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Research Article

A Sharp Double Inequality between Seiffert, Arithmetic, and Geometric Means

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For fixed $s \geq 1$ and any $t_1, t_2 \in (0, 1/2)$ we prove that the double inequality $G^s(t_1 a + (1-t_1)b, t_1 b + (1-t_1)a)A^{1-s}(a,b) < P(a,b) < G^s(t_2 a + (1-t_2)b, t_2 b + (1-t_2)a)A^{1-s}(a,b)$ holds for all $a, b > 0$ with $a \neq b$ if and only if $t_1 \leq (1 - \sqrt{1-(2/\pi)^{2/s}})/2$ and $t_2 \geq (1 - 1/\sqrt{3s})/2$. Here, $P(a,b)$, $A(a,b)$ and $G(a,b)$ denote the Seiffert, arithmetic, and geometric means of two positive numbers a and b, respectively.

1. Introduction

The Seiffert mean $P(a,b)$ [1] of two distinct positive numbers a and b is defined by

$$P(a,b) = \frac{a-b}{4\arctan\left(\sqrt{a/b}\right) - \pi}. \tag{1.1}$$

Recently, the Seiffert mean $P(a,b)$ has been the subject of intensive research. In particular, many remarkable inequalities for $P(a,b)$ can be found in the literature [2–17]. The Seiffert mean $P(a,b)$ can be rewritten as (see [6, (2.4)])

$$P(a,b) = \frac{a-b}{2\arcsin((a-b)/(a+b))}. \tag{1.2}$$

Let $A(a,b) = (a+b)/2$, $G(a,b) = \sqrt{ab}$ and $H(a,b) = 2ab/(a+b)$ be the classical arithmetic, geometric, and harmonic means of two positive numbers a and b, respectively. Then it is well known that inequalities $H(a,b) < G(a,b) < P(a,b) < A(a,b)$ hold for all $a,b > 0$ with $a \neq b$.

For $\alpha, \beta, \lambda, \mu \in (0, 1/2)$, Chu et al. [18, 19] proved that the double inequalities

$$\begin{gathered} G(\alpha a + (1-\alpha)b, \alpha b + (1-\alpha)a) < P(a,b) < G(\beta a + (1-\beta)b, \beta b + (1-\beta)a), \\ H(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a) < P(a,b) < H(\mu a + (1-\mu)b, \mu b + (1-\mu)a) \end{gathered} \tag{1.3}$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leq (1 - \sqrt{1 - 4/\pi^2})/2$, $\beta \geq (3 - \sqrt{3})/6$, $\lambda \leq (1 - \sqrt{1 - 2/\pi})/2$ and $\mu \geq (6 - \sqrt{6})/12$.

Let $t \in (0, 1/2)$, $s \geq 1$ and

$$Q_{t,s}(a,b) = G^s(ta + (1-t)b, tb + (1-t)a) A^{1-s}(a,b), \tag{1.4}$$

then it is not difficult to verify that

$$\begin{gathered} Q_{t,1}(a,b) = G(ta + (1-t)b, tb + (1-t)a), \\ Q_{t,2}(a,b) = H(ta + (1-t)b, tb + (1-t)a) \end{gathered} \tag{1.5}$$

and $Q_{t,s}(a,b)$ is strictly increasing with respect to $t \in (0, 1/2)$ for fixed $a,b > 0$ with $a \neq b$.

It is natural to ask what are the largest value $t_1 = t_1(s)$ and the least value $t_2 = t_2(s)$ in $(0, 1/2)$ such that the double inequality $Q_{t_1,s}(a,b) < P(a,b) < Q_{t_2,s}(a,b)$ holds for all $a,b > 0$ with $a \neq b$ and $s \geq 1$. The main purpose of this paper is to answer this question.

2. Main Result

In order to establish our main result we need two lemmas, which we present in the following.

Lemma 2.1. *If $s \geq 1$, then $1/(3s) + (2/\pi)^{2/s} < 1$.*

Proof. Consider the following:

$$f(s) = \frac{1}{3s} + \left(\frac{2}{\pi}\right)^{2/s}. \tag{2.1}$$

Then simple computations lead to

$$\lim_{s \to +\infty} f(s) = 1, \tag{2.2}$$

$$\begin{aligned} f'(s) &= \frac{2}{s^2} \log \frac{\pi}{2} \left[\left(\frac{2}{\pi} \right)^{2/s} - \frac{1}{6 \log(\pi/2)} \right] \\ &\geq \frac{2}{s^2} \log \frac{\pi}{2} \left[\left(\frac{2}{\pi} \right)^{2} - \frac{1}{6 \log(\pi/2)} \right] \\ &= \frac{24 \log(\pi/2) - \pi^2}{3\pi^2 s^2} \end{aligned} \tag{2.3}$$

for $s \geq 1$.

Computational and numerical experiments show that

$$24 \log\left(\frac{\pi}{2}\right) - \pi^2 = 0.968 \cdots > 0. \tag{2.4}$$

Inequalities (2.3) and (2.4) imply that $f(s)$ is strictly increasing in $[1, +\infty)$. Therefore, Lemma 2.1 follows from (2.1) and (2.2) together with the monotonicity of $f(s)$. □

Lemma 2.2. *Let* $0 \leq u \leq 1$, $s \geq 1$ *and*

$$f_{u,s}(x) = \frac{s}{2} \log\left(1 - ux^2\right) - \log x + \log(\arcsin x). \tag{2.5}$$

Then inequality $f_{u,s}(x) > 0$ *holds for all* $x \in (0,1)$ *if and only if* $3su \leq 1$, *and inequality* $f_{u,s}(x) < 0$ *holds for all* $x \in (0,1)$ *if and only if* $u + (2/\pi)^{2/s} \geq 1$.

Proof. If $u = 0$, then we clearly see that $3su \leq 1$, $u + (2/\pi)^{2/s} < 1$ and $f_{0,s}(x) = \log[(\arcsin x)/x] > 0$ for all $s \geq 1$ and $x \in (0,1)$. In the following discussion, we assume that $0 < u \leq 1$.

From (2.5) and simple computations we have

$$\lim_{x \to 0^+} f_{u,s}(x) = 0, \tag{2.6}$$

$$f'_{u,s}(x) = \frac{1}{\sqrt{1-x^2} \arcsin x} - \frac{1 + u(s-1)x^2}{x(1-ux^2)} = \frac{1 + u(s-1)x^2}{x(1-ux^2) \arcsin x} g_{u,s}(x), \tag{2.7}$$

where

$$g_{u,s}(x) = \frac{x(1-ux^2)}{\sqrt{1-x^2}[1+u(s-1)x^2]} - \arcsin x, \tag{2.8}$$

$$g_{u,s}(0) = 0, \tag{2.9}$$

$$g'_{u,s}(x) = \frac{x^2}{(1-x^2)^{3/2}[1+u(s-1)x^2]^2} h_{u,s}(x), \tag{2.10}$$

where

$$h_{u,s}(x) = u^2(s-1)^2x^4 + u\left(-s^2u + us + 4s - 2\right)x^2 + 1 - 3su, \tag{2.11}$$

$$h_{u,s}(0) = 1 - 3su, \tag{2.12}$$

$$h_{u,s}(1) = us(1-u) + (1-u)^2. \tag{2.13}$$

We divide the proof into four cases.

Case 1 ($3su \leq 1$). Then from (2.11) and (2.12) together with the fact that

$$-us^2 + us + 4s - 2 = 2(s-1) + s(u + 2su + 1) + s(1 - 3su) > 0, \tag{2.14}$$

we clearly see that

$$h_{u,s}(0) \geq 0, \tag{2.15}$$

and $h_{u,s}(x)$ is strictly increasing in $[0,1]$.

Equation (2.12) and the monotonicity of $h_{u,s}(x)$ imply that

$$h_{u,s}(x) > 0 \tag{2.16}$$

for $x \in (0,1]$.

Equation (2.10) and inequality (2.16) lead to the conclusion that $g_{u,s}(x)$ is strictly increasing in $[0,1)$. Then from (2.9) we know that

$$g_{u,s}(x) > 0 \tag{2.17}$$

for $x \in (0,1)$.

It follows from (2.7) and inequality (2.17) that $f_{u,s}(x)$ is strictly increasing in $(0,1]$.

Therefore, $f_{u,s}(x) > 0$ for all $x \in (0,1)$ follows from (2.6) and the monotonicity of $f_{u,s}(x)$.

Case 2 ($3su > 1$). Then (2.12) and the continuity of $h_{u,s}(x)$ imply that there exists $0 < \lambda < 1$ such that

$$h_{u,s}(x) < 0 \tag{2.18}$$

for $x \in [0,\lambda)$.

Therefore, $f_{u,s}(x) < 0$ for $x \in (0, \lambda)$ follows easily from (2.6), (2.7), (2.9) and (2.10) together with inequality (2.18).

Case 3 ($u + (2/\pi)^{2/s} \geq 1$). Then Lemma 2.1 and (2.12) lead to

$$h_{u,s}(0) = 1 - 3su \leq 1 - 3s\left[1 - \left(\frac{2}{\pi}\right)^{2/s}\right] < 0. \tag{2.19}$$

We divide the proof into two subcases.

Subcase 3.1 ($u = 1$). Then (2.13) becomes

$$h_{u,s}(1) = 0. \tag{2.20}$$

Let $t = x^2$, then from (2.11) we clearly see that the function $h_{u,s}$ is a quadratic function of variable t. It follows from inequality (2.19) and (2.20) that

$$h_{u,s}(x) < 0 \tag{2.21}$$

for all $x \in [0, 1)$.

Therefore, $f_{u,s}(x) < 0$ for $x \in (0, 1)$ follows easily from (2.6), (2.7), (2.9) and (2.10) together with inequality (2.21).

Subcase 3.2 ($0 < u < 1$). Then from (2.5), (2.8), and (2.13) we have

$$f_{u,s}(1) = \log\left[\frac{\pi}{2}(1 - u)^{s/2}\right] \leq 0, \tag{2.22}$$

$$\lim_{x \to 1^-} g_{u,s}(x) = +\infty, \tag{2.23}$$

$$h_{u,s}(1) > 0. \tag{2.24}$$

From (2.11), (2.19), and (2.24) we clearly see that there exists $0 < \lambda_1 < 1$ such that $h_{u,s}(x) < 0$ for $x \in [0, \lambda_1)$ and $h_{u,s}(x) > 0$ for $x \in (\lambda_1, 1]$. Then (2.10) implies that $g_{u,s}(x)$ is strictly decreasing in $[0, \lambda_1]$ and strictly increasing in $[\lambda_1, 1)$.

From (2.9) and (2.23) together with the piecewise monotonicity of $g_{u,s}(x)$ we clearly see that there exists $0 < \lambda_2 < 1$ such that $g_{u,s}(x) < 0$, for $x \in (0, \lambda_2)$ and $g_{u,s}(x) > 0$ for $x \in (\lambda_2, 1)$. Then (2.7) implies that $f_{u,s}(x)$ is strictly decreasing in $(0, \lambda_2]$ and strictly increasing in $[\lambda_2, 1]$.

Therefore, $f_{u,s}(x) < 0$ for $x \in (0, 1)$ follows from (2.6) and (2.22) together with the piecewise monotonicity of $f_{u,s}(x)$.

Case 4 ($u + (2/\pi)^{2/s} < 1$). Then (2.5) leads to

$$f_{u,s}(1) = \log\left[\frac{\pi}{2}(1 - u)^{s/2}\right] > 0. \tag{2.25}$$

From inequality (2.25) and the continuity of $f_{u,s}(x)$ we know that there exists $0 < \mu < 1$ such that $f_{u,s}(x) > 0$ for $x \in (\mu, 1]$. □

Theorem 2.3. *If $t_1, t_2 \in (0, 1/2)$ and $s \geq 1$, then the double inequality*

$$Q_{t_1,s}(a,b) < P(a,b) < Q_{t_2,s}(a,b) \tag{2.26}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $t_1 \leq (1 - \sqrt{1 - (2/\pi)^{2/s}})/2$ and $t_2 \geq (1 - 1/\sqrt{3s})/2$.

Proof. Since both $Q_{t,s}(a,b)$ and $P(a,b)$ are symmetric and homogeneous of degree 1. Without loss of generality, we assume that $a > b$. Let $x = (a-b)/(a+b) \in (0,1)$. Then from (1.2) and (1.4) we have

$$\begin{aligned}\log\left(\frac{Q_{t,s}(a,b)}{P(a,b)}\right) &= \log\left(\frac{Q_{t,s}(a,b)}{A(a,b)}\right) - \log\left(\frac{P(a,b)}{A(a,b)}\right) \\ &= \frac{s}{2}\log\left[1 - (1-2t)^2 x^2\right] - \log x + \log(\arcsin x).\end{aligned} \tag{2.27}$$

Therefore, Theorem 2.3 follows easily from Lemma 2.2 and (2.27). □

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