \left[ p + (a_1 + b_1)^{-1} b_{12} b_{21} (a_1 + b_1)^{-1} \right]^{-1} \right\| \\ &\leq \|a_1 + b_1\|^2 \left\| \left[ (a_1 + b_1)^2 + b_{12} b_{21} \right]^{-1} \right\| = T_1,\end{aligned}\quad (2.14)$$

where  $\theta = (a_1 + b_1)^{-1} b_{12} b_{21} (a_1 + b_1)^{-1}$ .

It shows from  $\|a^d b a a^d\| < 1$  (i.e.,  $\|a_1^{-1} b_1\| < 1$ ) that

$$\begin{aligned}\|\theta\| &= \left\| (a_1 + b_1)^{-1} b_{12} b_{21} (a_1 + b_1)^{-1} \right\| \\ &\leq \frac{\|a_1^{-1}\| \|b_{12} b_{21}\|}{1 - \|a_1^{-1} b_1\|} = T_2.\end{aligned}\quad (2.15)$$

From (2.13), (2.14), (2.15), and by  $\|a_1^{-1} b_1\| < 1$ , we obtain that

$$\begin{aligned}\|\eta - a_1^{-1}\| &= \left\| -\delta^{-1} \theta a_1^{-1} \delta^{-1} - a_1^{-1} \theta \delta^{-1} + \delta^{-1} \sum_{n=1}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} \delta^{-1} \right\| \\ &\leq \left\| \delta^{-1} \theta a_1^{-1} \delta^{-1} \right\| + \left\| a_1^{-1} \theta \delta^{-1} \right\| + \left\| \delta^{-1} \sum_{n=1}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} \delta^{-1} \right\| \\ &\leq \left( T_1 T_2 + T_2 + \frac{\|a_1^{-1} b_1\|}{1 - \|a_1^{-1} b_1\|} T_1 \right) T_1 \|a_1^{-1}\|.\end{aligned}\quad (2.16)$$

It follows from (2.12) that

$$(a + b)^{\#} - a^d = \begin{bmatrix} \eta - a_1^{-1} & \eta(a_1 + b_1)^{-1} b_{12} \\ b_{21}(a_1 + b_1)^{-1} \eta & b_{21}(a_1 + b_1)^{-1} \eta(a_1 + b_1)^{-1} b_{12} \end{bmatrix}_p. \quad (2.17)$$

Therefore, according to (2.14), (2.15), and (2.16), we obtain

$$\begin{aligned}\|(a + b)^{\#} - a^d\| &= \left\| \begin{bmatrix} \eta - a_1^{-1} & \eta(a_1 + b_1)^{-1} b_{12} \\ b_{21}(a_1 + b_1)^{-1} \eta & b_{21}(a_1 + b_1)^{-1} \eta(a_1 + b_1)^{-1} b_{12} \end{bmatrix}_p \right\| \\ &\leq \left\| \eta - a_1^{-1} \right\| + \left\| \eta(a_1 + b_1)^{-1} b_{12} \right\| + \left\| b_{21}(a_1 + b_1)^{-1} \eta \right\| \\ &\quad + \left\| b_{21}(a_1 + b_1)^{-1} \eta(a_1 + b_1)^{-1} b_{12} \right\| \\ &\leq \frac{\|a_1^{-1}\| T_1^2}{1 - \|a_1^{-1} b_1\|} (\|b_{12}\| + \|b_{21}\|) + T_1^2 \left( \frac{\|a_1^{-1}\|}{1 - \|a_1^{-1} b_1\|} \right)^4 \|b_{12}\| \|b_{21}\| \\ &\quad + \left( T_1 T_2 + T_2 + \frac{\|a_1^{-1} b_1\|}{1 - \|a_1^{-1} b_1\|} T_1 \right) T_1 \|a_1^{-1}\|.\end{aligned}\quad (2.18)$$

Since  $\|a_1^{-1}\| = \|a^d\|$  and by (2.18), it is easy to see that the conclusion holds.

Thus, we complete the proof.  $\square$

Let  $A, E \in B(X)$  be both bounded linear operators with  $B = A + E$  on Banach space, where  $X$  denotes Banach space. If  $\|A^D E\| + \|B^\pi - A^\pi\| < 1$  is satisfied, (it implies that  $\|A^D E\| < 1$  and  $\|A^D E A A^D\| < 1$ ), then we have the remark.

*Remark 2.6* (see [23, Theorem 4.2]). Let  $A, B \in B(X)$  be Drazin invertible and group invertible, respectively. If  $\|A^D E\| + \|B^\pi - A^\pi\| < 1$ , then

$$\frac{\|B^\# - A^D\|}{\|A^D\|} \leq \frac{\|A^D E\| + 2\|B^\pi - A^\pi\|}{1 - \|A^D E\| - \|B^\pi - A^\pi\|}. \quad (2.19)$$

Let  $a = a_1 \oplus a_2$  and  $a^d = [a_1]_p^{-1} = a_1^\#$ ; if we put  $\delta a = b + a_2$ , then  $1 + \delta a a$  is invertible in  $\mathcal{A}$  when  $\|a^d b a a^d\| < 1$ . From the Proposition 2.2 (5) of [20], we have  $a_2 + b_2 = b_{21}(a_1 + b_1)^{-1}b_{12}$  when  $(a_1 + \delta a)\mathcal{A} \cap (1 - a a^d)\mathcal{A} = \{0\}$  for  $[a_1]_p [a_1]_p^{-1} = p = a a^d$ . Therefore, for  $\|a_1^\#\| \|b + a_2\| < (1 + \|1 - a a^d\|)^{-1}$ , we arrive at [20, Theorem 4.2]. In fact, the following remark implies that Theorem 2.5 improves the upper bound of  $\|(a + b)^\# - a^\#\|$  of [20, Theorem 4.2].

*Remark 2.7* (see [20, Theorem 4.2]). Let  $a \in G(\mathcal{A})$  and let  $\bar{a} = a + \delta a \in \mathcal{A}$  with  $\mathcal{K}_\# \epsilon_a < (1 + \|a^\pi\|)$ . Assume that  $\bar{a}\mathcal{A} \cap (1 - a a^\#)\mathcal{A} = \{0\}$ . Then  $\bar{a} \in G(\mathcal{A})$  and

$$\|\bar{a}^\# - a^\#\| \leq \frac{(1 + 2\|a^\pi\|)\|a^\#\|\mathcal{K}_\#(a)\epsilon_a}{[1 - (1 + \|a^\pi\|)\|a^\#\|\mathcal{K}_\#(a)\epsilon_a]^2}, \quad (2.20)$$

where  $\mathcal{K}_\# = \|a\|\|a^\#\|$  and  $\epsilon_a = \|\delta a\|\|a\|^{-1}$ .

**Theorem 2.8.** Let  $a, b \in \mathcal{A}$  be generalized Drazin invertible and satisfy the conditions

$$\|a^d b a a^d\| < 1, \quad a^\pi b a = 0. \quad (2.21)$$

Then  $(a + b)^\#$  exists if and only if  $a^\pi(a + b)$  is group invertible. In this case,

$$\begin{aligned} \frac{\|(a + b)^\# - a^d\|}{\|a^d\|} &\leq \frac{\|a^d\|}{(1 - \|a^d b\|)^2} \left[ \|b\| + \|b a^\pi\| \sum_{n=0}^{\infty} \|a^{n+1}\| \|(b^d)^{n+1}\| \right] \\ &\quad + \left[ \frac{\|a^d\|}{(1 - \|a^d b\|)^2} \|b a^\pi b\| + \frac{\|b a^\pi\|}{1 - \|a^d b\|} \right] \sum_{n=0}^{\infty} \|a^n\| \|(b^d)^{n+1}\| \\ &\quad + \|a^\pi\| \|a^d\|^{-1} \sum_{n=0}^{\infty} \|a^n\| \|(b^d)^{n+1}\| + \frac{\|a^d b\|}{1 - \|a^d b\|}. \end{aligned} \quad (2.22)$$

*Proof.* Since  $a^d$  exists,  $a^d$  is defined as (2.1). Let  $b$  have the block matrix form as

$$b = \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix}_p. \quad (2.23)$$



Applying the condition  $a^\pi ba = 0$ , we have  $b_2 a_2 = 0$  and

$$a^\pi b a a^d a = \begin{bmatrix} 0 & 0 \\ b_4 a_1 & 0 \end{bmatrix}_p = 0. \quad (2.24)$$

It follows from (2.24) that  $a_1 \in p\mathcal{A}p$  is invertible,  $b_4 = 0$ , and

$$b = \begin{bmatrix} b_1 & b_3 \\ 0 & b_2 \end{bmatrix}_p. \quad (2.25)$$

Combining (2.1) and (2.25), we obtain

$$a + b = \begin{bmatrix} a_1 + b_1 & b_3 \\ 0 & a_2 + b_2 \end{bmatrix}_p. \quad (2.26)$$

The condition  $\|a^d b a a^d\| < 1$  implies  $\|a_1^{-1} b_1\| < 1$  in the subalgebra  $p\mathcal{A}p$ . Therefore, we conclude that  $a_1 + b_1 \in p\mathcal{A}p$  is invertible and  $\text{Ind}(a_1 + b_1) = 0$ . According to (2.26) and by Lemma 2.1, one observes that  $(a + b)^d$  exists if and only if  $(a_2 + b_2)^d$  also. Thus,  $(a + b)^\#$  exists if and only if  $a^\pi(a + b)$  is group invertible.

If  $a^\pi(a + b)$  is group invertible and by Lemma 2.1, we obtain

$$(a + b)^\# = \begin{bmatrix} (a_1 + b_1)^{-1} & x \\ 0 & (a_2 + b_2)^\# \end{bmatrix}_p, \quad (2.27)$$

where  $x = (a_1 + b_1)^{-2} b_3 (a_2 + b_2)^\pi - (a_1 + b_1)^{-1} b_3 (a_2 + b_2)^\#$ .

Since  $b_2 a_2 = 0$  and  $a_2$  is quasinilpotent, by Lemma 2.2, we obtain

$$(a_2 + b_2)^\# = \sum_{n=0}^{\infty} a_2^n (b_2^d)^{n+1} = a^\pi \sum_{n=0}^{\infty} a^n (b^d)^{n+1}. \quad (2.28)$$

From  $\|a^d b a a^d\| < 1$ , one easily has

$$\begin{aligned} (a_1 + b_1)^{-1} \oplus 0 &= \sum_{n=0}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} \oplus 0 = \sum_{n=0}^{\infty} (a^d b)^n a^d \\ &= a_1^{-1} (1 + b_1 a_1^{-1})^{-1} \oplus 0 = a^d \sum_{n=0}^{\infty} (b a^d)^n. \end{aligned} \quad (2.29)$$

It follows from (2.27) and (2.29) that

$$\begin{aligned}
 x &= (a_1 + b_1)^{-2} b_3 (a_2 + b_2)^\pi - (a_1 + b_1)^{-1} b_3 (a_2 + b_2)^\sharp \\
 &= \left( \sum_{n=0}^{\infty} (a^d b)^n a^d \right)^2 \left[ b - b a^\pi \sum_{n=0}^{\infty} a^{n+1} (b^d)^{n+1} \right] \\
 &\quad + \left[ \left( \sum_{n=0}^{\infty} (a^d b)^n a^d \right)^2 b a^\pi b - \sum_{n=0}^{\infty} (a^d b)^n a^d b a^\pi \right] \sum_{n=0}^{\infty} a^n (b^d)^{n+1}.
 \end{aligned} \tag{2.30}$$

Combining (2.27), (2.28), and (2.29), we obtain

$$\begin{aligned}
 (a + b)^\sharp &= \begin{bmatrix} \sum_{n=0}^{\infty} a_1^{-1} (b_1 a_1^{-1})^n & x \\ 0 & \sum_{n=0}^{\infty} a_2^n (b_2^d)^{n+1} \end{bmatrix}_p \\
 &= \left( \sum_{n=0}^{\infty} (a^d b a^d)^n \right)^2 \left[ b - b a^\pi \sum_{n=0}^{\infty} a^{n+1} (b^d)^{n+1} \right] \\
 &\quad + \left[ \left( \sum_{n=0}^{\infty} (a^d b)^n a^d \right)^2 b a^\pi b - \sum_{n=0}^{\infty} (a^d b)^n a^d b a^\pi \right] \sum_{n=0}^{\infty} a^n (b^d)^{n+1} \\
 &\quad + \sum_{n=0}^{\infty} (a^d b)^n a^d + a^\pi \sum_{n=0}^{\infty} a^n (b^d)^{n+1}.
 \end{aligned} \tag{2.31}$$

From (2.31), we derive

$$\begin{aligned}
 (a + b)^\sharp - a^d &= \left( \sum_{n=0}^{\infty} (a^d b)^n a^d \right)^2 \left[ b - b a^\pi \sum_{n=0}^{\infty} a^{n+1} (b^d)^{n+1} \right] \\
 &\quad + \left[ \left( \sum_{n=0}^{\infty} (a^d b)^n a^d \right)^2 b a^\pi b - \sum_{n=0}^{\infty} (a^d b)^n a^d b a^\pi \right] \sum_{n=0}^{\infty} a^n (b^d)^{n+1} \\
 &\quad + a^\pi \sum_{n=0}^{\infty} a^n (b^d)^{n+1} + \sum_{n=1}^{\infty} (a^d b)^n a^d.
 \end{aligned} \tag{2.32}$$

Moreover, by (2.32) we get

$$\begin{aligned}
\|(a+b)^{\sharp} - a^d\| &\leq \left(\frac{\|a^d\|}{1 - \|a^d b\|}\right)^2 \left[ \|b\| + \|ba^{\pi}\| \sum_{n=0}^{\infty} \|a^{n+1}\| \|(b^d)^{n+1}\| \right] \\
&\quad + \left[ \left(\frac{\|a^d\|}{1 - \|a^d b\|}\right)^2 \|ba^{\pi}b\| + \frac{\|a^d\| \|ba^{\pi}\|}{1 - \|a^d b\|} \right] \sum_{n=0}^{\infty} \|a^n\| \|(b^d)^{n+1}\| \\
&\quad + \|a^{\pi}\| \sum_{n=0}^{\infty} \|a^n\| \|(b^d)^{n+1}\| + \frac{\|a^d\| \|a^d b\|}{1 - \|a^d b\|}.
\end{aligned} \tag{2.33}$$

Finally, from (2.33) we easily finish the proof.  $\square$

**Corollary 2.9.** Let  $a \in \mathcal{A}_g$  and let  $b \in \mathcal{A}^d$ . If  $a, b$  satisfy the conditions

$$\|a^{\sharp}baa^{\sharp}\| < 1, \quad a^{\pi}ba = 0, \tag{2.34}$$

then  $(a+b)^{\sharp}$  exists if and only if  $a^{\pi}b$  is group invertible. In this case,

$$\begin{aligned}
\frac{\|(a+b)^{\sharp} - a^{\sharp}\|}{\|a^{\sharp}\|} &\leq \|b\| \|(a^{\pi}b)^{\pi}\| \frac{\|a^{\sharp}\|}{(1 - \|a^{\sharp}b\|)^2} \\
&\quad + \frac{\|ba^{\pi}b^{\sharp}\|}{1 - \|a^{\sharp}b\|} + \frac{\|a^{\pi}b^{\sharp}\|}{\|a^{\sharp}\|} + \frac{\|a^{\sharp}b\|}{1 - \|a^{\sharp}b\|}.
\end{aligned} \tag{2.35}$$

The conditions of Theorem 2.8  $\|a^d b a a^d\| < 1$ ,  $a^{\pi}ba = 0$  are weaker than the conditions  $(\mathcal{W})$  (see [12, Theorem 3.2] for finite dimensional cases and [8, Theorem 5.3.2 and Corollary 5.3.3] for Banach algebra). According to  $a^{\pi}ba = 0$ , we obtain that (2.26) holds. However, in view of  $(\mathcal{W})$ , we have

$$a + b = \begin{bmatrix} a_1 + b_1 & 0 \\ 0 & a_2 \end{bmatrix}_p. \tag{2.36}$$

Thus, by the conditions  $(\mathcal{W})$ , we know that  $a$  and  $a + b$  have the same Drazin invertible property (see [12, Theorem 3.1]). Thus, if  $a$  is group invertible, then  $(a+b)$  is group invertible. It is easy to see that  $\|a^d b a a^d\| < 1$ ,  $a^{\pi}ba = 0$  are weaker than the conditions  $(\mathcal{W})$ . From [8, Theorem 5.3.2 and Corollary 5.3.3], we easily state the following remark.

**Remark 2.10.** Let  $a \in \mathcal{A}_g$  and let  $b \in \mathcal{A}^d$ . If  $a, b$  satisfy the condition  $(\mathcal{W})$

$$\|a^{\sharp}baa^{\sharp}\| < 1, \quad b = aa^{\sharp}baa^{\sharp}, \tag{2.37}$$

then  $a + b$  is group invertible and

$$\frac{\|(a + b)^\# - a^\#\|}{\|a^\#\|} \leq \frac{\|a^\#b\|}{1 - \|a^\#b\|}. \quad (2.38)$$

**Theorem 2.11.** Let  $a, b \in \mathcal{A}$  be generalized Drazin invertible and satisfy the conditions

$$\max\left\{\|a^d b a a^d\|, \|a^\pi a\| \|a^\pi b^d\|\right\} < 1, \quad a^\pi b a = 0. \quad (2.39)$$

Then  $(a + b)^\#$  exists if and only if  $a^\pi(a + b)$  is group invertible. In this case,

$$\begin{aligned} \frac{\|(a + b)^\# - a^d\|}{\|a^d\|} &\leq \frac{\|a^d\|}{(1 - \|a^d b\|)^2} \left[ \|b\| + \|b a^\pi\| \frac{\|a\| \|b^d\|}{1 - \|a\| \|b^d\|} \right] \\ &+ \left[ \frac{\|a^d\|}{(1 - \|a^d b\|)^2} \|b a^\pi b\| + \frac{\|b a^\pi\|}{1 - \|a^d b\|} \right] \frac{\|a\| \|b^d\|}{1 - \|a\| \|b^d\|} \\ &+ \frac{\|a^d\|^{-1} \|a^\pi\| \|b^d\|}{1 - \|a\| \|b^d\|} + \frac{\|a^d b\|}{1 - \|a^d b\|}. \end{aligned} \quad (2.40)$$

*Proof.* The notations are taken as Theorem 2.8, and the rest of proof of theorem is similar to Theorem 2.8. Now, we only consider the perturbation of  $a_2 + b_2$ . From (2.28) and the first condition of (2.39), we have  $\|a_2\| \|b_2^d\| < 1$  and

$$\|(a_2 + b_2)^\#\| = \left\| \sum_{n=0}^{\infty} a_2^n (b_2^d)^{n+1} \right\| \leq \frac{\|a^\pi\| \|b^d\|}{1 - \|a\| \|b^d\|}. \quad (2.41)$$

Thus, from (2.41) we completed the proof.  $\square$

**Theorem 2.12.** Let  $a, b \in \mathcal{A}$  be generalized Drazin invertible and satisfy the conditions

$$\|a^d b a a^d\| < 1, \quad a^\pi b a = a b a^\pi. \quad (2.42)$$

Then  $(a + b)^\#$  exists if and only if  $a^\pi(a + b)$  is group invertible. In this case,

$$\frac{\|(a + b)^\# - a^d\|}{\|a^d\|} \leq \frac{\|a^d b\|}{1 - \|a^d b\|} + \|a^\pi\| \|a^d\|^{-1} \sum_{n=0}^{\infty} \|(b^d)^{n+1}\| \|a^n\|. \quad (2.43)$$

*Proof.* Letting  $p = aa^d$ , and it is similar to Theorem 2.8, we obtain that  $a$ ,  $a^d$ , and  $p$  have the matrix forms as (2.1). Here  $b$  is taken as (2.23) in the proof of Theorem 2.8. The condition  $a^\pi ba = aba^\pi$  implies that

$$\begin{aligned} a^\pi ba &= \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_p \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix}_p \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p = \begin{bmatrix} 0 & 0 \\ b_4 a_1 & b_2 a_2 \end{bmatrix}_p, \\ aba^\pi &= \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix}_p \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_p = \begin{bmatrix} 0 & a_1 b_3 \\ 0 & a_2 b_2 \end{bmatrix}_p. \end{aligned} \quad (2.44)$$

Thus, according to (2.44), we obtain  $b_4 a_1 = 0$ ,  $a_1 b_3 = 0$  and  $a_2 b_2 = b_2 a_2$ . Because  $a_1$  is invertible in subalgebra  $p\mathcal{A}p$ , we have  $b_3 = b_4 = 0$ . Thus,  $b$ ,  $a + b$  have the matrix forms as follows:

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_p, \quad a + b = \begin{bmatrix} a_1 + b_1 & 0 \\ 0 & a_2 + b_2 \end{bmatrix}_p. \quad (2.45)$$

It follows from the condition  $\|a^d b a a^d\| < 1$  that  $\|a_1^{-1} b_1\| < 1$ . Thus, it shows from  $\|a_1^{-1} b_1\| < 1$  that  $a_1 + b_1$  is invertible in subalgebra  $p\mathcal{A}p$ . Therefore, easily we observe that  $a + b$  is Drazin invertible if and only if  $a_2 + b_2 \in (1-p)\mathcal{A}(1-p)$  is Drazin invertible. That is,  $(a + b)^\#$  exists if and only if  $a^\pi(a + b)$  is group invertible.

In the following, we will consider the perturbation of  $a_2$ .

Let  $a_2 + b_2$  be group invertible. The condition  $a^\pi ba = aba^\pi$  implies that  $a_2 b_2 = b_2 a_2$  holds. Since  $a_2$  is quasinilpotent in subalgebra  $(1-p)\mathcal{A}(1-p)$  and by Lemma 2.3, we get

$$(a_2 + b_2)^\# = \sum_{n=0}^{\infty} (b_2^d)^{n+1} (-a_2)^n = a^\pi \sum_{n=0}^{\infty} (b^d)^{n+1} (-a)^n. \quad (2.46)$$

By virtue of  $\|a_1^{-1} b_1\| < 1$ , we get that

$$\left[ aa^d(a + b) \right]^{-1} = (a_1 + b_1)^{-1} = \sum_{n=0}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} = \sum_{n=0}^{\infty} (a^d b)^n a^d. \quad (2.47)$$

It follows from (2.46) and (2.47) that

$$\begin{aligned} \left\| \left[ aa^d(a + b) \right]^{-1} - \left[ aa^d a \right]_p^{-1} \right\| &= \left\| \sum_{n=1}^{\infty} (a^d b)^n a^d \right\| \leq \frac{\|a^d\| \|a^d b\|}{1 - \|a^d b\|}, \\ \left\| [a^\pi(a + b)]^d \right\| &= \left\| (a_2 + b_2)^\# \right\| \leq \|a^\pi\| \sum_{n=0}^{\infty} \left\| (b^d)^{n+1} \right\| \|a^n\|. \end{aligned} \quad (2.48)$$

Next, according to (2.48), we obtain

$$\left\| (a + b)^\# - a^d \right\| \leq \frac{\|a^d\| \|a^d b\|}{1 - \|a^d b\|} + \|a^\pi\| \sum_{n=1}^{\infty} \left\| (b^d)^{n+1} \right\| \|a^n\|. \quad (2.49)$$

Finally, using (2.49) the proof is finished.  $\square$

**Corollary 2.13.** Let  $a \in \mathcal{A}_g$  and let  $b \in \mathcal{A}^d$ . If  $a, b$  satisfy the conditions

$$\|a^\sharp b a a^\sharp\| < 1, \quad a^\pi b a = a b a^\pi. \quad (2.50)$$

Then  $(a + b)^\sharp$  exists if and only if  $a^\pi b$  is group invertible. In this case,

$$\frac{\|(a + b)^\sharp - a^\sharp\|}{\|a^\sharp\|} \leq \frac{\|a^\sharp b\|}{1 - \|a^\sharp b\|} + \frac{\|a^\pi\| \|b^d\|}{\|a^\sharp\|}. \quad (2.51)$$

Let  $A, E \in C^{n \times n}$  with  $B = A + E$ , and let

$$A = P^{-1} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} P, \quad E = P^{-1} \begin{bmatrix} E_1 & E_{12} \\ E_{21} & E_2 \end{bmatrix} P. \quad (2.52)$$

If  $B^\pi = A^\pi$  (see [10, Theorem 2.1]), then

$$B = P^{-1} \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} P, \quad A + E = P^{-1} \begin{bmatrix} A_1 + E_1 & 0 \\ 0 & A_2 + E_2 \end{bmatrix} P, \quad (2.53)$$

where  $B_1$  is invertible and  $B_2 = A_2$  is quasinilpotent (it follows that  $E_2 = 0$ ). It follows from (2.53) that  $A^\pi = B^\pi$  implies that  $A^\pi B A = A B A^\pi$ , (i.e.,  $A^\pi E A = A E A^\pi$ ). If  $A$  is group invertible, then  $B$  is group invertible and

$$B^\sharp = P^{-1} \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} P, \quad (2.54)$$

where  $B_1 = A_1 + E_1$ .

By virtue of  $A^\pi = B^\pi$  and  $\|A^D(B - A)\| < 1$  (see [10]), we give the following remark.

*Remark 2.14* (see [10, Theorem 3.1]). Let  $A, B \in C^{d \times d}$  with  $A^\pi = B^\pi$ . Then

$$\frac{\|A^D\|}{1 + \|A^D(B - A)\|} \leq \|B^D\|. \quad (2.55)$$

If  $\|A^D(B - A)\| < 1$ , then

$$\begin{aligned} \|B^D\| &\leq \frac{\|A^D\|}{1 - \|A^D(B - A)\|}, \\ \frac{\|B^D - A^D\|}{\|A^D\|} &\leq \frac{\|A^D(B - A)\|}{1 - \|A^D(B - A)\|}. \end{aligned} \quad (2.56)$$

**Theorem 2.15.** Let  $a, b \in \mathcal{A}$  be generalized Drazin invertible and satisfy the conditions

$$\max\left\{\left\|a^\pi ab^d\right\|, \left\|a^d baa^d\right\|\right\} < 1, \quad a^\pi ba = aba^\pi. \quad (2.57)$$

Then  $(a + b)^\sharp$  exists if and only if  $a^\pi(a + b)$  is group invertible. In this case,

$$\frac{\left\|(a + b)^\sharp - a^d\right\|}{\left\|a^d\right\|} \leq \frac{\left\|a^d b\right\|}{1 - \left\|a^d b\right\|} + \frac{\left\|a^\pi\right\|\left\|b^d\right\|}{\left\|a^d\right\|(1 - \left\|b^d a\right\|)}. \quad (2.58)$$

*Proof.* Similarly to Theorem 2.12, we have that the formulas (2.45) hold. The details will be omitted. In the following we only give the simple proof.

By the condition

$$\max\left\{\left\|a^\pi ab^d\right\|, \left\|a^d baa^d\right\|\right\} < 1, \quad (2.59)$$

it shows that  $\|a_1^{-1}b_1\| < 1$  and  $\|a_2b_2^d\| < 1$ . Thus, the first result shows that  $a_1 + b_1 \in p\mathcal{A}p$  is invertible. In view of Lemma 2.1, one concludes that  $a + b$  is Drazin invertible if and only if  $a_2 + b_2$  is Drazin invertible. That is,  $(a + b)^\sharp$  exists if and only if  $a^\pi(a + b)$  is group invertible.

In the following, we consider the perturbation of  $a_2$ .

After application of the hypothesis  $a^\pi ba = aba^\pi$ , we find that  $a_2b_2 = b_2a_2$ . It follows from Theorem 2.12 and Lemma 2.3 that

$$(a_2 + b_2)^\sharp = \sum_{n=0}^{\infty} \left(b_2^d\right)^{n+1} (-a_2)^n = a^\pi \sum_{n=0}^{\infty} (-1)^n \left(b^d\right)^{n+1} a^n. \quad (2.60)$$

It follows from the condition  $\|a_2b_2^d\| < 1$  and

$$\sigma\left(a_2b_2^d\right) \cup \{0\} = \sigma\left(b_2^da_2\right) \cup \{0\}. \quad (2.61)$$

It implies that  $\|b_2^da_2\| < 1$  and

$$\left\|(a_2 + b_2)^\sharp\right\| \leq \frac{\left\|a^\pi\right\|\left\|b^d\right\|}{1 - \left\|b^da\right\|}. \quad (2.62)$$

Therefore, combining (2.49) with (2.62), we have

$$\left\|(a + b)^\sharp - a^d\right\| \leq \frac{\left\|a^d\right\|\left\|a^db\right\|}{1 - \left\|a^db\right\|} + \frac{\left\|a^\pi\right\|\left\|b^d\right\|}{1 - \left\|b^da\right\|}. \quad (2.63)$$

Thus, by (2.63), we complete the proof.  $\square$

### 3. Perturbation Bound of the Generalized Schur Complement

The perturbation bounds of the Schur complement are investigated in [29–31]. In [29] Stewart gave perturbation bounds for the Schur complement of a positive definite matrix in a positive semidefinite matrix. In [30] Wei and Wang generalized the results in [29] and enrich the perturbation theory for the Schur complement. In [31] the authors derived some new norm upper bounds for Schur complements of a positive semidefinite operator matrix. In this section, we consider the perturbation bounds of the generalized Schur complement in Banach algebra.

Some notations of the generalized Schur complement over Banach algebra will be stated in the following.

Let  $a \in \mathcal{A}$ , and let it be written in the form as follows:

$$a = a_{11} + a_{12} + a_{21} + a_{22}. \quad (3.1)$$

It has the following matrix form:

$$\mathcal{M} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_s, \quad (3.2)$$

where  $s \in \mathcal{A}^\bullet$  is idempotent element in  $\mathcal{A}$  and  $a_{ij}$  is taken as (1.10).

The formulas (1.5) and (1.6) are written in Banach algebra, respectively:

$$\begin{aligned} s_1 &= a_{22} - a_{21} a_{11}^{-1} a_{12}, \\ \overline{\mathcal{M}} &= \begin{bmatrix} a_{11}^{-1} + a_{11}^{-1} a_{12} s_1^{-1} a_{21} a_{11}^{-1} & -a_{11}^{-1} a_{12} s_1^{-1} \\ -s_1^{-1} a_{21} a_{11}^{-1} & s_1^{-1} \end{bmatrix}_s. \end{aligned} \quad (3.3)$$

Similarly, the generalized Schur complement in (1.7) and (1.8) is defined in the following over Banach algebra, respectively:

$$\begin{aligned} s_1 &= a_{22} - a_{21} a_{11}^\# a_{12}, \\ s_1 &= a_{22} - a_{21} a_{11}^d a_{12}, \end{aligned} \quad (3.4)$$

where  $s_1$  denotes the generalized Schur complement of  $a_{11}$  in  $\mathcal{M}$ .

**Theorem 3.1.** Let  $\mathcal{M}$  be given as (3.2) let and

$$\overline{\mathcal{M}} = \begin{bmatrix} a_{11} + \Delta a_{11} & a_{12} + \Delta a_{12} \\ a_{21} + \Delta a_{21} & a_{22} + \Delta a_{22} \end{bmatrix}_s = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} \\ \overline{a}_{21} & \overline{a}_{22} \end{bmatrix}_s \quad (3.5)$$

be perturbed version of  $\mathcal{M}$ , and the following conditions are satisfied:

$$\|\Delta a_{11}\| \leq \epsilon \|a_{11}\|, \quad \|\Delta a_{12}\| \leq \epsilon \|a_{12}\|, \quad \|\Delta a_{21}\| \leq \epsilon \|a_{21}\|, \quad \|\Delta a_{22}\| \leq \epsilon \|a_{22}\|, \quad (3.6)$$



where  $\epsilon > 0$ . If  $a_{11}$ ,  $\Delta a_{11}$  and  $\bar{a}_{11}$  satisfy the conditions of Theorem 2.8, then

$$\begin{aligned} \|\bar{s}_1 - s_1\| &\leq \epsilon \|a_{22}\| + \|a_{21}\| \|a_{12}\| \left( \theta (2\epsilon + \epsilon^2) + \|a_{11}^\pi\| \eta_2 + \epsilon \|a_{11}^d\| \|a_{11}\| \eta_1 \right) \\ &\quad + \epsilon \|a_{21}\| \|a_{11}\| \|a_{12}\| \left( \eta_1^2 + \|a_{11}^\pi\| \eta_1 \eta_2 + (\epsilon + \epsilon^2) \|a_{11}^\pi\| \|a_{11}\| \eta_1^2 \eta_2 \right), \end{aligned} \quad (3.7)$$

where

$$\eta_1 = \frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|}, \quad \eta_2 = \sum_{n=0}^{\infty} \|a_{11}^{n+1}\| \left\| \left[ (\Delta a_{11})^d \right]^{n+1} \right\|, \quad (3.8)$$

$$\begin{aligned} \theta &= \epsilon \|a_{11}\| \eta_1^2 (1 + \|a_{11}^\pi\| \eta_2) + \epsilon \|a_{11}\| \|a_{11}^\pi\| \eta_1 \eta_2 (1 + \epsilon \eta_1 \|a_{11}\|) \\ &\quad + \|a_{11}^\pi\| \eta_2 + \epsilon \|a_{11}^d\| \|a_{11}\| \eta_1, \end{aligned} \quad (3.9)$$

and  $\bar{s}_1$  and  $s_1$  are Schur complement of  $a_{11}$  in  $\mathcal{M}$  and Schur complement of  $\bar{a}_{11}$  in  $\bar{\mathcal{M}}$ , respectively.

*Proof.* Since  $a_{11}$ ,  $\Delta a_{11}$  and  $\bar{a}_{11}$  satisfy Theorem 2.8, according to (2.31), we obtain

$$\begin{aligned} \bar{a}_{11}^\# &= \left( \sum_{n=0}^{\infty} (a_{11}^d \Delta a_{11})^n a_{11}^d \right)^2 \left[ \Delta a_{11} - \Delta a_{11} a_{11}^\pi \sum_{n=0}^{\infty} a_{11}^{n+1} \left[ (\Delta a_{11})^d \right]^{n+1} \right] \\ &\quad + \left[ \left( \sum_{n=0}^{\infty} (a_{11}^d \Delta a_{11})^n a_{11}^d \right)^2 \Delta a_{11} a_{11}^\pi \Delta a_{11} - \sum_{n=0}^{\infty} (a_{11}^d \Delta a_{11})^n a_{11}^d \Delta a_{11} a_{11}^\pi \right] \sum_{n=0}^{\infty} a_{11}^n \left[ (\Delta a_{11})^d \right]^{n+1} \\ &\quad + \sum_{n=0}^{\infty} (a_{11}^d \Delta a_{11})^n a_{11}^d + a_{11}^\pi \sum_{n=0}^{\infty} a_{11}^n \left[ (\Delta a_{11})^d \right]^{n+1}. \end{aligned} \quad (3.10)$$

Therefore, it is easy to see that

$$\begin{aligned} \bar{s}_1 &= a_{22} + \Delta a_{22} - (a_{21} + \Delta a_{21}) \bar{a}_{11}^\# (a_{12} + \Delta a_{12}) \\ &= a_{22} + \Delta a_{22} - a_{21} \bar{a}_{11}^\# a_{12} - \Delta a_{21} \bar{a}_{11}^\# a_{12} - a_{21} \bar{a}_{11}^\# \Delta a_{12} - \Delta a_{21} \bar{a}_{11}^\# \Delta a_{12} \\ &= s_1 + \Delta a_{22} - \Delta a_{21} \bar{a}_{11}^\# a_{12} - a_{21} \bar{a}_{11}^\# \Delta a_{12} - \Delta a_{21} \bar{a}_{11}^\# \Delta a_{12} \\ &\quad - a_{21} a_{11}^\pi \sum_{n=0}^{\infty} a_{11}^n \left[ (\Delta a_{11})^d \right]^{n+1} a_{12} + a_{21} \sum_{n=1}^{\infty} (a_{11}^d \Delta a_{11})^n a_{11}^d a_{12} \\ &\quad + a_{21} \left( \sum_{n=0}^{\infty} (a_{11}^d \Delta a_{11})^n a_{11}^d \right)^2 \Delta a_{11} \left[ 1 - a_{11}^\pi (a_{11} + \Delta a_{11}) \sum_{n=0}^{\infty} a_{11}^n \left[ (\Delta a_{11})^d \right]^{n+1} \right] a_{12} \\ &\quad - a_{21} \sum_{n=0}^{\infty} (a_{11}^d \Delta a_{11})^n a_{11}^d \Delta a_{11} a_{11}^\pi \sum_{n=0}^{\infty} a_{11}^n \left[ (\Delta a_{11})^d \right]^{n+1} a_{12}, \end{aligned}$$

$$\begin{aligned}
\|\bar{a}_{11}^\# \| &\leq \epsilon \|a_{11}\| \left( \frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|} \right)^2 \left[ 1 + \|a_{11}^\pi\| \sum_{n=0}^{\infty} \|a_{11}^{n+1}\| \left\| [(\Delta a_{11})^d]^{n+1} \right\| \right] \\
&+ \left[ \epsilon^2 \left( \frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|} \right)^2 \|a_{11}^\pi\| \|a_{11}\|^2 + \frac{\epsilon \|a_{11}^d\| \|a_{11}\| \|a_{11}^\pi\|}{1 - \|a_{11}^d \Delta a_{11}\|} \right] \sum_{n=0}^{\infty} \|a_{11}^n\| \left\| [(\Delta a_{11})^d]^{n+1} \right\| \\
&+ \|a_{11}^\pi\| \sum_{n=0}^{\infty} \|a_{11}^n\| \left\| [(\Delta a_{11})^d]^{n+1} \right\| + \frac{\epsilon \|a_{11}^d\|^2 \|a_{11}\|}{1 - \|a_{11}^d \Delta a_{11}\|}.
\end{aligned} \tag{3.11}$$

From (3.11) and by the conditions (3.6), we obtain

$$\begin{aligned}
\|\bar{s}_1 - s_1\| &\leq \|\Delta a_{22}\| + \theta (2\epsilon + \epsilon^2) \|a_{21}\| \|a_{12}\| \\
&+ \|a_{21}\| \|a_{12}\| \left( \|a_{11}^\pi\| \sum_{n=0}^{\infty} \|a_{11}^n\| \left\| [(\Delta a_{11})^d]^{n+1} \right\| + \frac{\epsilon \|a_{11}^d\|^2 \|a_{11}\|}{1 - \|a_{11}^d \Delta a_{11}\|} \right) \\
&+ \epsilon \|a_{21}\| \|a_{11}\| \|a_{12}\| \left( \frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|} \right)^2 \left[ 1 + (1 + \epsilon) \|a_{11}\| \sum_{n=0}^{\infty} \|a_{11}^n\| \left\| [(\Delta a_{11})^d]^{n+1} \right\| \right] \\
&+ \epsilon \|a_{21}\| \|a_{12}\| \frac{\|a_{11}^d\| \|a_{11}\| \|a_{11}^\pi\|}{1 - \|a_{11}^d \Delta a_{11}\|} \sum_{n=0}^{\infty} \|a_{11}^n\| \left\| [(\Delta a_{11})^d]^{n+1} \right\|,
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
\theta &= \epsilon \|a_{11}\| \left( \frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|} \right)^2 \left[ 1 + \|a_{11}^\pi\| \sum_{n=0}^{\infty} \|a_{11}^{n+1}\| \left\| [(\Delta a_{11})^d]^{n+1} \right\| \right] \\
&+ \frac{\epsilon \|a_{11}^d\| \|a_{11}\| \|a_{11}^\pi\|}{1 - \|a_{11}^d \Delta a_{11}\|} \left[ 1 + \frac{\epsilon \|a_{11}^d\| \|a_{11}\| \|a_{11}^\pi\|}{1 - \|a_{11}^d \Delta a_{11}\|} \right] \sum_{n=0}^{\infty} \|a_{11}^n\| \left\| [(\Delta a_{11})^d]^{n+1} \right\| \\
&+ \|a_{11}^\pi\| \sum_{n=0}^{\infty} \|a_{11}^n\| \left\| [(\Delta a_{11})^d]^{n+1} \right\| + \frac{\epsilon \|a_{11}^d\|^2 \|a_{11}\|}{1 - \|a_{11}^d \Delta a_{11}\|}.
\end{aligned} \tag{3.13}$$

Thus, we finish the proof.  $\square$

Similar to Theorem 3.1. It follows from the proof of Theorem 2.11 that the results are given as follow.

**Theorem 3.2.** Let  $\mathcal{M}$  and  $\overline{\mathcal{M}}$  be taken as Theorem 3.1, and let the relations in (3.6) be satisfied, where  $\epsilon > 0$ . If  $a_{11}$ ,  $\Delta a_{11}$  and  $\overline{a}_{11}$  satisfy the conditions of Theorem 2.11, then

$$\begin{aligned} \|\overline{s}_1 - s\| &\leq \epsilon \|a_{22}\| + \|a_{21}\| \|a_{12}\| \left( \theta (2\epsilon + \epsilon^2) + \|a_{11}^\pi\| \eta_2 + \epsilon \|a_{11}^d\| \|a_{11}\| \eta_1 \right) \\ &\quad + \epsilon \|a_{21}\| \|a_{11}\| \|a_{12}\| \left( \eta_1^2 + \|a_{11}^\pi\| \eta_1 \eta_2 + (\epsilon + \epsilon^2) \|a_{11}^\pi\| \|a_{11}\| \eta_1^2 \eta_2 \right), \end{aligned} \quad (3.14)$$

where  $\eta_1, \theta, \overline{s}_1$  and  $s_1$  are taken as Theorem 3.1.

**Theorem 3.3.** Let  $\mathcal{M}$  and  $\overline{\mathcal{M}}$  be taken as Theorem 3.1, and let the relations in (3.6) be satisfied, where  $\epsilon > 0$ . If  $a_{11}$ ,  $\Delta a_{11}$  and  $\overline{a}_{11}$  satisfy the conditions of Theorem 2.12, then

$$\|\overline{s}_1 - s\| \leq \epsilon \|a_{22}\| + \left[ (1 + 2\epsilon + \epsilon^2) \delta_1 + \epsilon \|a_{11}^d\|^2 \|a_{11}\| \right] \|a_{12}\| \|a_{21}\|, \quad (3.15)$$

where

$$\delta_1 = \left\{ \frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|} + \|a_{11}^\pi\| \sum_{n=0}^{\infty} \left\| \left[ (\Delta a_{11})^d \right]^{n+1} a_{11}^n \right\| \right\}, \quad (3.16)$$

and  $\overline{s}_1$  and  $s_1$  are taken as Theorem 3.1.

*Proof.* Similar to the proof of Theorem 3.1 the details are omitted. A simple proof is given as follows.

By (2.45), (2.46), and (2.47), we obtain

$$\overline{a}_{11}^\sharp = \sum_{n=0}^{\infty} \left( a_{11}^d \Delta a_{11} \right)^n a_{11}^d + a_{11}^\pi \sum_{n=0}^{\infty} \left[ (\Delta a_{11})^d \right]^{n+1} (-a_{11})^n. \quad (3.17)$$

In view of (3.17), we easily have

$$\begin{aligned} \overline{s}_1 &= a_{22} + \Delta a_{22} - (a_{21} + \Delta a_{21}) \overline{a}_{11}^\sharp (a_{12} + \Delta a_{12}) \\ &= s_1 + \Delta a_{22} - a_{21} \sum_{n=0}^{\infty} \left( a_{11}^d \Delta a_{11} \right)^n a_{11}^d a_{12} + a_{21} a_{11}^\pi \sum_{n=0}^{\infty} \left[ (\Delta a_{11})^d \right]^{n+1} (-a_{11})^n a_{12} \\ &\quad - a_{21} a_{11}^d \Delta a_{11} a_{11}^d a_{12} - \Delta a_{21} \overline{a}_{11}^\sharp a_{12} - a_{21} \overline{a}_{11}^\sharp \Delta a_{12} - \Delta a_{21} \overline{a}_{11}^\sharp \Delta a_{12}, \\ \|\overline{a}_{11}^\sharp\| &\leq \left\| \sum_{n=0}^{\infty} \left( a_{11}^d \Delta a_{11} \right)^n a_{11}^d \right\| + \left\| a_{11}^\pi \sum_{n=0}^{\infty} \left[ (\Delta a_{11})^d \right]^{n+1} (-a_{11})^n \right\| \\ &\leq \frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|} + \|a_{11}^\pi\| \sum_{n=0}^{\infty} \left\| \left[ (\Delta a_{11})^d \right]^{n+1} a_{11}^n \right\|. \end{aligned} \quad (3.18)$$

It shows from (3.18) that

$$\begin{aligned}
\|\bar{s}_1 - s_1\| &\leq \epsilon \|a_{22}\| + \frac{\|a_{11}^d\| \|a_{21}\| \|a_{12}\|}{1 - \|a_{11}^d \Delta a_{11}\|} + \|a_{21}\| \|a_{11}^\pi\| \sum_{n=0}^{\infty} \left\| \left[ (\Delta a_{11})^d \right]^{n+1} a_{11}^n \right\| \|a_{12}\| \\
&\quad + (2\epsilon + \epsilon^2) \|a_{21}\| \|\bar{a}_{11}^\# \| \|a_{12}\| + \epsilon \|a_{11}^d\| \|a_{11}\| \|a_{12}\| \|a_{21}\| \\
&\leq \epsilon \|a_{22}\| + \left\{ \epsilon \|a_{11}^d\| \|a_{11}\| + \frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|} + \|a_{11}^\pi\| \sum_{n=0}^{\infty} \left\| \left[ (\Delta a_{11})^d \right]^{n+1} a_{11}^n \right\| \right\} \|a_{21}\| \|a_{12}\| \\
&\quad + (2\epsilon + \epsilon^2) \|a_{21}\| \|a_{12}\| \left\{ \|a_{11}^\pi\| \sum_{n=0}^{\infty} \left\| \left[ (\Delta a_{11})^d \right]^{n+1} a_{11}^n \right\| + \frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|} \right\} \\
&= \epsilon \|a_{22}\| + \|a_{21}\| \|a_{12}\| \left[ (1 + 2\epsilon + \epsilon^2) \delta_1 + \epsilon \|a_{11}^d\| \|a_{11}\| \right],
\end{aligned} \tag{3.19}$$

where

$$\delta_1 = \left\{ \frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|} + \|a_{11}^\pi\| \sum_{n=0}^{\infty} \left\| \left[ (\Delta a_{11})^d \right]^{n+1} a_{11}^n \right\| \right\}. \tag{3.20}$$

Therefore, we complete the proof.  $\square$

Similar to Theorems 3.1 and 3.3. The proof of the following theorem follows from Theorem 2.15.

**Theorem 3.4.** Let  $\mathcal{M}$  and  $\overline{\mathcal{M}}$  be taken as Theorem 3.1, and let the relations in (3.6) be satisfied, where  $\epsilon > 0$ . If  $a_{11}$ ,  $\Delta a_{11}$  and  $\bar{a}_{11}$  satisfy the conditions of Theorem 2.15, then

$$\|\bar{s}_1 - s_1\| \leq \epsilon \|a_{22}\| + (1 + 2\epsilon + \epsilon^2) \|a_{21}\| \|a_{12}\| \delta_1, \tag{3.21}$$

where

$$\delta_1 = \left\{ \frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|} + \frac{\|a_{11}^\pi\| \|(\Delta a_{11})^d\|}{1 - \|(\Delta a_{11})^d a_{11}\|} \right\}, \tag{3.22}$$

and  $\bar{s}_1$  and  $s_1$  are taken as Theorem 3.1.

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