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Research Article

Approximate Cubic $*$-Derivations on Banach $*$-Algebras

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We study the stability of cubic $*$-derivations on Banach $*$-algebras. We also prove the superstability of cubic $*$-derivations on a Banach $*$-algebra A, which is left approximately unital.

1. Introduction

In [1], Ulam proposed the stability problems for functional equations concerning the stability of group homomorphisms. In fact, a functional equation is called *stable* if any approximately solution to the functional equation is near a true solution of that functional equation and is *superstable* if every approximate solution is an exact solution to it. In [2], Hyers considered the case of approximate additive mappings in Banach spaces and satisfying the well-known weak Hyers inequality controlled by a positive constant. Bourgin [3] was the second author to treat this problem for additive mappings (see also [4]). In [5], Rassias provided a generalization of Hyers Theorem, which allows the Cauchy difference to be unbounded. Găvruţa then generalized the Rassias' result in [6] for the unbounded Cauchy difference. Subsequently, various approaches to the problem have been studied by a number of authors (see, e.g., [7–11]).

Recall that a Banach $*$-algebra is a Banach algebra (complete normed algebra) which has an isometric involution. For a locally compact group G, the algebraic group algebra $L^1(G)$ is a Banach $*$-algebra. The bounded operators on Hilbert space $\mathscr{H}$ is also a Banach $*$-algebra. In general, all C^*-algebras are Banach $*$-algebra. A left- (right-) bounded approximate identity for a normed algebra $\mathcal{A}$ is a bounded net $(e_j)_j$ in $\mathcal{A}$ such that $\lim_j e_j a = a$ ($\lim_j a e_j = a$) for each $a \in \mathcal{A}$. A bounded approximate identity for $\mathcal{A}$ is a bounded net $(e_j)_j$, which is both a left- and a right-bounded approximate identity. Every group algebra and every C^*-algebra has a bounded approximate identity.

The stability of functional equations of $*$-derivations and of quadratic $*$-derivations with the Cauchy functional equation and the Jensen functional equation on Banach $*$-algebras is investigated in [12]. The author also proved the superstability of $*$-derivations and of quadratic $*$-derivations on C^*-algebras.

In 2003, Cădariu and Radu employed the fixed point method to the investigation of the Jensen functional equation. They presented a short and a simple proof (different from the "*direct method*," initiated by Hyers in 1941) for the Cauchy functional equation [13] and for the quadratic functional equation [14] (see also [15–18]).

The functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \tag{1.1}$$

which is called cubic functional equation. In addition, every solution of functional equation (1.1) is said to be a *cubic mapping*. It is easy to check that function $f(x) = ax^3$ is a solution of (1.1).

In [19], Bodaghi et al. proved the generalized Hyers-Ulam stability and the superstability for the functional equation (1.1) by using the alternative fixed point (Theorem 3.1) under certain conditions on Banach algebras. Also, the stability and the superstability of homomorphisms on C^*-algebras by using the same fixed point method was proved in [20]. The generalized Hyers-Ulam-Rassias stability of $*$-homomorphisms between unital C^*-algebras associated with the Trif functional equation and of linear $*$-derivations on unital C^*-algebras has earlier been proved by Park and Hou in [21].

In this paper, we prove the stability and the superstability of cubic $*$-derivations on Banach $*$-algebras. We also show that these functional equations, under some mild conditions, are superstable. We also establish the stability and the superstability of cubic $*$-derivations on a Banach $*$-algebra with a left-bounded approximate identity.

2. Stability of Cubic $*$-Derivation

Throughout this paper, we assume that A is a Banach $*$-algebra. A mapping $D : A \to A$ is a cubic derivation if D is a cubic homogeneous mapping, that is, D is cubic and $D(\mu a) = \mu^3 D(a)$ for all $a \in A$ and $\mu \in \mathbb{C}$, and $D(ab) = D(a)b^3 + a^3 D(b)$ for all $a, b \in A$. In addition, if D satisfies in condition $D(a^*) = D(a)^*$ for all $a \in A$, then it is called the cubic $*$-derivation. An example of cubic derivations on Banach algebras is given in [22].

Let $\mu \in \mathbb{C}$. For the given mapping $f : A \to A$, we consider

$$\begin{gathered} \mathfrak{D}_\mu f(a,b) := f(2\mu a + \mu b) + f(2\mu a - \mu b) - 2\mu^3 f(a+b) - 2\mu^3 f(a-b) - 12\mu^3 f(a), \\ \mathfrak{D} f(a,b) = f(ab) - f(a)b^3 - a^3 f(b) \end{gathered} \tag{2.1}$$

for all $a, b \in A$.

Theorem 2.1. *Suppose that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : A^5 \to [0,\infty)$ such that*

$$\widetilde{\varphi}(a,b,x,y,z) := \sum_{k=0}^{\infty} \frac{1}{8^k}\varphi\left(2^k a, 2^k b, 2^k x, 2^k y, 2^k z\right) < \infty, \tag{2.2}$$

$$\left\|\mathfrak{D}_\mu f(a,b)\right\| \le \varphi(a,b,0,0,0), \tag{2.3}$$

$$\left\|\mathfrak{D} f(x,y) + f(z^*) - f(z)^*\right\| \le \varphi(0,0,x,y,z), \tag{2.4}$$

for all $\mu \in \mathbb{T}^1_{1/n_0} = \{e^{i\theta} : 0 \le \theta \le 2\pi/n_0\}$ and all $a,b,x,y,z \in A$ in which $n_0 \in \mathbb{N}$. Also, if for each fixed $a \in A$ the mapping $t \mapsto f(ta)$ from $\mathbb{R}$ to A is continuous, then there exists a unique cubic $*$-derivation D on A satisfying

$$\left\|f(a) - D(a)\right\| \le \frac{1}{16}\widetilde{\varphi}(a), \quad (a \in A), \tag{2.5}$$

in which $\widetilde{\varphi}(a) = \widetilde{\varphi}(a,0,0,0,0)$.

Proof. Putting $b = 0$ and $\mu = 1$ in (2.3), we have

$$\left\|\frac{1}{8}f(2a) - f(a)\right\| \le \frac{1}{16}\varphi(a) \tag{2.6}$$

for all $a \in A$ in which $\varphi(a) = \varphi(a,0,0,0,0)$. We can use induction to show that

$$\left\|\frac{f(2^n a)}{8^n} - \frac{f(2^m a)}{8^m}\right\| \le \frac{1}{16}\sum_{k=m}^{n-1}\frac{\varphi(2^k a)}{8^k} \tag{2.7}$$

for all $a \in A$ and $n > m \ge 0$. On the other hand,

$$\left\|\frac{f(2^n a)}{8^n} - f(a)\right\| \le \frac{1}{16}\sum_{k=0}^{n-1}\frac{\varphi(2^k a)}{8^k} \tag{2.8}$$

for all $a \in A$ and $n > 0$. It follows from (2.2) and (2.7) that the sequence $\{f(2^n a)/8^n\}$ is a Cauchy sequence. Since A is a Banach algebra, this sequence converges to the map D, that is,

$$\lim_{n\to\infty}\frac{f(2^n a)}{8^n} = D(a). \tag{2.9}$$

Thus the inequalities (2.2) and (2.8) show that (2.5) holds. Substituting a,b by $2^n a$, $2^n b$, respectively, in (2.3), we get

$$\left\|\mathfrak{D}_\mu D(a,b)\right\| = \lim_{n\to\infty}\frac{1}{8^n}\left\|\mathfrak{D}_\mu f(2^n a, 2^n b)\right\| \le \lim_{n\to\infty}\frac{\varphi(2^n a, 2^n b, 0,0,0)}{8^n} = 0 \tag{2.10}$$

for all $a, b \in A$ and $\mu \in \mathbb{T}^1_{1/n_0}$. Since $\mathfrak{D}_1 D(a,b) = 0$, the mapping D is cubic. The equality $\mathfrak{D}_\mu D(a,0) = 0$ implies that $D(\mu a) = \mu^3 D(a)$ for all $a \in A$ and $\mu \in \mathbb{T}^1_{1/n_0}$. Now, let $\mu \in \mathbb{T}^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ such that $\mu = e^{i\theta}$ in which $0 \le \theta < 2\pi$. We set $\mu_1 = e^{i\theta/n_0}$, thus μ_1 belongs to $\mathbb{T}^1_{1/n_0}$ and $D(\mu a) = D(\mu_1^{n_0} a) = \mu_1^{3n_0} D(a) = \mu^3 D(a)$ for all $a \in A$. Now, suppose that $\mathcal{F}$ is any continuous linear functional on A and a is a fixed element in A. Define the mapping $g : \mathbb{R} \to \mathbb{R}$ via $g(\mu) = \mathcal{F}[D(\mu a)]$ for each $\mu \in \mathbb{R}$. Obviously, g is a cubic function. Under the hypothesis that $f(ta)$ is continuous in $t \in \mathbb{R}$ for each fixed $a \in A$, the function g is the pointwise limit of the sequence of measurable functions $\{g_n\}$ in which $g_n(\mu) = \mathcal{F}(2^n \mu a)/8^n$, $n \in \mathbb{N}$, $\mu \in \mathbb{R}$. Hence, g is a continuous function and has the form $g(\mu) = \mu^3 g(1)$ for all $\mu \in \mathbb{R}$. Therefore,

$$\mathcal{F}[D(\mu a)] = g(\mu) = \mu^3 g(1) = \mu^3 \mathcal{F}[D(a)] = \mathcal{F}\left[\mu^3 D(a)\right]. \tag{2.11}$$

Since $\mathcal{F}$ is an arbitrary continuous linear functional on A, $D(\mu a) = \mu^3 D(a)$ for all $\mu \in \mathbb{R}$ and $a \in A$. Thus

$$D(\mu a) = D\left(\frac{\mu}{|\mu|}|\mu| a\right) = \frac{\mu^3}{|\mu|^3} D(|\mu| a) = \frac{\mu^3}{|\mu|^3} |\mu|^3 D(a) = \mu^3 D(a) \tag{2.12}$$

for all $a \in A$ and $\mu \in \mathbb{C}$ $(\mu \neq 0)$. Therefore, D is a cubic homogeneous. If we replace x, y by $2^n x$, $2^n y$, respectively, and put $z = 0$ in (2.4), we have

$$\frac{1}{8^{2n}} \|\mathfrak{D} f(2^n x, 2^n y)\| \le \frac{\varphi(0,0,2^n x, 2^n y, 0)}{8^{2n}} \le \frac{\varphi(0,0,2^n x, 2^n y, 0)}{8^n} \tag{2.13}$$

for all $x, y \in A$. Taking the limit as n tends to infinity, we get $\mathfrak{D} D(x,y) = 0$, for all $x, y \in A$. Putting $x = y = 0$ and substituting z by $2^n z$ in (2.4) and then dividing the both sides of the obtained inequality by 8^n, then we get

$$\left\| \frac{f(2^n z^*)}{8^n} - \frac{f(2^n z)^*}{8^n} \right\| \le \frac{\varphi(0,0,0,0,2^n z)}{8^n} \tag{2.14}$$

for all $z \in A$. Passing to the limit as $n \to \infty$ in (2.14), we get $D(z^*) = D(z)^*$ for all $z \in A$. This shows that D is a cubic $*$-derivation.

Now, let $D' : A \to A$ be another cubic $*$-derivation satisfying (2.5). Then we have

$$\begin{aligned} \|D(a) - D'(a)\| &= \frac{1}{8^n} \|D(2^n a) - D'(2^n a)\| \\ &\le \frac{1}{8^n} \left(\|D(2^n a) - f(2^n a)\| + \|f(2^n a) - D'(2^n a)\| \right) \\ &\le \frac{1}{8^{n+1}} \widetilde{\varphi}(2^n a) = \frac{1}{8} \sum_{k=n}^{\infty} \frac{1}{8^k} \varphi\left(2^k a\right), \end{aligned} \tag{2.15}$$

which tends to zero as $n \to \infty$ for all $a \in$ A. So we can conclude that $D(a) = D'(a)$ for all $a \in A$. This proves the uniqueness of D. □

We have the following theorem, which is analogous to Theorem 2.1. Since the proof is similar, it is omitted.

Theorem 2.2. *Suppose that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : A^5 \to [0,\infty)$ satisfying* (2.3), (2.4), *and*

$$\tilde{\varphi}(a,b,x,y,z) := \sum_{k=1}^{\infty} 8^k \varphi\left(2^{-k}a, 2^{-k}b, 2^{-k}x, 2^{-k}y, 2^{-k}z\right) < \infty \tag{2.16}$$

for all $a,b,x,y,z \in A$. Also, if for each fixed $a \in A$ the mappings $t \mapsto f(ta)$ from $\mathbb{R}$ to A is continuous, then there exists a unique cubic $*$-derivation D on A satisfying

$$\|f(a) - D(a)\| \le \frac{1}{16}\tilde{\varphi}(a), \quad (a \in A), \tag{2.17}$$

where $\tilde{\varphi}(a) = \tilde{\varphi}(a,0,0,0,0)$.

Corollary 2.3. *Let θ, r be positive real numbers with $r \neq 3$, and let $f : A \to A$ be a mapping with $f(0) = 0$ such that*

$$\begin{gathered} \|\mathfrak{D}_\mu f(a,b)\| \le \theta(\|a\|^r + \|b\|^r), \\ \|\mathfrak{D} f(x,y) + f(z^*) - f(z)^*\| \le \theta(\|x\|^r + \|y\|^r + \|z\|^r), \end{gathered} \tag{2.18}$$

for all $\mu \in \mathbb{T}^1_{1/n_0}$ and all $a,b,x,y,z \in A$. Then there exists a unique cubic $*$-derivation D on A satisfying

$$\|f(a) - D(a)\| \le \frac{\theta\|a\|^r}{|16 - 2^{r+1}|}, \tag{2.19}$$

for all $a \in A$.

Proof. We can obtain the result from Theorem 2.1 and Theorem 2.2 by taking

$$\varphi(a,b,x,y,z) = \theta(\|a\|^r + \|b\|^r + \|x\|^r + \|y\|^r + \|z\|^r) \tag{2.20}$$

for all $a,b,x,y,z \in A$. □

In the next theorem, we investigate the superstability of cubic $*$-derivations of Banach $*$-algebras with a left-bounded approximate identity.

Theorem 2.4. *Suppose that A is a Banach $*$-algebra with a left-bounded approximate identity and $s \in \{-1, 1\}$. Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A \times A \to [0, \infty)$ such that*

$$\lim_{n\to\infty} n^{-3s}\varphi(n^s a, b) = \lim_{n\to\infty} n^{-3s}\varphi(a, n^s b) = 0, \tag{2.21}$$

$$\left\| a^3 f(b) - f(a) b^3 \right\| \le \varphi(a, b), \tag{2.22}$$

$$\left\| f(c)(ab)^3 - c^3\left[f(a) b^3 + a^3 f(b)\right] \right\| \le \varphi(c, ab), \tag{2.23}$$

$$\left\| a^3 f(b^*) - f(a)\left(b^3\right)^* \right\| \le \varphi(a, b) \tag{2.24}$$

for all $a, b, c \in A$. Then f is a cubic $*$-derivation on A.

Proof. First, we show that f is cubic. For each $a, b, c \in A$, we have

$$\begin{aligned}
&\left\| c^3 \left[f(2a+b) + f(2a-b) - 2f(a+b) - 2f(a-b) - 12 f(a) \right] \right\| \\
&\quad = n^{-3s}\left\| n^{3s}c^3 f(2a+b) + n^{3s}c^3 f(2a-b) - 2n^{3s}c^3 f(a+b) - 2n^{3s}c^3 f(a-b) - 12n^{3s}c^3 f(a) \right\| \\
&\quad \le n^{-3s}\Big[\left\| n^{3s}c^3 f(2a+b) - f\left(n^{3s}c^3\right)(2a+b)^3 \right\| + \left\| n^{3s}c^3 f(2a-b) - f\left(n^{3s}c^3\right)(2a-b)^3 \right\| \\
&\qquad\qquad + 2\left\| n^{3s}c^3 f(a+b) - f\left(n^{3s}c^3\right)(a+b)^3 \right\| \\
&\qquad\qquad + 2\left\| n^{3s}c^3 f(a-b) - f\left(n^{3s}c^3\right)(a-b)^3 \right\| \\
&\qquad\qquad + 12\left\| n^{3s}c^3 f(a) - f\left(n^{3s}c^3\right)a^3 \right\| \Big] \\
&\quad \le n^{-3s}\left[\varphi(n^s c, 2a+b) + \varphi(n^s c, 2a-b) + 2\varphi(n^s c, a+b) + 2\varphi(n^s c, a-b) + 12\varphi(n^s c, a) \right].
\end{aligned} \tag{2.25}$$

Taking the limit from the right side as n tends to infinity and using (2.21), we get

$$c^3\left[f(2a+b) + f(2a-b) - 2f(a+b) - 2f(a-b) - 12f(a) \right] = 0 \tag{2.26}$$

for all $a, b, c \in A$. If (e_j) is a left-bounded approximate identity for A, then so is (e_j^3). Now, it follows from (2.26) that f is cubic. For being cubic homogeneous of f, we have

$$\begin{aligned}
\left\| n^{3s}b^3\left[f(\mu a) - \mu^3 f(a)\right] \right\| &\le \left\| n^{3s}b^3 f(\mu a) - f(n^s b)(\mu a)^3 \right\| \\
&\quad + \left\| (\mu a)^3 f(n^s b) - n^{3s}(\mu b)^3 f(a) \right\| \\
&\le \varphi(n^s b, \mu a) + |\mu|^3 \varphi(n^s b, a).
\end{aligned} \tag{2.27}$$

Thus $\|b^3[f(\mu a) - \mu^3 f(a)]\| \le n^{-3s}\psi(n^s b, \mu a) + n^{-3s}|\mu|^3\psi(n^s b, a)$. By the same reasoning as in the above, f is cubic homogeneous. For each $a, b, c \in A$, we have

$$\begin{aligned}\left\|c^3\left[f(ab) - f(a)b^3 - a^3 f(b)\right]\right\| &= n^{-3s}\left\|n^{3s}c^3\left[f(ab) - f(a)b^3 - a^3 f(b)\right]\right\| \\ &\le n^{-3s}\left\|n^{3s}c^3 f(ab) - f(n^s c)(ab)^3\right\| \\ &\quad + n^{-3s}\left\|f(n^s c)(ab)^3 - n^{3s}c^3 f(a)b^3 - n^{3s}c^3 a^3 f(b)\right\| \\ &\le 2n^{-3s}\psi(n^s c, ab).\end{aligned} \tag{2.28}$$

The above inequality and (2.21), (2.22), and (2.23) show that $f(ab) = f(a)b^3 + a^3 f(b)$ for all $a, b \in A$. Finally, we have

$$\begin{aligned}\left\|b^3[f(a^*) - f(a)^*]\right\| &= n^{-3s}\left\|n^{3s}b^3 f(a^*) - n^{3s}b^3 f(a)^*\right\| \\ &\le n^{-3s}\left\|n^{3s}b^3 f(a^*) - f(n^s b)\left(a^3\right)^*\right\| \\ &\quad + n^{-3s}\left\|f(n^s b)\left(a^3\right)^* - n^{3s}b^3 f(a)^*\right\| \\ &\le n^{-3s}\psi(n^s b, a^*) + n^{-3s}\psi(n^s b, a)\end{aligned} \tag{2.29}$$

for all $a, b \in A$. Note that in the last inequality we have used (2.22) and (2.24). This completes the proof. □

Corollary 2.5. *Let r, δ be the nonnegative real numbers with $r \neq 3$, and let A be a Banach $*$-algebra with a left bounded approximate identity. Suppose that $f : A \to A$ is a mapping satisfying*

$$\begin{aligned}\left\|a^3 f(b) - f(a)b^3\right\| &\le \delta\left(\|a\|^r\|b\|^r\right), \\ \left\|f(c)(ab)^3 - c^3\left[f(a)b^3 + a^3 f(b)\right]\right\| &\le \delta\left(\|ab\|^r\|c\|^r\right), \\ \left\|a^3 f(b^*) - f(a)\left(b^3\right)^*\right\| &\le \delta\left(\|a\|^r\|b\|^r\right)\end{aligned} \tag{2.30}$$

for all all $a, b, c \in A$. Then f is a cubic $*$-derivation on A.

Proof. Using Theorem 2.4 with $\psi(a, b) = \delta(\|a\|^r\|b\|^r)$, we get the desired result. □

3. A Fixed Point Approach

Before proceeding to the main results in this section, we bring the upcoming theorem, which is useful to our purpose (For an extension of the result see [23]).

Theorem 3.1 (The fixed point alternative [24]). *Let (Ω, d) be a complete generalized metric space and $\mathcal{T} : \Omega \to \Omega$ a mapping with Lipschitz constant $L < 1$. Then, for each element $\alpha \in \Omega$, either $d(\mathcal{T}^n\alpha, \mathcal{T}^{n+1}\alpha) = \infty$ for all $n \ge 0$, or there exists a natural number n_0 such that:*

(i) *$d(\mathcal{T}^n\alpha, \mathcal{T}^{n+1}\alpha) < \infty$ for all $n \ge n_0$;*

(ii) *the sequence* $\{\mathfrak{T}^n\alpha\}$ *is convergent to a fixed point* β^* *of* $\mathfrak{T}$;

(iii) β^* *is the unique fixed point of* $\mathfrak{T}$ *in the set* $\Lambda = \{\beta \in \Omega : d(\mathfrak{T}^{n_0}\alpha, \beta) < \infty\}$;

(iv) $d(\beta, \beta^*) \leq 1/(1-L)d(\beta, \mathfrak{T}\beta)$ *for all* $\beta \in \Lambda$.

Theorem 3.2. *Let* $f : A \to A$ *be a continuous mapping with* $f(0) = 0$, *and let* $\varphi : A^4 \to [0, \infty)$ *be a continuous function such that*

$$\left\| \mathfrak{D}_\mu f(a,b) + \mathfrak{D} f(c,d) \right\| \leq \varphi(a,b,c,d), \tag{3.1}$$

$$\left\| f(a^*) - f(a)^* \right\| \leq \varphi(a,a,a,a) \tag{3.2}$$

for all $\mu \in \mathbb{T}^1_{1/n_0}$ *and all* $a,b,c,d \in A$. *If there exists a constant* $k \in (0,1)$ *such that*

$$\varphi(2a, 2b, 2c, 2d) \leq 8k\varphi(a,b,c,d) \tag{3.3}$$

for all $a,b,c,d \in A$, *then there exists a unique cubic* $*$*-derivation* D *on* A *satisfying*

$$\left\| f(a) - D(a) \right\| \leq \frac{1}{16(1-k)} \widetilde{\varphi}(a) \quad (a \in A), \tag{3.4}$$

in which $\widetilde{\varphi}(a) = \varphi(a, 0, 0, 0)$.

Proof. First, we wish to provide the conditions of Theorem 3.1. We consider the set

$$\Omega = \{ g : A \longrightarrow A \mid g(0) = 0 \} \tag{3.5}$$

and define the mapping d on $\Omega \times \Omega$ as follows:

$$d(g_1, g_2) := \inf\{ C \in (0, \infty) : \left\| g_1(a) - g_2(a) \right\| \leq C\widetilde{\varphi}(a),\ (\forall a \in A) \} \tag{3.6}$$

if there exist such constant C and $d(g_1, g_2) = \infty$, otherwise. It is easy to check that $d(g, g) = 0$ and $d(g_1, g_2) = d(g_2, g_1)$, for all $g, g_1, g_2 \in \Omega$. For each $g_1,\ g_2,\ g_3 \in \Omega$, we have

$$\begin{aligned} &\inf\{ C \in (0,\infty) : \left\| g_1(a) - g_2(a) \right\| \leq C\widetilde{\varphi}(a)\ \forall a \in A \} \\ &\quad \leq \inf\{ C \in (0,\infty) : \left\| g_1(a) - g_3(a) \right\| \leq C\widetilde{\varphi}(a)\ \forall a \in A \} \\ &\qquad + \inf\{ C \in (0,\infty) : \left\| g_3(a) - g_2(a) \right\| \leq C\widetilde{\varphi}(a)\ \forall a \in A \}. \end{aligned} \tag{3.7}$$

Hence $d(g_1, g_2) \leq d(g_1, g_3) + d(g_3, g_2)$. If $d(g_1, g_2) = 0$, then for every fixed $a_0 \in A$, we have $\|g_1(a_0) - g_2(a_0)\| \leq C\widetilde{\varphi}(a_0)$ for all $C > 0$. This implies $g_1 = g_2$. Let $\{g_n\}$ be a d-Cauchy sequence in Ω. Then $d(g_m, g_n) \to 0$, and thus $\|g_m(a) - g_n(a)\| \to 0$ for all $a \in A$. Since A is complete, then there exists $g \in \Omega$ such that $g_n \xrightarrow{d} g$ in Ω. Therefore, d is a generalized metric on Ω and the metric space (Ω, d) is complete. Now, we define the mapping $\mathcal{T} : \Omega \to \Omega$ by

$$\mathcal{T}g(a) = \frac{1}{8}g(2a), \quad (a \in A). \tag{3.8}$$

If $g_1, g_2 \in \Omega$ such that $d(g_1, g_2) < C$, by definition of d and $\mathcal{T}$, we have

$$\left\|\frac{1}{8}g_1(2a) - \frac{1}{8}g_2(2a)\right\| \leq \frac{1}{8}C\varphi(2a, 0, 0, 0) \tag{3.9}$$

for all $a \in A$. By using (3.3), we get

$$\left\|\frac{1}{8}g_1(2a) - \frac{1}{8}g_2(2a)\right\| \leq Ck\varphi(a, 0, 0, 0) \tag{3.10}$$

for all $a \in A$. The above inequality shows that $d(\mathcal{T}g_1, \mathcal{T}g_2) \leq kd(g_1, g_2)$ for all $g_1, g_2 \in \Omega$. Hence, $\mathcal{T}$ is a strictly contractive mapping on Ω with a Lipschitz constant k. To achieve inequality (3.4), we prove that $d(\mathcal{T}f, f) < \infty$. Putting $b = c = d = 0$ and $\mu = 1$ in (3.1), we obtain

$$\|2f(2a) - 16f(a)\| \leq \tilde{\varphi}(a) \tag{3.11}$$

for all $a \in A$. Hence

$$\left\|\frac{1}{8}f(2a) - f(a)\right\| \leq \frac{1}{16}\tilde{\varphi}(a) \tag{3.12}$$

for all $a \in A$. We conclude from (3.12) that $d(\mathcal{T}f, f) \leq 1/16$. It follows from Theorem 3.1 that $d(\mathcal{T}^n g, \mathcal{T}^{n+1} g) < \infty$ for all $n \geq 0$, and thus in this theorem we have $n_0 = 0$. Therefore, the parts (iii) and (iv) of Theorem 3.1 hold on the whole Ω. Hence there exists a unique mapping $D : A \to A$ such that D is a fixed point of $\mathcal{T}$ and that $\mathcal{T}^n f \to D$ as $n \to \infty$. Thus

$$\lim_{n \to \infty} \frac{f(2^n a)}{8^n} = D(a) \tag{3.13}$$

for all $a \in A$, hence

$$d(f, D) \leq \frac{1}{1-k}d(\mathcal{T}f, f) \leq \frac{1}{16(1-k)}. \tag{3.14}$$

The above equalities show that (3.4) is true for all $a \in A$. It follows from (3.3) that

$$\lim_{n \to \infty} \frac{\varphi(2^n a, 2^n b, 2^n c, 2^n d)}{8^n} = 0. \tag{3.15}$$

Putting $c = d = 0$ and substituting a, b by $2^n a,\ 2^n b$, respectively, in (3.1), we get

$$\frac{1}{8^n}\left\|\mathfrak{D}_\mu f(2^n a, 2^n b)\right\| \le \frac{\varphi(2^n a, 2^n b, 0, 0)}{8^n}. \tag{3.16}$$

Taking the limit as n tend to infinity, we obtain $\mathfrak{D}_\mu D(a,b) = 0$ for all $a, b \in A$ and all $\mu \in \mathbb{T}^1_{1/n_0}$. Similar to the proof of Theorem 2.1, we have $D(\mu a) = \mu^3 D(a)$ for all $a \in A$ and $\mu \in \mathbb{T}^1$. Since $\mathfrak{D}_1 D(a,b) = 0$, we can show that $D(ra) = r^3 D(a)$ for any rational number r. The continuity of f and φ imply that $D(\mu a) = \mu^3 D(a)$, for all $a \in A$ and $\mu \in \mathbb{R}$. Hence $D(\mu a) = \mu^3 D(a)$, for all $a \in A$ and $\mu \in \mathbb{C}$ $(\mu \neq 0)$. Therefore, D is a cubic homogeneous. If we put $a = b = 0$ and replace c, d by $2^n c, 2^n d$, respectively, in (3.1), we have

$$\frac{1}{8^{2n}}\left\|\mathfrak{D} f(2^n c, 2^n d)\right\| \le \frac{\varphi(0,0,2^n c, 2^n d)}{8^{2n}} \le \frac{\varphi(0,0,2^n c, 2^n d)}{8^n} \tag{3.17}$$

for all $c, d \in A$. By letting $n \to \infty$ in the preceding inequality, we find $\mathfrak{D} D(c,d) = 0$ for all $c, d \in A$. Substituting a by $2^n a$ in (3.2) and then dividing the both sides of the obtained inequality by 8^n, we get

$$\left\|\frac{f(2^n a^*)}{8^n} - \frac{f(2^n a)^*}{8^n}\right\| \le \frac{\varphi(2^n a, 2^n a, 2^n a, 2^n a)}{8^n} \tag{3.18}$$

for all $a \in A$. Passing to the limit as $n \to \infty$ in (3.18) and applying (3.13), we conclude that $D(a^*) = D(a)^*$ for all $a \in A$. This shows that D is a unique cubic $*$-derivation. □

Corollary 3.3. *Let θ, r be positive real numbers with $r < 3$, and let $f : A \to A$ be a mapping with $f(0) = 0$ such that*

$$\begin{gathered}\left\|\mathfrak{D}_\mu f(a,b) + \mathfrak{D} f(c,d)\right\| \le \theta\left(\|a\|^r + \|b\|^r + \|c\|^r + \|d\|^r\right), \\ \left\|f(a^*) - f(a)^*\right\| \le 4\theta\|a\|^r\end{gathered} \tag{3.19}$$

for all $\mu \in \mathbb{T}^1_{1/n_0}$ and all $a, b, c, d \in A$. Then there exists a unique cubic $*$-derivation D on A satisfying

$$\left\|f(a) - D(a)\right\| \le \frac{\theta}{2(8 - 2^r)}\|a\|^r \tag{3.20}$$

for all $a \in A$.

Proof. The result follows from Theorem 3.2 by letting

$$\varphi(a,b,c,d) = \theta(\|a\|^r + \|b\|^r + \|c\|^r + \|d\|^r). \tag{3.21}$$

□

In the following corollary, we show the superstability for cubic $*$-derivations.

Corollary 3.4. *Let r_j $(1 \leq j \leq 4)$ θ be nonnegative real numbers with $0 < \sum_{j=1}^{4} r_j \neq 3$, and let $f : A \to A$ be a mapping such that*

$$\left\| \mathfrak{D}_\mu f(a,b) + \mathfrak{D} f(c,d) \right\| \leq \theta\left(\|a\|^{r_1} \|b\|^{r_2} \|c\|^{r_3} \|d\|^{r_4} \right), \tag{3.22}$$

$$\left\| f(a^*) - f(a)^* \right\| \leq \theta \|a\|^{\sum_{j=1}^{4} r_j} \tag{3.23}$$

for all $\mu \in \mathbb{T}^1_{1/n_0}$ and all $a,\ b,\ c,\ d \in A$. Then f is a cubic $*$-derivation on A.

Proof. Putting $a = b = c = d = 0$ in (3.22), we get $f(0) = 0$. Now, if we put $b = c = d = 0$, $\mu = 1$ in (3.22), then we have $f(2a) = 8f(a)$ for all $a \in A$. It is easy to see by induction that $f(2^n a) = 8^n f(a)$, and thus $f(a) = f(2^n a)/8^n$ for all $a \in A$ and $n \in \mathbb{N}$. It follows from Theorem 3.2 that f is a cubic mapping. Now, by putting $\varphi(a,b,c,d) = \theta(\|a\|^{r_1}\|b\|^{r_2}\|c\|^{r_3}\|d\|^{r_4})$ in Theorem 3.2, we can obtain the desired result. □

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