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Research Article

Stability and Hopf Bifurcation in a Computer Virus Model with Multistate Antivirus

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By considering that people may immunize their computers with countermeasures in susceptible state, exposed state and using anti-virus software may take a period of time, a computer virus model with time delay based on an SEIR model is proposed. We regard time delay as bifurcating parameter to study the dynamical behaviors which include local asymptotical stability and local Hopf bifurcation. By analyzing the associated characteristic equation, Hopf bifurcation occurs when time delay passes through a sequence of critical value. The linerized model and stability of the bifurcating periodic solutions are also derived by applying the normal form theory and the center manifold theorem. Finally, an illustrative example is also given to support the theoretical results.

1. Introduction

As globalization and development of communication networks have made computers more and more present in our daily life, the threat of computer viruses also becomes an increasingly important issue of concern. In 2003, a virus, called worm king, rapidly spread and attacked the global world, which results the network of the internet to be seriously congested and server to be paralyzed [1]. In 2010, the report of pestilence about computer virus in China revealed that more than 90% computers in China are infected computer virus.

Computer viruses are small programs developed to damage the computer systems erasing data, stealing information. Their action throughout a network can be studied by using classical epidemiological models for disease propagation [2–6]. In [7–9], based on SIR classical epidemic model, Mark had proposed the dynamical models for the computer

virus propagation, which provided estimations for temporal evolutions of infected nodes depending on network parameters [10–12]. In [13], Richard and Mark propose a modified propagation model named SEIR (susceptible-exposed-infected-recover) model to simulate virus propagation. In [14], on this basis of the SIR model, Yao et al. proposed a SIDQV model with time delay which add a quarantine state to clean the virus. However, both above models assume the viruses are cleaned in the infective state. In fact, in addition to clean viruses in state I, people may immunize their computers with countermeasures in state S and state E in the real world. Moreover there may be a time lag when the node uses antivirus software to clean the virus.

In this paper, in order to overcome the above-mentioned limitation, we present a new computer virus model with time delay which is depending on the SEIR model [15]; time delay can be considered the period of the node uses antivirus software to clean the virus. This model provides an opportunity for us to study the behaviors of virus propagation in the presence of antivirus countermeasures, which are very important and desirable for understanding of the virus spread patterns, as well as for management and control of the spread. The remainder of this paper is organized as follows. In Section 2, the stability of trivial solutions and the existence of Hopf bifurcation are discussed. In Section 3, a formula for determining the direction of Hopf bifurcation and the stability of bifurcating periodic solutions will be given by using the normal form and center manifold theorem introduced by Hassard et al. in [16]. In Section 4, numerical simulations aimed at justifying the theoretical analysis will be reported.

2. Mathematical Model Formulation

Our model is based on the traditional SEIR model [7–9, 15, 17]. The SEIR model has four states: susceptible, exposed (infected but not yet infectious), infectious, and recovered. Our assumptions on the dynamical model are as follows.

(1) In the real world, in addition cleaning viruses in state I, people may immunize their computers with countermeasures in state S and state E (after virus being cleaned), which may result in new state transition paths in comparison with SIR model:

S-R: using countermeasure of real-time immunization,

E-R: using real-time immunization after virus codes cleaning.

(2) In state S, when people install the antivirus software on their computer, we assume that their computer can be immunized at a unit time.

(3) In state E, since the computer is infected by the virus, the antivirus software may use a period to search the document and clean the viruses.

(4) Denote the period of time of killing viruses when users find that their computers are infected by viruses.

(5) While the computer is installed the antivirus software, it will not be quarantine or replacement. On the basis of the above hypotheses (1)–(5), the dynamical model

can be formulated by the following equations:

$$\begin{aligned}
\frac{dS(t)}{dt} &= uN - \beta I(t)S(t) - (\rho_{SR} + \mu)S(t), \\
\frac{dE(t)}{dt} &= \beta I(t)S(t) - (\alpha + \mu)E(t) - \rho_{ER}E(t-\tau), \\
\frac{dI(t)}{dt} &= \alpha E(t) - (\gamma + \mu)I(t), \\
\frac{dR(t)}{dt} &= \rho_{SR}S(t) + \rho_{ER}E(t-\tau) + \gamma I(t-\tau) - \mu R(t),
\end{aligned} \tag{2.1}$$

where ρ_{SR} describes the impact of implementing real-time immunization, ρ_{ER} describes the impact of cleaning the virus and immunizing the nodes, and μ describes the impact of quarantine or replacement. α is the transition rate from E to I, and γ is the recovery rate from I to R. τ is the time delay that the node usees antivirus software to clean the virus. β is the transition rate from S to E.

3. Local Stability of the Equilibrium and Existence of Hopf Bifurcation

We may see that the first three equations in (2.1) are independent of the fourth equation, and therefore, the fourth equation can be omitted without loss of generality. Hence, model (2.1) can be rewritten as

$$\begin{aligned}
\frac{dS(t)}{dt} &= uN - \beta I(t)S(t) - (\rho_{SR} + \mu)S(t), \\
\frac{dE(t)}{dt} &= \beta I(t)S(t) - (\alpha + \mu)E(t) - \rho_{ER}E(t-\tau), \\
\frac{dI(t)}{dt} &= \alpha E(t) - (\gamma + \mu)I(t).
\end{aligned} \tag{3.1}$$

For the convenience of description, we define the basic reproduction number of the infection as

$$R_0 = \frac{\mu N \beta \alpha}{(\rho_{SR} + \mu)(\alpha + \rho_{ER} + \mu)(\gamma + \mu)}. \tag{3.2}$$

Clearly, we have the following results with respect to the stable state of system (3.1). Here, the proof is omitted (see [17] for the details).

Theorem 3.1. *If $R_0 < 1$, system* (3.1) *has only the disease-free equilibrium $E_0 = (\mu N/(\rho_{SR}+\mu), 0, 0)$ and is globally asymptotically stable. If $R_0 > 1$, E_0 becomes unstable and there exists a unique positive equilibrium E_{ve}, where $E_{ve} = (\mu N/(\rho_{SR} + \mu)R_0,\ \mu N(R_0 - 1)/R_0(\alpha + \mu + \rho_{ER}),\ \alpha E^*/(\gamma + \mu))$. Furthermore, for any $\tau > 0$, E_0 is asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.*

To investigate the qualitative properties of the positive equilibrium E^* with $\tau > 0$, it is necessary to make the following assumption:

(H1) $R_0 > 1$.

Under hypothesis (H1), the Jacobian matrix of the system (3.1) about E_{ve} is given by

$$J(E_{ve}) = \begin{bmatrix} -a_1 & 0 & -a_2 \\ a_3 & -a_4 - a_7 e^{-\lambda\tau} & a_2 \\ 0 & a_5 & -a_6 \end{bmatrix}, \tag{3.3}$$

where $a_1 = \beta I + \rho_{SR} + \mu,\ a_2 = \beta S,\ a_3 = \beta I,\ a_4 = \alpha + \mu,\ a_5 = \alpha,\ a_6 = \gamma + \mu,\ a_7 = \rho_{ER}$.

We can obtain the following characteristic equation:

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 + e^{-\lambda\tau}\left(b_4\lambda^2 + b_5\lambda + b_6\right) = 0, \tag{3.4}$$

where

$$\begin{gathered} b_1 = a_1 + a_4 + a_6, \qquad b_2 = a_1a_6 + a_4(a_1 + a_6) - a_2a_5, \qquad b_3 = a_1a_4a_6 - a_1a_2a_5 + a_2a_3a_5, \\ b_4 = a_7, \qquad b_5 = a_7(a_1 + a_6), \qquad b_6 = a_1a_6a_7. \end{gathered} \tag{3.5}$$

If $i\omega$ $(\omega > 0)$ is a root of (3.4), then

$$-i\omega^3 - b_1\omega^2 + b_2 i\omega + b_3 + e^{-i\omega\tau}\left(-\omega^2 b_4 + b_5 i\omega + b_6\right) = 0. \tag{3.6}$$

Separating the real and imaginary parts of (3.6), we have

$$\begin{aligned} b_5\omega\sin\omega\tau + \left(b_6 - b_4\omega^2\right)\cos\omega\tau &= b_1\omega^2 - b_3, \\ b_5\omega\cos\omega\tau - \left(b_6 - b_4\omega^2\right)\sin\omega\tau &= \omega^3 - b_2\omega. \end{aligned} \tag{3.7}$$

Adding up the squares of (3.7) yields

$$\omega^6 + \left(b_1^2 - 2b_2 - b_4^2\right)\omega^4 + \left(b_2^2 - 2b_1b_3 + 2b_4b_6 - b_5^2\right)\omega^2 + \left(b_3^2 - b_6^2\right) = 0. \tag{3.8}$$

Letting $z = \omega^2,\ c_1 = b_1^2 - 2b_2 - b_4^2,\ c_2 = b_2^2 - 2b_1b_3 + 2b_4b_6 - b_5^2,\ c_3 = b_3^2 - b_6^2$, then (3.8) becomes

$$z^3 + c_1z^2 + c_2z + c_3 = 0. \tag{3.9}$$

Letting $z^* = (1/3)(-c_1 + \sqrt{c_1^2 - 3c_2})$, $h(z^*) = (z^*)^3 + c_1(z^*)^2 + c_2z^* + c_3$, then we have the following results (see [18–22] for details) about the distributions of the positive roots of (3.9).

Lemma 3.2 (see [18–22]). (i) *If $c_3 < 0$, then* (3.9) *has at least one positive root.*

(ii) *If $c_3 \geq 0$ and $c_1^2 - 3c_2 \leq 0$, then* (3.9) *has no positive root.*

(iii) *If $c_3 \geq 0$ and $c_1^2 - 3c_2 > 0$, then* (3.9) *has positive roots if and only if $z^* > 0$ and $h(z^*) \leq 0$.*

Suppose (3.9) has positive roots; without loss of generality, we assume that it has three positive roots defined by $\omega_k = \sqrt{z_k}$, $k = 1, 2, 3$. By (3.7), we have

$$\cos(\omega_k \tau) = \frac{(b_1\omega_k^2 - b_3)(b_6 - b_4\omega_k^2) + b_5\omega_k^2(\omega_k^2 - b_2)}{b_5^2\omega_k^2 + (b_6 - b_4\omega_k^2)^2}. \tag{3.10}$$

Thus, denoting

$$\tau_k^j = \frac{1}{\omega_k}\arccos\frac{(b_1\omega_k^2 - b_3)(b_6 - b_4\omega_k^2) + b_5\omega_k^2(\omega_k^2 - b_2)}{b_5^2\omega_k^2 + (b_6 - b_4\omega_k^2)^2} + \frac{2j\pi}{\omega_k}, \tag{3.11}$$

where $k = 1, 2, 3$; $j = 0, 1, \ldots$, then $\pm i\omega$ is a pair of purely imaginary roots of (3.4) with τ_k^j. Define

$$\tau_0 = \tau_{k_0}^0 = \min_{k=1,2,3}\left\{\tau_{k_0}^0\right\}, \qquad \omega_0 = \omega_{k_0}. \tag{3.12}$$

Note that when $\tau = 0$, (3.4) becomes

$$\lambda^3 + (b_1 + b_4)\lambda^2 + (b_2 + b_5)\lambda + (b_3 + b_6) = 0. \tag{3.13}$$

In addition, Routh-Hurwitz criterion [13] implies that, if the following condition holds, then all roots of (3.13) have negative real parts.

(H2) $(b_1 + b_4) > 0$, $(b_1 + b_4)(b_2 + b_5) - (b_3 + b_6) > 0$.

Till now, we can employ a result from Ruan and Wei [23] to analyze (3.4), which is, for the convenience of the reader, stated as follows.

Lemma 3.3 (see [23]). *Consider the exponential polynomial*

$$\begin{aligned} P\left(\lambda, e^{-\lambda\tau}, \ldots, e^{-\lambda\tau_m}\right) &= \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} + \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right]e^{-\lambda\tau} \\ &\quad + \cdots + \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right]e^{-\lambda\tau_m}, \end{aligned} \tag{3.14}$$

where $\tau_i \geq 0$ $(i = 1, 2, \ldots, m)$ and $p_j^{(i)}$ $(j = 1, 2, \ldots, m)$ are constants. As $(\tau_1, \tau_2, \ldots, \tau_m)$ vary, the sum of the order of the zeros of $P(\lambda, e^{-\lambda\tau}, \ldots, e^{-\lambda\tau_m})$ on the open right half plane can change only if a zero appears on or crosses the imaginary axis.

Using Lemmas 3.2 and 3.3 we can easily obtain the following results on the distribution of roots of the transcendental (3.4).

Lemma 3.4. (2.1) *If $c_3 > 0$ and $c_1^2 - 3c_2 \leq 0$, then all roots with positive real parts of* (3.4) *have the same sum as those of the polynomial* (3.13) *for all $\tau \geq 0$.*

(3.1) *If either $c_3 < 0$ or $c_3 \geq 0$ and $c_1^2 - 3c_2 > 0$, $z^* > 0$, $h(z^*) \leq 0$, then all roots with positive real parts of* (3.4) *have the same sum as those of the polynomial* (3.13) *for $\tau \in [0, \tau_0)$.*

Lemma 3.5. *If $3w_k^4 + c_1 w_k^1 + c_2 \neq 0$, then the following transversality condition holds:*

$$\operatorname{sgn}\left\{\operatorname{Re}\left\{\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}\right\} \neq 0 \quad \text{when } \tau = \tau_0. \tag{3.15}$$

Proof. Differentiating (3.4) with respect to τ yields

$$\left[3\lambda^2 + 2b_1\lambda + b_2 + \left(2b_4\lambda + b_5 - \tau\left(b_4\lambda^2 + b_5\lambda + b_6\right)e^{-\lambda\tau}\right)\right]\frac{d\lambda}{d\tau} = \lambda\left(b_4\lambda^2 + b_5\lambda + b_6\right)e^{-\lambda\tau}. \tag{3.16}$$

For the sake of simplicity, denoting ω_0 and τ_0 by ω, τ respectively, then

$$\begin{aligned}\left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{3\lambda^2 + 2b_1\lambda + b_2}{\lambda(b_4\lambda^2 + b_5\lambda + b_6)e^{-\lambda\tau}} + \frac{2b_4\lambda + b_5}{\lambda(b_4\lambda^2 + b_5\lambda + b_6)} - \frac{\tau}{\lambda} \\ &= -\frac{2\lambda^3 + b_1\lambda^2 - b_3}{\lambda^2(\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3)} + \frac{b_4\lambda^2 - b_6}{\lambda(b_4\lambda^2 + b_5\lambda + b_6)} - \frac{\tau}{\lambda} \\ &= \frac{-2i\omega^3 - b_1\omega^2 - b_3}{\omega^2(b_3 - b_1\omega^2 - i(\omega^3 - b_2\omega))} + \frac{b_4\omega^2 + b_6}{\omega^2(b_6 - b_3\omega^2 + b_4 i\omega)} - \frac{\tau}{i\omega}.\end{aligned} \tag{3.17}$$

Then we get

$$\begin{aligned}\operatorname{Re}\left\{\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\} &= -\frac{1}{\omega^2}\left[\frac{b_3^3 - 2\omega^6 - (b_1^2 - 2b_2)\omega^4}{(b_3 - b_1\omega^2)^2 + (\omega^3 - b_2\omega)^2} + \frac{b_6^2 - b_4^2\omega^4}{(b_6 - b_4\omega^2)^2 + b_5^2\omega^2}\right] \\ &= \frac{2\omega^6 + c_1\omega^4 - c_3}{\omega^2\left((b_6 - b_4\omega^2)^2 + b_5^2\omega^2\right)} = \frac{3\omega^4 + c_1\omega^2 + c_2}{\left((b_6 - b_4\omega^2)^2 + b_5^2\omega^2\right)}.\end{aligned} \tag{3.18}$$

Then, if $3\omega^4 + c_1\omega^2 + c_2 \neq 0$, we have $\operatorname{sgn}\{\operatorname{Re}\{(d\lambda/d\tau)^{-1}\}\} \neq 0$, we complete proof. □

Thus from Lemmas 3.2, 3.3, 3.4, and 3.5, and we have the following.

Theorem 3.6. *Suppose that (H1) and (H2) hold, then the following results hold.*

(1) *The positive equilibrium of* (3.1) *is asymptotically stable, if $c_3 > 0$ and $c_1^2 - 3c_2 \leq 0$;*

(2) *if either $c_3 < 0$ or $c_3 \geq 0$ and $c_1^2 - 3c_2 > 0$, $z^* > 0$, $h(z^*) \leq 0$, system* (3.1) *is asymptotically stable for $\tau \in [0, \tau_0)$ and system* (3.1) *undergoes a Hopf bifurcation at the origin when $\tau = \tau_0$.*

4. Direction of the Hopf Bifurcation

In this section, we derive explicit formulae for computing the direction of the Hopf bifurcation and the stability of bifurcation periodic solution at critical values τ_0 by using the normal form theory and center manifold reduction.

Letting $x_1 = S - S^*$, $x_2 = E - E^*$, $x_3 = I - I^*$, $\overline{x}_i(t) = x_i(\tau t)$, $\tau = \tau_0 + \mu$, and dropping the bars for simplification of notation, system (3.1) is transformed into an FDE as

$$\dot{x}(t) = L_\mu(x_t) + f(\mu, x_t), \tag{4.1}$$

with

$$L_\mu \varphi = (\tau_0 + \mu)\left[B_1\varphi(0) + B_2\varphi(-1)\right], \tag{4.2}$$

where

$$\begin{gathered} B_1 = \begin{bmatrix} -a_1 & 0 & -a_2 \\ a_3 & -a_4 & a_2 \\ 0 & a_5 & -a_6 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -a_7 e^{-\lambda\tau} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ f(\mu, \varphi) = (\tau_0 + \mu)\begin{pmatrix} -\beta\varphi_1(0)\varphi_2(0) \\ \beta\varphi_1(0)\varphi_2(0) \\ 0 \end{pmatrix}. \end{gathered} \tag{4.3}$$

Using the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$, such that

$$L_\mu \varphi = \int_{-1}^{0} d\eta(\theta, \mu)\varphi(\theta) \quad \varphi \in C. \tag{4.4}$$

In fact, we can choose

$$\eta(\theta, \mu) = (\tau_0 + \mu)[B_1\delta(\theta) + B_2\delta(\theta + 1)], \tag{4.5}$$

where $\delta(\theta)$ is Dirac delta function.

In the next, for $\varphi \in [-1, 0]$, we define

$$A(\mu)\varphi = \begin{cases} \dfrac{d\varphi}{d\theta}, & \theta \in [-1, 0), \\ \displaystyle\int_{-1}^{0} d\eta(\theta, \mu)\varphi(\theta), & \theta = 0, \end{cases} \tag{4.6}$$

$$R(\mu)\varphi = \begin{cases} 0, & \theta \in [-1, 0], \\ f(\mu, \varphi), & \theta = 0. \end{cases} \tag{4.7}$$

Then system (4.2) can be rewritten as

$$\dot{x}(t) = A(\mu)x_t + R(\mu)x_t, \tag{4.8}$$

where $x_t(\theta) = x(t+\theta)$.

The adjoint operator A^* of A is defined by

$$A^*(\mu)\psi = \begin{cases} -\dfrac{d\psi(s)}{d\theta}, & s \in (0,1], \\ \displaystyle\int_{-1}^{0} d\eta^T(t,0)\psi(-t), & s = 0, \end{cases} \tag{4.9}$$

where η^T is the transpose of the matrix η.

For $\varphi \in C^1[-1,0]$ and $\psi \in C^1[0,1]$, we define

$$\langle \psi, \varphi \rangle = \overline{\psi}(0) \cdot \varphi(0) - \int_{\theta=-1}^{0} \int_{\xi=0}^{\theta} \overline{\psi}(\xi - \theta) d\eta(\theta)\varphi(\xi) d\xi, \tag{4.10}$$

where $\eta(\theta) = \eta(\theta, 0)$. We know that $\pm i\tau_0\omega_0$ is an eigenvalue of $A(0)$, so $\pm i\tau_0\omega_0$ is also an eigenvalue of $A^*(0)$. We can get

$$q(\theta) = \begin{pmatrix} 1 \\ q_1 \\ q_2 \end{pmatrix} e^{i\tau_0\omega_0\theta}, \quad -1 < \theta \leq 0. \tag{4.11}$$

From the above discussion, it is easy to know that

$$Aq(0) = i\tau_0\omega_0 q(0). \tag{4.12}$$

Hence we obtain

$$\begin{aligned} q_1 &= \frac{i\omega_0 q_2}{a_5}, \\ q_2 &= -\frac{i\omega_0 + a_1}{a_2}. \end{aligned} \tag{4.13}$$

Suppose that the eigenvector q^* of A^* is

$$q^*(s) = \begin{pmatrix} 1 \\ {q_1}^* \\ {q_2}^* \end{pmatrix} e^{i\tau_0\omega_0 s}, \tag{4.14}$$

Then the following relationship is obtained:

$$A^* q(0) = -i\tau_0\omega_0 q^*(0). \tag{4.15}$$

Hence we obtain

$$q_1^* = \frac{a_1 - i\omega_0}{a_3},$$
$$q_2^* = \frac{a_4 + a_7 e^{i\omega_0\tau_0}}{a_5} q_1^*. \tag{4.16}$$

Let

$$\langle q^*, q\rangle = 1. \tag{4.17}$$

One can obtain

$$\begin{aligned}
\langle q^*, q\rangle &= \overline{q}^*(0)\cdot q(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \overline{q}^{*T}(\xi-\theta)\,d\eta(\theta)\varphi(\xi)\,d\xi \\
&= \frac{1}{\rho}\left(1 + q_1\overline{q}_1{}^* + q_2\overline{q}_2{}^*\right) \\
&\quad - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \tau_0 \frac{1}{\rho}\begin{pmatrix}1 & \overline{q}_1{}^* & \overline{q}_2{}^*\end{pmatrix}\left[\begin{pmatrix}-a_1 & 0 & -a_2\\ a_3 & -a_4 & a_2\\ 0 & a_5 & -a_6\end{pmatrix}\delta(\theta)\right. \\
&\qquad\qquad \left. + \begin{pmatrix}0 & 0 & 0\\ 0 & -a_7 & 0\\ 0 & 0 & 0\end{pmatrix}\delta(\theta+1)\right]\begin{pmatrix}1\\ q_1\\ q_2\end{pmatrix} e^{i\tau_0\omega_0\theta}\,d\xi\,d\theta \\
&= \frac{1}{\rho}\left(1 + q_1\overline{q}_1{}^* + q_2\overline{q}_2{}^*\right) - \frac{1}{\rho}\tau_0 e^{-i\omega_0\tau_0} a_7 q_1 \overline{q}_1{}^*.
\end{aligned} \tag{4.18}$$

Hence we obtain

$$\rho = \left(1 + q_1\overline{q}_1{}^* + q_2\overline{q}_2{}^*\right) - \tau_0 e^{-i\omega_0\tau_0} a_7 q_1 \overline{q}_1{}^*. \tag{4.19}$$

In the remainder of this section, by using the same notations as in Hassard et al. [16], we first compute the coordinates for describing the center manifold C_0 at $\mu = 0$. Leting x_t be the solution of (4.1) with $\mu = 0$, we define

$$z(t) = \langle q^*, x_t\rangle,$$
$$W(t,\theta) = x_t - 2\,\mathrm{Re}\{z(t)q(\theta)\}. \tag{4.20}$$

On the center manifold C_0 we have

$$W(t,\theta) = W(z,\overline{z},t), \tag{4.21}$$

where

$$W(z,\overline{z},t) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)\frac{z\overline{z}}{2} + W_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots. \tag{4.22}$$

In fact, z and $\overline{z}$ are local coordinate for C_0 in the direction of q and q^*. Note that, if x_t is, we will deal with real solutions only. Since $\mu = 0$

$$\begin{aligned}\dot{z}(t) &= \langle q^*, \dot{x}_t\rangle = \langle q^*, A(\mu)x_t + R(\mu)x_t\rangle = \langle q^*, Ax_t\rangle + \langle q^*, Rx_t\rangle \\ &= i\tau_0 w_0 z + \overline{q}^*(0)\cdot f\big(0, W(t,0) + 2\operatorname{Re}\big[z(t)q(0)\big]\big).\end{aligned} \tag{4.23}$$

Rewrite (4.23) as

$$\dot{z}(t) = i\tau_0\omega_0 z + g(z,\overline{z}), \tag{4.24}$$

where

$$g(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \tag{4.25}$$

From (4.1) and (4.24), we have

$$\dot{W} = \dot{x}_t - \dot{z}q - \dot{\overline{z}}\overline{q} = \begin{cases} AW - 2\operatorname{Re}\big[\overline{q}^*(0)\cdot f(z,\overline{z})q(\theta)\big], & \theta\in[-2\tau,0), \\ AW - 2\operatorname{Re}\big[\overline{q}^*(0)\cdot f(z,\overline{z})q(\theta)\big] + f_0(z,\overline{z}), & \theta = 0. \end{cases} \tag{4.26}$$

Let

$$\dot{W} = AW + H(z,\overline{z},\theta), \tag{4.27}$$

where

$$H(z,\overline{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)\frac{z\overline{z}}{2} + H_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots. \tag{4.28}$$

Expanding the above series and comparing the corresponding coefficients, we obtain

$$(A - 2iw_0)W_{20}(\theta) = -H_{20}(\theta), \qquad AW_{11}(\theta) = -H_{11}(\theta), \qquad (A + 2iw_0)W_{02}(\theta) = -H_{02}(\theta). \tag{4.29}$$

Since $x_t = x(t+\theta) = W(z,\overline{z},\theta) + zq + \overline{z}\cdot\overline{q}$, we have

$$x_t = \begin{pmatrix} W^{(1)}(z,\overline{z},\theta) \\ W^{(2)}(z,\overline{z},\theta) \\ W^{(3)}(z,\overline{z},\theta) \end{pmatrix} + z\begin{pmatrix} 1 \\ q_1 \\ q_2 \end{pmatrix} e^{i\omega_0\theta} + \overline{z}\begin{pmatrix} 1 \\ \overline{q}_1 \\ \overline{q}_2 \end{pmatrix} e^{-i\omega_0\theta}. \tag{4.30}$$

Thus, we can obtain

$$\begin{aligned}\varphi_1(0) &= z + \overline{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2},\\ \varphi_2(0) &= zq_1 + \overline{z}\overline{q}_1^* + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2}.\end{aligned} \tag{4.31}$$

So

$$\varphi_1(0)\varphi_2(0) = q_1z^2 + \overline{q}_1\overline{z}^2 + (q_1+q_2)z\overline{z} + \left(W_{11}^{(2)} + \frac{1}{2}W_{20}^{(2)} + W_{11}^{(1)}q_1 + \frac{1}{2}W_{20}^{(1)}\overline{q}\right)z^2\overline{z}. \tag{4.32}$$

It follows from (4.24) and (4.25) that

$$f(\varphi,\mu) = \begin{pmatrix} K_{11}z^2 + K_{12}z\overline{z} + K_{13}\overline{z}^2 + K_{14}z^2\overline{z} \\ K_{21}z^2 + K_{22}z\overline{z} + K_{23}\overline{z}^2 + K_{24}z^2\overline{z} \\ 0 \end{pmatrix}, \tag{4.33}$$

where

$$\begin{gathered} K_{11} = -\beta q_1, \qquad K_{12} = -\beta\overline{q}_1, \qquad K_{13} = -\beta(q_1+\overline{q}_1),\\ K_{14} = -\beta\left(W_{11}^{(2)} + \frac{1}{2}W_{20}^{(2)} + W_{11}^{(1)}q_1 + \frac{1}{2}W_{20}^{(1)}\overline{q}\right),\\ K_{21} = \beta q_1, \qquad K_{22} = \beta\overline{q}_1, \qquad K_{23} = \beta(q_1+\overline{q}_1),\\ K_{24} = \beta\left(W_{11}^{(2)} + \frac{1}{2}W_{20}^{(2)} + W_{11}^{(1)}q_1 + \frac{1}{2}W_{20}^{(1)}\overline{q}\right). \end{gathered} \tag{4.34}$$

Since $q^*(0) = (1/\rho)(1,\overline{q_1^*},\overline{q_2^*})^T$, we have

$$g(z,\overline{z}) = \frac{1}{\rho}\left(1,\overline{q_1^*},\overline{q_2^*}\right)\begin{pmatrix} K_{11}z^2 + K_{12}z\overline{z} + K_{13}\overline{z}^2 + K_{14}z^2\overline{z} \\ K_{21}z^2 + K_{22}z\overline{z} + K_{23}\overline{z}^2 + K_{24}z^2\overline{z} \\ 0 \end{pmatrix}. \tag{4.35}$$

Comparing the coefficients of the above equation with those in (4.27), we have

$$\begin{aligned} g_{20} &= \frac{1}{\rho}\left(K_{11} + K_{21}\overline{q_1^*}\right), \qquad g_{11} = \frac{1}{\rho}\left(K_{12} + K_{22}\overline{q_1^*}\right),\\ g_{02} &= \frac{1}{\rho}\left(K_{13} + K_{23}\overline{q_1^*}\right), \qquad g_{21} = \frac{1}{\rho}\left(K_{14} + K_{24}\overline{q_1^*}\right). \end{aligned} \tag{4.36}$$

In what follows, we focus on the computation of $W_{20}(\theta)$ and $W_{11}(\theta)$. For the expression of g_{21}, we have

$$\begin{aligned} H(z,\overline{z},\theta) &= -2\operatorname{Re}\left[\overline{q}^{*}(0)\cdot f(z,\overline{z})q(\theta)\right] \\ &= -\left(g_{20}\frac{z^2}{2}+g_{11}z\overline{z}+g_{02}\frac{\overline{z}^2}{2}+\cdots\right)q(\theta)-\left(\overline{g}_{20}\frac{\overline{z}^2}{2}+\overline{g}_{11}z\overline{z}+\overline{g}_{02}\frac{z^2}{2}+\cdots\right)q(\theta). \end{aligned} \tag{4.37}$$

Comparing the coefficients of the above equation, we can obtain that

$$H_{20}(\theta) = -g_{20}q(\theta)-\overline{g}_{02}\overline{q}(\theta), \quad \theta\in[-1,0), \tag{4.38}$$

$$H_{11}(\theta) = -g_{11}q(\theta)-\overline{g}_{11}\overline{q}(\theta), \quad \theta\in[-1,0). \tag{4.39}$$

Substituting (4.39) into (4.27) and (4.38) into (4.27), respectively, we get

$$\begin{aligned} \dot{W}_{20}(\theta) &= 2i\tau_0\omega_0W_{20}(\theta)+g_{20}q(\theta)+\overline{g}_{20}\overline{q}(\theta), \\ \dot{W}_{11}(\theta) &= +g_{11}q(\theta)+\overline{g}_{11}\overline{q}(\theta). \end{aligned} \tag{4.40}$$

So

$$\begin{aligned} W_{20}(\theta) &= \frac{ig_{20}}{\tau_0\omega_0}q(0)e^{i\tau_0\omega_0\theta}-\frac{\overline{g}_{02}}{3i\tau_0\omega_0}\overline{q}(0)e^{-i\tau_0\omega_0\theta}+E_1e^{2i\tau_0\omega_0\theta}, \\ W_{11}(\theta) &= \frac{g_{11}}{i\tau_0\omega_0}q(0)e^{i\tau_0\omega_0\theta}-\frac{\overline{g}_{11}}{i\tau_0\omega_0}\overline{q}(0)e^{-i\tau_0\omega_0\theta}+E_2. \end{aligned} \tag{4.41}$$

In the sequel, we will determine E_1 and E_2. Form the definition of A in (4.8), we have

$$\int_{-1}^{0}d\eta(\theta)W_{20}(\theta) = 2i\tau_0\omega_0W_{20}(0)-H_{20}(0), \tag{4.42}$$

$$\int_{-1}^{0}d\eta(\theta)W_{11}(\theta) = -H_{11}(0). \tag{4.43}$$

From (4.6) and (4.38)-(4.39), we have

$$H_{20}(\theta) = -g_{20}q(\theta)-\overline{g}_{02}\overline{q}(\theta)+(K_{11},K_{21},0)^T, \tag{4.44}$$

$$H_{11}(\theta) = -g_{11}q(\theta)-\overline{g}_{11}\overline{q}(\theta)+(K_{12},K_{22},0)^T. \tag{4.45}$$

Substituting (4.41) and (4.44) into (4.42) and noticing that

$$\begin{gathered}\left(i\omega_0 I - \int_{-1}^{0} e^{i\omega_0\theta} d\eta(\theta)\right) q(0) = 0, \\ \left(-i\omega_0 I - \int_{-1}^{0} e^{-i\omega_0\theta} d\eta(\theta)\right) \overline{q}(0) = 0,\end{gathered} \tag{4.46}$$

we can obtain

$$\left(2i\omega_0 I - \int_{-1}^{0} e^{2i\tau_0\omega_0\theta} d\eta(\theta)\right) E_1 = \begin{pmatrix} K_{11} & K_{21} & 0 \end{pmatrix}^T, \tag{4.47}$$

which leads to

$$\begin{gathered}\begin{pmatrix} 2i\omega_0 + a_1 & 0 & a_2 \\ -a_3 & 2i\omega_0 + a_4 + a_7 e^{-2i\omega_0\tau_0} & -a_2 \\ 0 & -a_5 & 2i\omega_0 + a_6 \end{pmatrix} \begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \\ E_1^{(3)} \end{pmatrix} = \begin{pmatrix} K_{11} \\ K_{21} \\ 0 \end{pmatrix}, \\ \begin{pmatrix} a_1 & 0 & a_2 \\ -a_3 & a_4 + a_7 & -a_2 \\ 0 & -a_5 & a_6 \end{pmatrix} \begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \\ E_2^{(3)} \end{pmatrix} = \begin{pmatrix} K_{12} \\ K_{22} \\ 0 \end{pmatrix}.\end{gathered} \tag{4.48}$$

It follows that

$$\begin{gathered}E_1^{(1)} = \frac{K_{11} - a_2 E_1^{(3)}}{2i\omega_0 + a_1}, \\ E_1^{(2)} = \frac{2i\omega_0 + a_6}{a_5} E_1^{(3)}, \\ E_1^{(3)} = \frac{K_{22} - K_{11}/(2i\omega_0 + a_1)}{a_2 a_3/(2i\omega_0 + a_1) + (2i\omega_0 + a_4 + a_7 e^{-2i\omega_0\tau_0})((2i\omega_0 + a_6)/a_5) - 2i\omega_0 - a_6}, \\ E_2^{(1)} = \frac{K_{12} - a_2 E_2^{(3)}}{a_1}, \qquad E_2^{(2)} = \frac{a_6}{a_5} E_2^{(3)}, \qquad E_2^{(3)} = \frac{a_1 a_5 K_{22} - a_5 K_{12}}{a_2 a_3 a_5 + (a_4 + a_7) a_1 a_6 - a_1 a_5 a_6}.\end{gathered} \tag{4.49}$$

Based on the above analysis, we can see each g_{ij} in (4.37) is determined by parameters and delays in (3.1). Thus, we can compute the following quantities:

$$\begin{gathered}\mu_2 = -\frac{\operatorname{Re} C_1(0)}{\operatorname{Re} \lambda'(\tau_0)}, \\ T_2 = -\frac{\operatorname{Im} C_1(0) + \mu_2 \operatorname{Im} \lambda'(0)}{\omega_0}, \\ \beta_2 = -2 \operatorname{Re} C_1(0).\end{gathered} \tag{4.50}$$

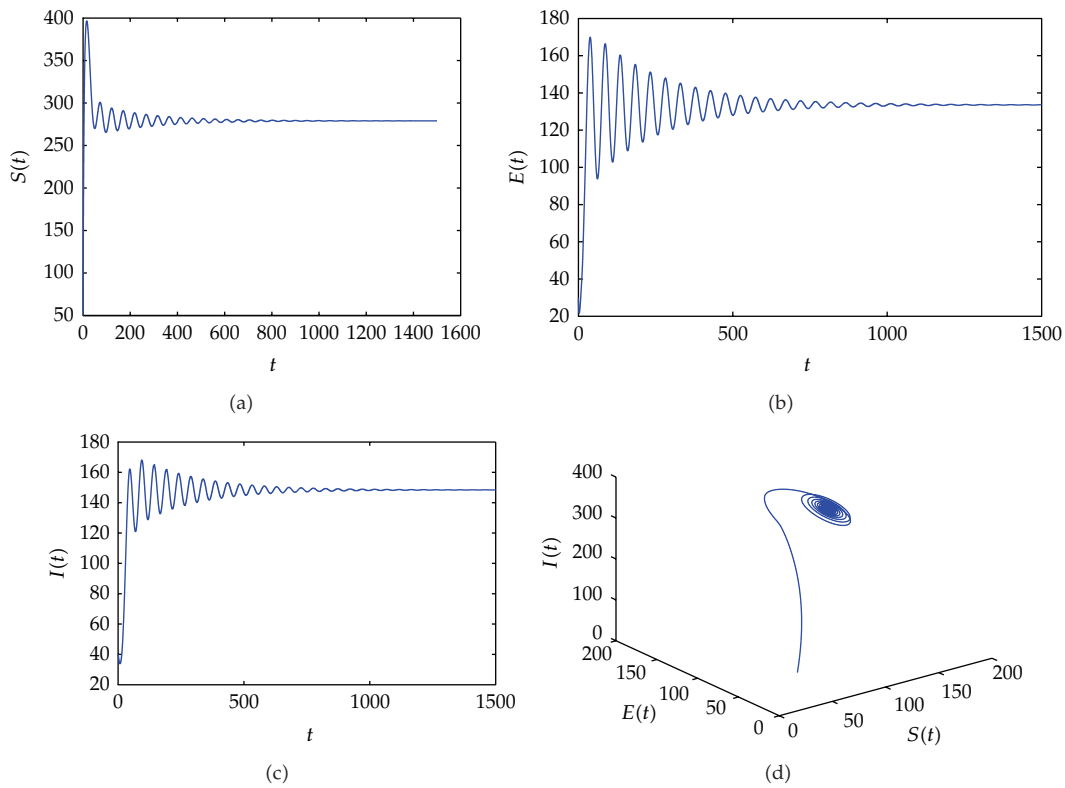

Figure 1: $\tau = 13 < \tau_0$. The positive equilibrium E^* of system (3.1) is asymptotically stable.

Theorem 4.1. *In* (4.50), *the following results hold.*

(1) *The sign of* μ_2 *determines the directions of the Hopf bifurcation: if* $\mu_2 > 0$ $(\mu_2 < 0)$ *then the Hopf bifurcation is forward (backward) and the bifurcating periodic solutions exist for* $\tau > \tau_0$ $(\tau < \tau_0)$.

(2) *The sign of* β_2 *determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable (unstable) if* $\beta_2 < 0$ $(\beta_2 > 0)$.

(3) *The sign of* T_2 *determines the period of the bifurcating periodic solutions: the period increases (decreases) if* $T_2 > 0$ $(T_2 < 0)$.

5. Numerical Examples

In this section, some numerical results of system (3.1) are presented to justify the Previous theorem above. As an example, considering the following parameters: $\mu = 0.01$, $N = 10000$, $\gamma = 0.08$, $\alpha = 0.1$, $\beta = 0.01$, $\rho_{SR} = 0.2$, $\rho_{ER} = 0.2$, then $R_0 = 1.706$, $c_3 = -3.6284e - 5$, and $E^* = (279, 133.6, 148.4)$. According to the Lemma 3.2, (3.9) has one positive real root $\omega = 0.1194$. Correspondingly, by (3.13), we obtain $\tau_0 = 14.05$. First, we choose $\tau = 13 < \tau_0$, the corresponding wave form and phase plots are shown in Figure 1; it is easy to see from Figure 1 that system (3.1) is asymptotically stable. Finally, we choose $\tau = 14.15 > \tau_0$ the

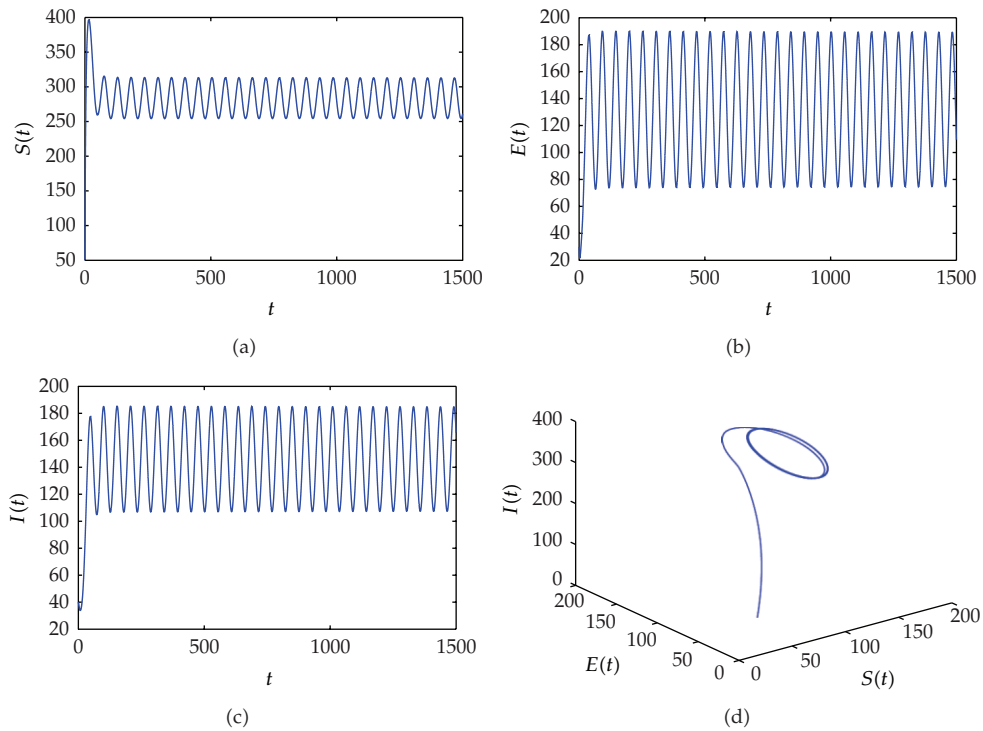

Figure 2: $\tau = 14.15 > \tau_0$. The bifurcation periodic solution for system (3.1) is stable.

corresponding wave form and phase plots are shown in Figure 2; it is easy to see that Figure 2 undergoes a Hopf bifurcation.

6. Conclusions

In this paper, considering that in addition to cleaning viruses in state I, people may immunize their computers with countermeasures in state S and state E, and since using antivirus software will take a period of time, we have constructed a computer virus model with time delay depending on the SEIR model. The theoretical analyses for the computer virus models are given. Furthermore, we have proved that when time cross through the critical value, the system exist a Hopf bifurcation. Finally, simulation clarifies our results.

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