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Retraction

Retracted: On the Weak Relatively Nonexpansive Multivalued Mappings in Banach Spaces

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This article has been retracted as it is found to contain a substantial amount of material, without referencing, from the paper "On the Weak Relatively Nonexpansive Mappings in Banach Spaces," Yongchun Xu and Yongfu Su, Fixed Point Theory and Applications, Volume 2010, Article ID 189751, 7 pages. doi:10.1155/2010/189751 [1].

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[1] Y. Su, "On the weak relatively nonexpansive multivalued mappings in Banach spaces," *Abstract and Applied Analysis*, vol. 2012, Article ID 730619, 7 pages, 2012.

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Review Article

On the Weak Relatively Nonexpansive Multivalued Mappings in Banach Spaces

Yongfu Su

Department of Mathematics, Tianjin Polytechnic University, Tianjin 300387, China

Correspondence should be addressed to Yongfu Su, suyongfu@tjpu.edu.cn

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In recent years, the definition of relatively nonexpansive multivalued mapping and the definition of weak relatively nonexpansive multivalued mapping have been presented and studied by many authors. In this paper, we give some results about weak relatively nonexpansive multivalued mappings and give two examples which are weak relatively nonexpansive multivalued mappings but not relatively nonexpansive multivalued mappings in Banach space l^2 and $L^p[0,1](1 < p < +\infty)$.

1. Introduction

Let E be a smooth Banach space and let C be a nonempty closed convex subset of E. We denote by ϕ the function defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \text{for } x,y \in E. \tag{1.1}$$

Following Alber [1], the generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) = \arg\min_{y\in C} \phi(y,x), \quad \forall x \in E. \tag{1.2}$$

The generalized projection Π_C from E onto C is well defined, single-value, and satisfies

$$(\|x\| - \|y\|)^2 \le \phi(x,y) \le (\|x\| + \|y\|)^2, \quad \text{for } x,y \in E. \tag{1.3}$$

If E is a Hilbert space, then $\phi(y,x) = \|y-x\|^2$ and Π_C is the metric projection of E onto C.

In recent years, the definition of relatively nonexpansive multivalued mapping and the definition of weak relatively nonexpansive multivalued mapping have been presented and studied by many authors (see [1]). In this paper, we give some results about weak relatively nonexpansive multivalued mappings and give two examples which are weak relatively nonexpansive multivalued mappings but not relatively nonexpansive multivalued mappings in Banach space l^2 and $L^p[0,1](1 < p < +\infty)$.

Remark 1.1. The definition of relatively nonexpansive multivalued mapping presented in this paper and the definition of [2] are different.

Let C be a closed convex subset of E, and let T be a multivalued mapping from C into itself. We denote by $F(T)$ the set of fixed points of T, that is,

$$F(T) = \{x \in C : x \in Tx\}. \tag{1.4}$$

A point p in C is said to be an asymptotic fixed point (strong asymptotic fixed point) of T [3–5] if C contains a sequence $\{x_n\}$ which converges weakly (strongly) to p and there exists a sequence $\{y_n\}$ such that $y_n \in Tx_n$, $\lim_{n\to\infty}\|y_n - x_n\| = 0$. The set of asymptotic fixed point (the set of strong asymptotic fixed point) of T will be denoted by $\widehat{F}(T)(\widetilde{F}(T))$.

A multivalued mapping T of C into itself is said to be relatively nonexpansive multivalued mapping (weak relatively nonexpansive multivalued mapping) if the following conditions are satisfied:

(1) $F(T)$ is nonempty;

(2) $\phi(u,v) \le \phi(u,x), \forall u \in F(T),\ \forall x \in C,\ \exists v \in Tx$;

(3) $\widehat{F}(T) = F(T)(\widetilde{F}(T) = F(T))$.

A multivalued mapping T of C into itself is said to be relatively uniformly nonexpansive multivalued mapping (weak relatively uniformly nonexpansive multivalued mapping) if the following conditions are satisfied:

(1) $F(T)$ is nonempty;

(2) $\phi(u,v) \le \phi(u,x), \forall u \in F(T),\ \forall x \in C,\ \forall v \in Tx$;

(3) $\widehat{F}(T) = F(T)(\widetilde{F}(T) = F(T))$.

Following Matsushita and Takahashi [3], a mapping T of C into itself is said to be relatively nonexpansive mapping if the following conditions are satisfied:

(1) $F(T)$ is nonempty;

(2) $\phi(u,Tx) \le \phi(u,x), \forall u \in F(T), x \in C$;

(3) $\widehat{F}(T) = F(T)$.

The hybrid algorithms for fixed point of relatively nonexpansive mappings and applications have been studied by many authors, for example, [3–8].

In recent years, the definition of weak relatively nonexpansive mapping has been presented and studied by many authors [6–9].

A mapping T from C into itself is said to be weak relatively nonexpansive mapping if

(1) $F(T)$ is nonempty;

(2) $\phi(u, Tx) \leq \phi(u, x), \forall u \in F(T), x \in C$;

(3) $\widetilde{F}(T) = F(T)$.

Remark 1.2. In [7], the weak relatively nonexpansive mapping is also said to be relatively weak nonexpansive mapping.

Remark 1.3. In [8], the authors have given the definition of hemirelatively nonexpansive mapping as follows. A mapping T from C into itself is called hemirelatively nonexpansive if

(1) $F(T)$ is nonempty;

(2) $\phi(u, Tx) \leq \phi(u, x), \forall u \in F(T), x \in C$.

The following conclusion is obvious.

Conclusion 1. A mapping is closed hemirelatively nonexpansive if and only if it is weak relatively nonexpansive.

If E is strictly convex and reflexive Banach space, and $A \subset E \times E^*$ is a continuous monotone mapping with $A^{-1}(0) \neq \emptyset$, then it is proved in [3] that $J_r := (J + rA)^{-1} J$, for $r > 0$ is relatively nonexpansive. Moreover, if $T : E \to E$ is relatively nonexpansive, then using the definition of ϕ one can show that $F(T)$ is closed and convex. It is obvious that relatively nonexpansive mapping is weak relatively nonexpansive mapping. In fact, for any mapping $T : C \to C$, we have $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$. Therefore, if T is relatively nonexpansive mapping, then $F(T) = \widetilde{F}(T) = \widehat{F}(T)$.

2. Results for Weak Relatively Multivalued Nonexpansive Mappings in Banach Space

Theorem 2.1. *Let E be a smooth Banach space and C a nonempty closed convex and balanced subset of E. Let $\{x_n\}$ be a sequence in C such that $\{x_n\}$ converges weakly to $x_0 \neq 0$ and $\|x_n - x_m\| \geq r > 0$ for all $n \neq m$. Define a mapping $T : C \to C$ as follows:*

$$T(x) = \begin{cases} \left\{ kx_n : k = \dfrac{n}{n+\lambda}, 0 < \lambda \leq M < +\infty \right\}, & \text{if } x = x_n (\exists n \geq 1), \\ -x, & \text{if } x \neq x_n (\forall n \geq 1). \end{cases} \tag{2.1}$$

Then, T is a weak relatively uniformly nonexpansive multivalued mapping but not relatively uniformly nonexpansive multivalued mapping.

Proof. It is obvious that T has a unique fixed point 0, that is, $F(T) = \{0\}$. Firstly, we show that x_0 is an asymptotic fixed point of T. In fact that, since $\{x_n\}$ converges weakly to x_0 and

$$\|Tx_n - x_n\| = \left\| \frac{n}{n+\lambda} x_n - x_n \right\| = \frac{\lambda}{n+\lambda} \|x_n\| \longrightarrow 0, \tag{2.2}$$

as $n \to \infty$, so that x_0 is an asymptotic fixed point of T. Secondly, we show that T has a unique strong asymptotic fixed point 0, so that $F(T) = \widetilde{F}(T)$. In fact that, for any strong convergent

sequence $\{z_n\} \subset C$ such that $z_n \to z_0$ and $\|z_n - y_n\| \to 0, y_n \in Tx_n$ as $n \to \infty$, from the conditions of Theorem 2.1, there exist sufficiently large nature number N such that $z_n \neq x_m$, for any $n, m > N$. Then, $Tz_n = \{-z_n\}$ for $n > N$, it follows from $\|z_n - y_n\| \to 0,\ y_n \in Tx_n$ that $2z_n \to 0$ and hence $z_n \to z_0 = 0$. On the other hand, observe that

$$\phi(0, v) = \|v\|^2 \leq \|x\|^2 = \phi(0, x), \qquad \forall x \in C,\ \forall v \in Tx. \tag{2.3}$$

Then, T is a weak relatively uniformly nonexpansive multivalued mapping. On the other hand, since x_0 is an asymptotic fixed point of T but not fixed point, hence T is not a relatively uniformly nonexpansive multivalued mapping. □

Taking any fixed number $\lambda_0 \in (0, M)$, we have the following result.

Theorem 2.2. *Let E be a smooth Banach space and C a nonempty closed convex and balanced subset of E. Let $\{x_n\}$ be a sequence in C such that $\{x_n\}$ converges weakly to $x_0 \neq 0$ and $\|x_n - x_m\| \geq r > 0$ for all $n \neq m$. Define a mapping $T : C \to C$ as follows:*

$$T(x) = \begin{cases} \dfrac{n}{n + \lambda_0} x_n, & \text{if } x = x_n (\exists n \geq 1), \\ -x, & \text{if } x \neq x_n (\forall n \geq 1). \end{cases} \tag{2.4}$$

Then, T is a weak relatively nonexpansive mapping but not relatively nonexpansive mapping.

3. An Example in Banach Space l^2

In this section, we will give an example which is a weak relatively nonexpansive mapping but not a relatively nonexpansive mapping.

Example 3.1. Let $E = l^2$, where

$$l^2 = \left\{ \xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots) : \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\},$$

$$\|\xi\| = \left(\sum_{n=1}^{\infty} |\xi_n|^2 \right)^{1/2}, \quad \forall \xi \in l^2, \tag{3.1}$$

$$\langle \xi, \eta \rangle = \sum_{n=1}^{\infty} \xi_n \eta_n, \quad \forall \xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots),\ \eta = (\eta_1, \eta_2, \eta_3, \ldots, \eta_n \cdots .) \in l^2.$$

It is well known that l^2 is a Hilbert space, so that $(l^2)^* = l^2$. Let $\{x_n\} \subset E$ be a sequence defined by

$$\begin{aligned} x_0 &= (1,0,0,0,\ldots) \\ x_1 &= (1,1,0,0,\ldots) \\ x_2 &= (1,0,1,0,0,\ldots) \\ x_3 &= (1,0,0,1,0,0,\ldots) \\ &\vdots \\ x_n &= (\xi_{n,1},\xi_{n,2},\xi_{n,3},\ldots,\xi_{n,k},\ldots) \\ &\vdots, \end{aligned} \tag{3.2}$$

where

$$\xi_{n,k} = \begin{cases} 1, & \text{if } k = 1, n+1, \\ 0 & \text{if } k \neq 1, k \neq n+1, \end{cases} \tag{3.3}$$

for all $n \geq 1$. Define a mapping $T : E \to E$ as follows:

$$T(x) = \begin{cases} \left\{ kx_n : k = \dfrac{n}{n+\lambda}, 0 < \lambda \leq M < +\infty \right\}, & \text{if } x = x_n (\exists n \geq 1), \\ -x, & \text{if } x \neq x_n (\forall n \geq 1). \end{cases} \tag{3.4}$$

Conclusion 2. $\{x_n\}$ converges weakly to x_0.

Proof. For any $f = (\zeta_1, \zeta_2, \zeta_3, \ldots, \zeta_k, \ldots) \in l^2 = (l^2)^*$, we have

$$f(x_n - x_0) = \langle f, x_n - x_0 \rangle = \sum_{k=2}^{\infty} \zeta_k \xi_{n,k} = \zeta_{n+1} \longrightarrow 0, \tag{3.5}$$

as $n \to \infty$. That is, $\{x_n\}$ converges weakly to x_0. □

The following conclusion is obvious.

Conclusion 3. $\|x_n - x_m\| = \sqrt{2}$ for any $n \neq m$.

It follows from Theorem 2.1 and the above two conclusions that T is a weak relatively uniformly nonexpansive multivalued mapping but not relatively uniformly nonexpansive multivalued mapping.

4. An Example in Banach Space $L^p[0,1](1 < p < +\infty)$

Let $E = L^p[0,1], (1 < p < +\infty)$ and

$$x_n = 1 - \frac{1}{2^n}, \quad n = 1, 2, 3, \ldots. \tag{4.1}$$

Define a sequence of functions in $L^p[0,1]$ by the following expression:

$$f_n(x) = \begin{cases} \dfrac{2}{x_{n+1} - x_n}, & \text{if } x_n \le x < \dfrac{x_{n+1} + x_n}{2}, \\ \dfrac{-2}{x_{n+1} - x_n}, & \text{if } \dfrac{x_{n+1} + x_n}{2} \le x < x_{n+1} \\ 0 & \text{otherwise}, \end{cases} \tag{4.2}$$

for all $n \ge 1$. Firstly, we can see, for any $x \in [0,1]$, that

$$\int_0^x f_n(t)dt \longrightarrow 0 = \int_0^x f_0(t)dt, \tag{4.3}$$

where $f_0(x) \equiv 0$. It is well known that the above relation (4.3) is equivalent to $\{f_n(x)\}$ which converges weakly to $f_0(x)$ in uniformly smooth Banach space $L^p[0,1](1 < p < +\infty)$. On the other hand, for any $n \neq m$, we have

$$\begin{aligned} \|f_n - f_m\| &= \left(\int_0^1 |f_n(x) - f_m(x)|^p dx \right)^{1/p} \\ &= \left(\int_{x_n}^{x_{n+1}} |f_n(x) - f_m(x)|^p dx + \int_{x_m}^{x_{m+1}} |f_n(x) - f_m(x)|^p dx \right)^{1/p} \\ &= \left(\int_{x_n}^{x_{n+1}} |f_n(x)|^p dx + \int_{x_m}^{x_{m+1}} |f_m(x)|^p dx \right)^{1/p} \\ &= \left(\left(\frac{2}{x_{n+1} - x_n} \right)^p (x_{n+1} - x_n) + \left(\frac{2}{x_{m+1} - x_m} \right)^p (x_{m+1} - x_m) \right)^{1/p} \\ &= \left(\frac{2^p}{(x_{n+1} - x_n)^{p-1}} + \frac{2^p}{(x_{m+1} - x_m)^{p-1}} \right)^{1/p} \\ &\ge (2^p + 2^p)^{1/p} > 0. \end{aligned} \tag{4.4}$$

Let

$$u_n(x) = f_n(x) + 1, \quad \forall n \ge 1. \tag{4.5}$$

It is obvious that u_n converges weakly to $u_0(x) \equiv 1$ and

$$\|u_n - u_m\| = \|f_n - f_m\| \geq (2^p + 2^p)^{1/p} > 0, \quad \forall n \geq 1. \tag{4.6}$$

Define a mapping $T : E \to E$ as follows:

$$T(x) = \begin{cases} \left\{ ku_n : k = \dfrac{n}{n+\lambda}, 0 < \lambda \leq M < +\infty \right\}, & \text{if } x = u_n (\exists n \geq 1), \\ -x, & \text{if } x \neq u_n (\forall n \geq 1). \end{cases} \tag{4.7}$$

Since (4.6) holds, by using Theorem 2.1, we know that T is a weak relatively uniformly nonexpansive multivalued mapping but not relatively uniformly nonexpansive multivalued mapping.

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