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Research Article

Strong Convergence Theorems for Asymptotically Weak G-Pseudo-Ψ-Contractive Non-Self-Mappings with the Generalized Projection in Banach Spaces

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A new concept of the asymptotically weak G-pseudo-Ψ-contractive non-self-mapping $T : G \mapsto B$ is introduced and some strong convergence theorems for the mapping are proved by using the generalized projection method combined with the modified successive approximation method or with the modified Mann iterative sequence method in a uniformly and smooth Banach space. The proof methods are also different from some past common methods.

1. Introduction

Let B be a real Banach space with the norm $\|\cdot\|$, B^* its dual space with the norm $\|\cdot\|_*$. As usually, we introduce a dual product in $B^* \times B$ by $\langle x^*, x\rangle$, where $x^* \in B^*$ and $x \in B$. Let $J : B \mapsto B^*$ be the normalized duality mapping J in B defined as

$$Jx = \left\{ f \in B^* : \langle f, x\rangle = \|f\|_* \|x\| = \|x\|^2 \right\}, \quad \forall x \in B. \tag{1.1}$$

It is clear that the operator J is well defined in a Banach space by the famous Hahn-Banach theorem.

The concept of asymptotically nonexpansive mappings was first introduced by Goebel and Kirk [1] in 1972 and then Schu [2] introduced the asymptotically pseudocontractive mappings in 1991.

Definition 1.1. Let G be a nonempty subset of a real Banach space B and $T : G \mapsto G$ be a mapping.

(1) The mapping T is said to be asymptotically nonexpansive, if there exists a number sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \tag{1.2}$$

for all $x, y \in G$ and $n \geq 1$.

(2) The mapping T is said to be asymptotically pseudocontractive, if for all $x, y \in G$, there exists a number sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - T^n y, j(x - y)\rangle \leq k_n \|x - y\|^2. \tag{1.3}$$

(3) The mapping T is said to be asymptotically demi-pseudocontractive, if for all $x \in G$, $p \in F(T)$, there exists a number sequence k_n in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - p, j(x - p)\rangle \leq k_n \|x - p\|^2, \tag{1.4}$$

where $F(T) \neq \emptyset$ is the set of all fixed points of the mapping T.

The iterative approximation problems for asymptotically nonexpansive and pseudocontractive mapping T were studied by many authors and we always assume that the fixed point set $F(T)$ of the operator T is nonempty, such as see [1–6]. In 2011, Qin et al. [7] introduced a new concept of the asymptotically strict quasi-Φ-pseudocontractive mapping $T : G \to G$. They combined the generalized projection Π_G to give a new iterative sequence for the T and proved that the sequence converges strongly to a point $x' = \Pi_{F(T)} x_0$.

But, all these arguments are not enough if the operator T acts from G to B, which we called non-self-mappings, and the iterative methods we used to be, such as Mann iterative method and its some modifications, can not be used. Under this condition, it is natural for us to try to consider the metric projection operator $P_G : B \mapsto G$ and the generalized projection operator $\Pi_G : B \mapsto G$, and some authors have given relevant results and applications of the operator P_G and π_G (see [8–11]).

Very recently, in 2012, Yao et al. [12] and Liou et al. [13] considered the non-self-mapping $T : G \subseteq H \to H$ in a Hilbert space H. They also proved their new iterative sequence for the T combined with the metric projection P_G converges strongly to a point $x' = P_{VI(G,T)}(0)$ and the unique solution of a variational inequality, respectively.

Motivated and inspired by the said above, we first introduce a new concept of the asymptotically weak G-pseudo-Ψ-contractive non-self-mapping $T : G \mapsto B$. Then, in a uniformly convex and smooth Banach space, we prove some strong convergence theorems for the mapping by using the generalized projection method and the modified successive approximation method

$$x_{n+1} = (\Pi_G T)^n x_n, \quad n = 1, 2, \ldots, x_1 \in G, \tag{1.5}$$

or the modified Mann iterative sequence method

$$x_{n+1} = Q_G\Big((1-\alpha_n)x_n + \alpha_n T(\Pi_G T)^{n-1}x_n\Big), \quad n = 1, 2, \ldots, x_1 \in G, \tag{1.6}$$

where $Q_G : B \mapsto G$ is a sunny nonexpansive retraction. So, in some ways, our results extend and improve some results of other authors (such as, see [1–5, 7, 9–13]), from self mappings to non-self-mappings, from Hilbert spaces to Banach spaces.

2. Preliminaries

In the sequel, we will assume that B is a real uniformly convex and uniformly smooth (hence reflexive) Banach space, then B^* will be the same. If we denote by $\delta_B(\varepsilon)$ the modulus of convexity of the Banach space B and by $\rho_B(\tau)$ its modulus of smoothness, then

$$\delta_B(\varepsilon), \rho_B(\tau), g_B(\varepsilon) = \varepsilon^{-1}\delta_B(\varepsilon), \qquad h_B(\tau) = \tau^{-1}\rho_B(\tau) \tag{2.1}$$

are all continuous and increasing on their domains, respectively, and $\delta_B(0) = \rho_B(0) = g_B(0) = h_B(0) = 0$ (see [9]). Also, under the conditions the normalized duality operator

$$J : Jx = \frac{1}{2}\text{grad}\Big\{\|x\|^2\Big\} \tag{2.2}$$

is single-valued, strictly monotone, continuous, coercive, bounded, and homogeneous, but not addible. In a Hilbert space, J is the Identity operator $I : Ix = x$.

Definition 2.1 (see [10, 11]). The operator $P_G : B \mapsto G \subseteq B$ is called metric projection operator if it assigns to each $x \in B$ its nearest point $\overline{x} \in G$, that is, the solution $\overline{x}$ for the minimization problem

$$P_G x = \overline{x}; \quad \overline{x} : \|x - \overline{x}\| = \inf_{\xi \in G}\|x - \xi\|. \tag{2.3}$$

The operator $\Pi_G : B \mapsto G \subseteq B$ is called the generalized projection operator if it assigns to each $x \in B$ a minimum point $\widehat{x} \in G$ of the Lapunov function $V(x, \xi) : B \times B \mapsto [0, \infty)$:

$$V(x, \xi) = \|x\|^2 - 2\langle Jx, \xi\rangle + \|\xi\|^2, \tag{2.4}$$

that is, a solution of the following minimization problem:

$$\Pi_G x = \widehat{x}; \quad \widehat{x} : V(x, \widehat{x}) = \inf_{\xi \in G} V(x, \xi). \tag{2.5}$$

Lemma 2.2 (see [10, 11]). *The point $\overline{x} = P_G x$ is the metric projection of $x \in B$ on $G \subseteq B$ if and only if the following inequality is satisfied:*

$$\langle J(x - \overline{x}), \overline{x} - \xi\rangle \geq 0, \quad \forall \xi \in G, \tag{2.6}$$

and the operator P_G is nonexpansive in Hilbert spaces.

The point $\hat{x} = \Pi_G x$ *is the generalized projection of* $x \in B$ *on* $G \subseteq B$ *if and only if the following inequality is satisfied:*

$$\langle Jx - J\hat{x}, \hat{x} - \xi \rangle \geq 0, \quad \forall \xi \in G. \tag{2.7}$$

Furthermore, the inequality below also holds:

$$V(\Pi_G x, \xi) \leq V(x, \xi) - V(x, \Pi_G x), \quad \forall \xi \in G. \tag{2.8}$$

And thus, we have

$$V(\Pi_G x, \xi) \leq V(x, \xi), \quad \forall \xi \in G. \tag{2.9}$$

Lemma 2.3 (see [8]). *For all* $x, y \in B$, *if* $\|x\| \leq R$ *and* $\|y\| \leq R$, *then the following inequality is satisfied:*

$$(2L)^{-1} R^2 \delta_B\left(\frac{\|x-y\|}{4R}\right) \leq V(x,y) \leq 4LR^2 \rho_B\left(\frac{4\|x-y\|}{R}\right), \tag{2.10}$$

where $L : 1 < L < 1.7$ *is a constant.*

In general, the operator P_G and Π_G are not nonexpansive in Banach spaces. It is easy to see $P_G = \Pi_G$ in Hilbert spaces because of $J = I$. In a uniformly convex and uniformly smooth Banach space, P_G is well defined on a closed convex set G and Π_G is also well defined on a closed convex set G from the properties of the functional $V(x, \xi)$ and strict monotonicity of the mapping J. More properties of the mappings J, V, P_G, and Π_G and some of their applications can be found in [8–11].

Definition 2.4 (see [14]). Let B be a real Banach space, $G \subseteq B$ be a subset. The operator $Q_G : B \mapsto G$ is called sunny nonexpansive retract if Q_G is nonexpansive, $Q_G^2 = Q_G$, and for any $x \in G$, $t > 0$, $tx + (1-t)Q_G x \in G$ holds $Q_G(tx + (1-t)Q_G x) = Q_G x$.

If B is a uniformly smooth Banach space and $G \subset B$, is a closed convex set, then the unique sunny nonexpansive retract Q_G exists.

Definition 2.5. Let B be a real Banach space, G be a nonempty subset of B, and $T : G \mapsto B$ be a non-self-mapping. If there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and a continuous increasing function $\Psi(t)$ for all $t > 0$ with $\Psi(0) = 0$, $\lim_{t\to\infty}\Psi(t) = \infty$, it is shown as follows, respectively:

(1) The mapping T is said to be asymptotically weak G-Ψ-contractive mapping, if

$$V\left(T(\Pi_G T)^{n-1}x, T(\Pi_G T)^{n-1}y\right) \leq k_n V(x,y) - \Psi(V(x,y)). \tag{2.11}$$

(2) The mapping T is said to be asymptotically weak G-quasi-Ψ-contractive mapping, if

$$V\left(T(\Pi_G T)^{n-1}x, x^*\right) \le k_n V(x, x^*) - \Psi(V(x, x^*)), \tag{2.12}$$

for all $x, y \in G$, $x^* \in F(T)$, and $n \ge 1$, where $F(T) \ne \emptyset$.

(3) The mapping T is said to be asymptotically weak G-Ψ-pseudocontractive mapping, if

$$\left\langle T(\Pi_G T)^{n-1}x - T(\Pi_G T)^{n-1}y, j(x-y) \right\rangle \le k_n \|x-y\|^2 - \Psi\left(\|x-y\|^2\right), \tag{2.13}$$

for all $x, y \in G$, $j(x-y) \in J(x-y)$.

(4) The mapping T is said to be asymptotically weak G-quasi-Ψ-pseudocontractive mapping, if

$$\left\langle T(\Pi_G T)^{n-1}x - x^*, j(x-x^*) \right\rangle \le k_n \|x-x^*\|^2 - \Psi\left(\|x-x^*\|^2\right), \tag{2.14}$$

for all $x, y \in G$, $x^* \in F(T)$, $j(x-x^*) \in J(x-x^*)$, and $n \ge 1$, where $F(T) \ne \emptyset$.

Remark 2.6. It is clear that one can omit each operator Π_G in (2.11)–(2.14) if the mapping T acts from G to G, that is, $T : G \mapsto G \subseteq B$ is a self-mapping. So, the class of asymptotically weak G-Ψ-contractive mappings contains that of asymptotically nonexpansive mappings and the class of asymptotically weak G-Ψ-pseudocontractive mappings contains that of asymptotically pseudocontractive mappings. Therefore, all the results and applications of asymptotically nonexpansive mappings can be as a part of the asymptotically weak G-Ψ-contractive mappings.

In order to prove our main results, we also need the following lemmas.

Lemma 2.7 (see [15]). *Let $\{\lambda_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be sequences of nonnegative numbers satisfying the following conditions:*

$$\alpha_n > 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \sum_{n=1}^{\infty} \beta_n < \infty, \qquad \lim_{n\to\infty} \frac{\gamma_n}{\alpha_n} = 0. \tag{2.15}$$

Suppose the following recurse inequality holds:

$$\lambda_{n+1} \le (1+\beta_n)\lambda_n - \alpha_n \psi(\lambda_n) + \gamma_n, \quad n = 1, 2, \ldots, \tag{2.16}$$

where $\psi(t)$ is a continuous strictly increasing function for all $t > 0$ with $\psi(0) = 0$, $\lim_{t\to\infty}\psi(t) = \infty$. Then $\lambda_n \to 0$ as $n \to \infty$.

Lemma 2.8 (see [16]). *Let B be a real Banach space and J be the normalized duality mapping. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \tag{2.17}$$

for all $x, y \in B$ and $j(x+y) \in J(x+y)$.

3. Main Results

Theorem 3.1. *Let B be a uniformly convex and uniformly smooth Banach space, G be a closed convex subset of B, $T : G \mapsto B$ be an asymptotically weak G-quasi-Ψ-contractive mapping with a sequence $\{k_n\} \subseteq [1,\infty)$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, and $x^* \in G$ is its fixed point. Then the iterative sequence $\{x_n\}$ generated by the modified successive approximation method* (1.5) *is bounded for all $n \geq 1$ and converges strongly to x^*.*

Proof. If $x^* \in F(T)$ is the fixed point of T in G, that is, $Tx^* = x^*$, then we get by (1.5) and (2.8) in Lemma 2.2 for $x^* \in G$,

$$\begin{aligned} V(x_{n+1}, x^*) = V\left((\Pi_G T)^n x_n, x^*\right) &= V\left(\Pi_G T(\Pi_G T)^{n-1} x_n, x^*\right) \\ &\leq V\left(T(\Pi_G T)^{n-1} x_n, x^*\right). \end{aligned} \tag{3.1}$$

We use the condition (2.12) of asymptotically weak G-quasi-Ψ-contractive of the operator T and get

$$\begin{aligned} V(x_{n+1}, x^*) &\leq k_n V(x_n, x^*) - \psi(V(x_n, x^*)) \leq k_n V(x_n, x^*) \\ &\leq k_n k_{n-1} V(x_{n-1}, x^*) \leq \cdots \leq k_n k_{n-1} \cdots k_1 V_1(x_1, x^*). \end{aligned} \tag{3.2}$$

Because $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, we know $\lim_{n\to\infty} k_n k_{n-1} \cdots k_1 = \text{constant}$ and $\{k_n k_{n-1} \cdots k_1\}$ is bounded, say by $K : 1 \leq k_n k_{n-1} \cdots k_1 \leq K$ for all $n \geq 1$.

It is obviously that $V(x,\xi) = \|x\|^2 - 2\langle Jx, \xi\rangle + \|\xi\|^2$ satisfies the inequality

$$(\|x\| - \|\xi\|)^2 \leq V(x,\xi) \leq (\|x\| + \|\xi\|)^2. \tag{3.3}$$

Therefore by (3.2) and (3.3), we have

$$\begin{aligned} &\|x_n\| - \|x^*\| \leq \sqrt{k_n k_{n-1} \cdots k_1 V_1(x_1, x^*)} \leq \sqrt{K}(\|x_1\| + \|x^*\|), \\ &\|x_n\| \leq \sqrt{K}\|x_1\| + \left(1 + \sqrt{K}\right)\|x^*\|, \end{aligned} \tag{3.4}$$

for all $n \geq 1$, that is, the sequence $\{x_n\}$ is bounded.

The sequence of positive number $\{\lambda_n\}$ defined by $\lambda_n = V(x_n, x^*)$ are bounded and from (3.2) it satisfies the following inequality:

$$\lambda_{n+1} \leq (1+\beta_n)\lambda_n - \alpha_n \Psi(\lambda_n) + \gamma_n, \quad n = 1, 2, \ldots, \tag{3.5}$$

where $\beta_n = k_n - 1$, $\sum_{n=1}^{\infty} \beta_n < \infty$, $\alpha_n = 1$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\gamma_n = 0$. So, using Lemma 2.7 we get

$$\lim_{n \to \infty} V(x_n, x^*) = 0. \tag{3.6}$$

Because $R = \sqrt{K}\|x_1\| + (1 + \sqrt{K})\|x^*\|$ is a constant and $\|x_n\| \leq R$, $\|y_n\| \leq R$, we obtain from the left part of the estimate of (2.10) in Lemma 2.3 the following:

$$0 \leq \lim_{n \to \infty} 2L^{-1} R^2 \delta_B \left(\frac{\|x_n - x^*\|}{4R} \right) \leq \lim_{n \to \infty} V(x_n, x^*) = 0. \tag{3.7}$$

By the properties of $\delta_B(\varepsilon)$, this implies

$$\lim_{n \to \infty} \|x_n - x^*\| = 0, \tag{3.8}$$

that is, the sequence $\{x_n\}$ converges strongly to fixed point x^*. □

Corollary 3.2. *Let G be a closed convex set in B, $T : G \mapsto B$ be an asymptotically weak G-Ψ-contractive mapping with a sequence $\{k_n\} \subseteq [1, \infty)$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, and $x^* \in G$ its fixed point. Then the iterative sequence $\{x_n\}$ defined by modified successive approximation method* (1.5) *converges strongly to x^*.*

Proof. If we take $y = x^* \in G$ as the fixed point of T, then we have

$$\begin{aligned} T(\pi_G T)^{n-1} x^* &= T(\pi_G T)^{n-2}(\pi_G T x^*) = T(\pi_G T)^{n-2}(\pi_G x^*) \\ &= T(\pi_G T)^{n-2} x^* = \cdots = T x^* = x^*. \end{aligned} \tag{3.9}$$

□

So the asymptotically weak G-Ψ-contractive mapping $T : G \mapsto B$ is also an asymptotically weak G-quasi-Ψ-contractive mapping and the results of Theorem 3.1 still hold.

Theorem 3.3. *Let B be a real uniformly convex and uniformly smooth Banach space, G be a nonempty closed convex subset of B, $T : G \mapsto B$ be an asymptotically weak G-quasi-Ψ-pseudocontractive mapping with a sequence $\{k_n\} \subseteq [1, \infty)$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, and $x^* \in G$ its fixed point. Consider the iterative sequence $\{x_n\}$ defined by the modified Mann iterative sequence method* (1.6). *Suppose the sequence $\{x_n\}$ and $\{T(\Pi_G T)^{n-1} x_n\}$ are bounded, $\{\alpha_n\}$ is a number sequence in $(0, 1]$ satisfing the conditions below:*

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \sum_{n=1}^{\infty} \alpha_n^2 < \infty, \tag{3.10}$$

where $Q_G : B \mapsto G$ is a sunny nonexpansive retraction. Then the iterative sequence $\{x_n\}$ converges strongly to x^*.

Proof. By the virtue of (2.17) in Lemma 2.8, it follows that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\| Q_G\Big((1-\alpha_n)x_n + \alpha_n T(\Pi_G T)^{n-1}x_n\Big) - Q_G x^* \right\|^2 \\
&\leq \left\|(1-\alpha_n)x_n + \alpha_n T(\Pi_G T)^{n-1}x_n - x^*\right\|^2 \\
&= \left\|(1-\alpha_n)(x_n - x^*) + \alpha_n\Big(T(\Pi_G T)^{n-1}x_n - x^*\Big)\right\|^2 \\
&\leq (1-\alpha_n)^2\|(x_n - x^*)\|^2 + 2\alpha_n\Big\langle T(\Pi_G T)^{n-1}x_n - x^*, j(x_{n+1} - x^*)\Big\rangle \\
&= (1-\alpha_n)^2\|(x_n - x^*)\|^2 + 2\alpha_n\Big\langle T(\Pi_G T)^{n-1}x_n - x^*, j(x_n - x^*)\Big\rangle \\
&\quad + 2\alpha_n\Big\langle T(\Pi_G T)^{n-1}x_n - x^*, j(x_{n+1} - x^*) - j(x_n - x^*)\Big\rangle.
\end{aligned} \tag{3.11}$$

Since $\{x_n - T(P_G T)^{n-1}x_n\}$ is bounded, say by K, we have

$$\begin{aligned}
\|(x_{n+1} - x^*) - (x_n - x^*)\| &= \left\| Q_G\Big((1-\alpha_n)x_n + \alpha_n T(P_G T)^{n-1}x_n\Big) - Q_G x_n \right\| \\
&\leq \left\|(1-\alpha_n)x_n + \alpha_n T(P_G T)^{n-1}x_n - x_n\right\| \\
&= \alpha_n\left\|x_n - T(P_G T)^{n-1}x_n\right\| \\
&\leq \alpha_n K.
\end{aligned} \tag{3.12}$$

□

From (3.10) we know $\lim_{n\to\infty}\alpha_n = 0$ and then one gets

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\|(x_{n+1} - x^*) - (x_n - x^*)\| = 0. \tag{3.13}$$

By using the uniform continuity of $j = J$ in the uniformly convex and uniform smooth Banach space B and the bound of the sequence $\{T(\Pi_G T)^{n-1}x_n - x^*\}$, we have

$$\gamma_n := \Big\langle T(\Pi_G T)^{n-1}x_n - x^*, j(x_{n+1} - x^*) - j(x_n - x^*)\Big\rangle \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.14}$$

Substituting (3.14) into (3.11) and using (2.14), we get

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\quad\le (1-\alpha_n)^2\|x_n-x^*\|^2+2\alpha_n\left\langle T(\Pi_G T)^{n-1}x_n-x^*, j(x_n-x^*)\right\rangle+2\alpha_n\gamma_n \\
&\quad\le (1-\alpha_n)^2\|x_n-x^*\|^2+2\alpha_n\left(k_n\|x_n-x^*\|^2-\Psi\left(\|x_n-x^*\|^2\right)\right)+2\alpha_n\gamma_n \\
&\quad= \left(1+2\alpha_n(k_n-1)+\alpha_n^2\right)\|x_n-x^*\|^2-2\alpha_n\Psi\left(\|x_n-x^*\|^2\right)+2\alpha_n\gamma_n \\
&\quad\le \left(1+2(k_n-1)+\alpha_n^2\right)\|x_n-x^*\|^2-2\alpha_n\Psi\left(\|x_n-x^*\|^2\right)+2\alpha_n\gamma_n.
\end{aligned} \tag{3.15}$$

Thus, the sequence of positive number $\{\lambda_n\}_{n=1}^{\infty}$ defined by $\lambda_n=\|x_n-x^*\|^2$ satisfies the recursive inequality

$$\lambda_{n+1}\le(1+\beta_n)\lambda_n-2\alpha_n\psi\left(\|x_n-x^*\|^2\right)+2\alpha_n\gamma_n, \tag{3.16}$$

where $\beta_n=2(k_n-1)+\alpha_n^2$, $\sum_{n=1}^{\infty}\beta_n=2\sum_{n=1}^{\infty}(k_n-1)+\sum_{n=1}^{\infty}\alpha_n^2<\infty$, $\sum_{n=1}^{\infty}(2\alpha_n)=\infty$, $\gamma_n\to 0$ as $n\to\infty$. Therefore by the virtue of Lemma 2.7, it is clear that the assertion $\lambda_n\to 0$ as $n\to\infty$ holds, that is,

$$\lim_{n\to\infty}\|x_n-x^*\|^2=0, \qquad \lim_{n\to\infty}\|x_n-x^*\|=0. \tag{3.17}$$

Corollary 3.4. *Let B be a real uniformly convex and uniformly smooth Banach space, G be a nonempty closed convex subset of B, $T: G\mapsto B$ be an asymptotically weak G-Ψ-pseudocontractive mapping with a sequence $\{k_n\}\subseteq[1,\infty)$, $\sum_{n=1}^{\infty}(k_n-1)<\infty$, and $x^*\in G$ its fixed point. Consider the iterative sequence $\{x_n\}$ defined by the modified Mann iterative sequence* (1.6)*. Suppose the sequence $\{x_n\}$ and $\{T(\Pi_G T)^{n-1}x_n\}$ are bounded, $\{\alpha_n\}$ is a real number sequence in $(0,1]$ satisfying the conditions* (3.10)*. Then one has*

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0, \tag{3.18}$$

and the iterative sequence $\{x_n\}$ converges in norm to x^*.

Proof. Following Theorem 3.3, we can have the assertions of the corollary. □

Remark 3.5. Because a Hilbert space must be a uniformly convex and uniformly smooth Banach space, the above results still hold in a Hilbert space. In fact, if we notice $\Pi_G=P_G$ in Hilbert spaces, we can abate some conditions in Corollary 3.4 and have the following theorem.

Theorem 3.6. *Let G be a closed convex set of a Hilbert space H. $T : G \mapsto H$ is said to be an asymptotically weak G-quasi-Ψ-pseudocontractive mapping with a sequence $\{k_n\} \subseteq [1,\infty)$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, and $x^* \in G$ its fixed point, if*

$$\left\| T(P_G T)^{n-1} x - x^* \right\| \leq k_n \|x - x^*\| - \psi(\|x - x^*\|), \tag{3.19}$$

where Ψ is a continuous increasing function for all $t > 0$ with $\Psi(0) = 0$, $\lim_{t\to\infty}\Psi(t) = \infty$. Consider the new modified Mann iterative sequence $\{x_n\}$ defined by the modified Mann iterative sequence (1.6). *If the number sequence $\{\alpha_n\}$ satisfies the conditions*

$$0 < \alpha_n \leq 1, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \tag{3.20}$$

then the iterative sequence $\{x_n\}$ converges strongly to x^*.

Proof. Because $P_G = \Pi_G$ is nonexpansive in Hilbert spaces, $\{\alpha_n\}$ satisfies (3.20) and the operator T satisfies (3.19), we get

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \left\| P_G\left((1-\alpha_n)x_n + \alpha_n T(P_G T)^{n-1} x_n\right) - P_G x^* \right\| \\
&\leq \left\| (1-\alpha_n)x_n + \alpha_n T(P_G T)^{n-1} x_n - x^* \right\| \\
&= \left\| (1-\alpha_n)(x_n - x^*) + \alpha_n\left(T(P_G T)^{n-1} x_n - x^*\right) \right\| \\
&\leq (1-\alpha_n)\|(x_n - x^*)\| + \alpha_n \left\| T(P_G T)^{n-1} x_n - x^* \right\| \\
&\leq (1-\alpha_n)\|(x_n - x^*)\| + \alpha_n \left(k_n \|(x_n - x^*)\| - \psi(\|(x_n - x^*)\|)\right) \\
&= (1 + \alpha_n(k_n - 1))\|(x_n - x^*)\| - \alpha_n \psi(\|(x_n - x^*)\|) \\
&\leq k_n \|(x_n - x^*)\| - \alpha_n \psi(\|(x_n - x^*)\|).
\end{aligned} \tag{3.21}$$

Denote $\lambda_n = \|x_n - x^*\|$ and we have the following inequality:

$$\lambda_{n+1} \leq (1 + \beta_n)\lambda_n - \alpha_n \psi(\lambda_n) + \gamma_n, \tag{3.22}$$

where $\beta_n = k_n - 1$, $\sum_{n=1}^{\infty} \beta_n < \infty$, $\alpha_n \in (0,1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\gamma_n = 0$. Therefore we know $\lambda_n \to 0$ as $n \to \infty$ by using Lemma 2.7, that is,

$$\lim_{n\to\infty} \|x_n - x^*\| = 0. \tag{3.23}$$

□

Remark 3.7. It is clear that the above results, in some ways, extend and improve some results of other authors (such as, see [1–5, 7, 9–13]), from self mappings to non-self-mappings, from

Hilbert spaces to Banach spaces. And in the proof process, our methods are different from some past common methods.

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