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Research Article

The Well-Posedness of Solutions for a Generalized Shallow Water Wave Equation

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A nonlinear partial differential equation containing the famous Camassa-Holm and Degasperis-Procesi equations as special cases is investigated. The Kato theorem for abstract differential equations is applied to establish the local well-posedness of solutions for the equation in the Sobolev space $H^s(R)$ with $s > 3/2$. Although the H^1-norm of the solutions to the nonlinear model does not remain constant, the existence of its weak solutions in the lower-order Sobolev space H^s with $1 \leq s \leq 3/2$ is proved under the assumptions $u_0 \in H^s$ and $\|u_{0x}\|_{L^\infty} < \infty$.

1. Introduction

Constantin and Lannes [1] derived the shallow water equation

$$u_t + u_x + \frac{3}{2}\rho u u_x + \mu(\alpha u_{xxx} + \beta u_{txx}) = \rho\mu(\gamma u_x u_{xx} + \delta u u_{xxx}), \tag{1.1}$$

where the constants $\alpha, \beta, \gamma, \delta, \rho$, and μ satisfy certain conditions. Under several restrictions on the coefficients of model (1.1), the large time well-posedness was established on a time scale $O(|\rho|^{-1})$ provided that the initial value u_0 belongs to $H^s(R)$ with $s > 5/2$, and the wave-breaking phenomena were also discussed in [1]. As stated in [1], using suitable mathematical transformations, one can turn (1.1) into the form

$$u_t - u_{txx} + 2ku_x + muu_x = au_x u_{xx} + buu_{xxx}, \tag{1.2}$$

where a, b, k, and m are constants. Obviously, (1.2) is a generalization of both the Camassa-Holm equation [2]

$$u_t - u_{txx} + 2ku_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad k \text{ is a constant} \tag{1.3}$$

and the Degasperis-Procesi model [3]

$$u_t - u_{xxt} + 2ku_x + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{1.4}$$

Equations (1.3) and (1.4) are bi-Hamiltonian and arise in the modeling of shallow water waves. These two equations pertain to waves of medium amplitude (cf. the discussions in [1, 4]) and accommodate wave-breaking phenomena. Moreover, the Camassa-Holm and Degasperis-Procesi models admit peaked (periodic as well as solitary) traveling waves capturing the main feature of the exact traveling wave solutions of the greatest height of the governing equations for water waves (cf. [5, 6]). For other dynamic properties about (1.3) and (1.4), the reader is referred to [7–20].

Recently, Lai and Wu [21] investigate (1.2) in the case where $k = 0$, $m = a+b$, $a > 0$, and $b > 0$. The well-posedness of global solutions is established in [21] in Sobolev space $H^s(R)$ with $s > 3/2$ under certain assumptions on the initial value. The local strong and weak solutions for (1.2) are discussed in [22] in the case where $b > 0$, a, k, and m are arbitrary constants.

Motivated by the desire to extend the work in [22], we investigate the following generalized model of (1.2):

$$u_t - u_{txx} + 2ku_x + muu_x = au_x u_{xx} + buu_{xxx} + \beta u_x^n, \tag{1.5}$$

where $b > 0$, a, k, m, and β are arbitrary constants, and n is a positive integer.

The aim of this paper is to investigate (1.5). Since a, b, m, and β are arbitrary constants, we do not have the result that the H^1 norm of the solution of (1.5) remains constant. We will apply the Kato theorem [23] to prove the existence and uniqueness of local solutions for (1.5) in the space $C([0,T), H^s(R)) \bigcap C^1([0,T), H^{s-1}(R))(s > 3/2)$ provided that the initial value $u_0(x)$ belongs to $H^s(R)(s > 3/2)$. Moreover, it is shown that there exists a weak solution of (1.5) in lower-order Sobolev space $H^s(R)$ with $1 \le s \le 3/2$.

The structure of this paper is as follows. The main results are given in Section 2. The existence and uniqueness of the local strong solution for the Cauchy problem (1.5) are proved in Section 3. The existence of weak solutions is established in Section 4.

2. Main Results

Firstly, we give some notations.

The space of all infinitely differentiable functions $\phi(t,x)$ with compact support in $[0,+\infty) \times R$ is denoted by C_0^∞. We let $L^p = L^p(R)(1 \le p < +\infty)$ be the space of all measurable functions h such that $\|h\|_{L^p}^p = \int_R |h(t,x)|^p dx < \infty$. We define $L^\infty = L^\infty(R)$ with the standard norm $\|h\|_{L^\infty} = \inf_{m(e)=0} \sup_{x\in R\setminus e} |h(t,x)|$. For any real number s, we let $H^s = H^s(R)$ denote the Sobolev space with the norm defined by

$$\|h\|_{H^s} = \left(\int_R \left(1 + |\xi|^2\right)^s \left|\widehat{h}(t,\xi)\right|^2 d\xi \right)^{1/2} < \infty, \tag{2.1}$$

where $\widehat{h}(t,\xi) = \int_R e^{-ix\xi} h(t,x)dx$. Here, we note that the norms $\|\cdot\|_{L^p}^p$, $\|\cdot\|_{L^\infty}$, and $\|\cdot\|_{H^s}$ depend on variable t.

For $T > 0$ and nonnegative number s, let $C([0,T); H^s(R))$ denote the space of functions $u : [0,T) \times R \to R$ with the properties that $u(t,\cdot) \in H^s(R)$ for each $t \in [0,T]$, and the mapping $u : [0,T) \to H^s(R)$ is continuous and bounded.

For simplicity, throughout this paper, we let c denote any positive constant which is independent of parameter ε and set $\Lambda = (1 - \partial_x^2)^{1/2}$.

In order to study the existence of solutions for (1.5), we consider its Cauchy problem in the form

$$\begin{gathered} u_t - u_{txx} = -\partial_x\left(2ku + \frac{m}{2}u^2\right) + au_x u_{xx} + buu_{xxx} + \beta u_x^n, \\ u(0,x) = u_0(x), \end{gathered} \tag{2.2}$$

where $b > 0$, a, k, m, β, and n are arbitrary constants. Now, we give the theorem to describe the local well-posedness of solutions for problem (2.2).

Theorem 2.1. *Let $u_0(x) \in H^s(R)$ with $s > 3/2$, then the Cauchy problem* (2.2) *has a unique solution $u(t,x) \in C([0,T); H^s(R)) \bigcap C^1([0,T); H^{s-1}(R))$ where $T > 0$ depends on $\|u_0\|_{H^s(R)}$.*

For a real number s with $s > 0$, suppose that the function $u_0(x)$ is in $H^s(R)$, and let $u_{\varepsilon 0}$ be the convolution $u_{\varepsilon 0} = \phi_\varepsilon \star u_0$ of the function $\phi_\varepsilon(x) = \varepsilon^{-1/4}\phi(\varepsilon^{-1/4}x)$ and u_0 such that the Fourier transform $\widehat{\phi}$ of ϕ satisfies $\widehat{\phi} \in C_0^\infty$, $\widehat{\phi(\xi)} \geq 0$, and $\widehat{\phi(\xi)} = 1$ for any $\xi \in (-1,1)$. Thus one has $u_{\varepsilon 0}(x) \in C^\infty$. It follows from Theorem 2.1 that for each ε satisfying $0 < \varepsilon < 1/4$, the Cauchy problem

$$\begin{gathered} u_t - u_{txx} = -\partial_x\left(2ku + \frac{m}{2}u^2\right) + au_x u_{xx} + buu_{xxx} + \beta u_x^n, \\ u(0,x) = u_{\varepsilon 0}(x), \quad x \in R \end{gathered} \tag{2.3}$$

has a unique solution $u_\varepsilon(t,x) \in C^\infty([0,T_\varepsilon); H^\infty)$, in which T_ε may depend on ε. However, one will show that under certain assumptions, there exist two constants c and $T > 0$, both independent of ε, such that the solution of problem (2.3) satisfies $\|u_{\varepsilon x}\|_{L^\infty} \leq c$ for any $t \in [0,T)$, and there exists a weak solution $u(t,x) \in L^2([0,T], H^s)$ for problem (2.2). These results are summarized in the following two theorems.

Theorem 2.2. *If $u_0(x) \in H^s(R)$ with $s \in [1, 3/2]$ such that $\|u_{0x}\|_{L^\infty} < \infty$, let $u_{\varepsilon 0}$ be defined as in system* (2.3)*, then there exist two constants c and $T > 0$, which are independent of ε, such that the solution u_ε of problem* (2.3) *satisfies $\|u_{\varepsilon x}\|_{L^\infty} \leq c$ for any $t \in [0,T)$.*

Theorem 2.3. *Suppose that $u_0(x) \in H^s$ with $1 \leq s \leq 3/2$ and $\|u_{0x}\|_{L^\infty} < \infty$, then there exists a $T > 0$ such that problem* (2.2) *has a weak solution $u(t,x) \in L^2([0,T], H^s(R))$ in the sense of distribution and $u_x \in L^\infty([0,T] \times R)$.*

3. Proof of Theorem 2.1

Consider the abstract quasilinear evolution equation

$$\frac{dv}{dt} + A(v)v = f(v), \quad t \geq 0, \ v(0) = v_0. \tag{3.1}$$

Let X and Y be Hilbert spaces such that Y is continuously and densely embedded in X, and let $Q : Y \to X$ be a topological isomorphism. Let $L(Y, X)$ be the space of all bounded linear operators from Y to X. If $X = Y$, we denote this space by $L(X)$. We state the following conditions in which ρ_1, ρ_2, ρ_3, and ρ_4 are constants depending only on $\max\{\|y\|_Y, \|z\|_Y\}$:

(I) $A(y) \in L(Y, X)$ for $y \in X$ with

$$\left\|(A(y) - A(z))w\right\|_X \leq \rho_1 \left\|y - z\right\|_X \|w\|_Y, \quad y, z, \ w \in Y, \tag{3.2}$$

and $A(y) \in G(X, 1, \beta)$ (i.e., $A(y)$ is quasi-m-accretive), uniformly on bounded sets in Y.

(II) $QA(y)Q^{-1} = A(y) + B(y)$, where $B(y) \in L(X)$ is bounded, uniformly on bounded sets in Y. Moreover,

$$\left\|(B(y) - B(z))w\right\|_X \leq \rho_2 \left\|y - z\right\|_Y \|w\|_X, \quad y, z \in Y, \ w \in X. \tag{3.3}$$

(III) $f : Y \to Y$ extends to a map from X into X, is bounded on bounded sets in Y, and satisfies

$$\begin{aligned} &\left\|f(y) - f(z)\right\|_Y \leq \rho_3 \left\|y - z\right\|_Y, \quad y, \ z \in Y, \\ &\left\|f(y) - f(z)\right\|_X \leq \rho_4 \left\|y - z\right\|_X, \quad y, \ z \in Y. \end{aligned} \tag{3.4}$$

Kato Theorem (see [23])

Assume that (I), (II), and (III) hold. If $v_0 \in Y$, there is a maximal $T > 0$ depending only on $\|v_0\|_Y$ and a unique solution v to problem (3.1) such that

$$v = v(\cdot, v_0) \in C([0, T); Y) \bigcap C^1([0, T); X). \tag{3.5}$$

Moreover, the map $v_0 \to v(\cdot, v_0)$ is a continuous map from Y to the space

$$C([0, T); Y) \bigcap C^1([0, T); X). \tag{3.6}$$

In fact, problem (2.2) can be written as

$$\begin{aligned} u_t - u_{txx} &= -\left[2ku + \frac{m}{2}u^2\right]_x + \frac{b}{2}\partial_x^3 u^2 - \frac{3b - a}{2}\partial_x\left(u_x^2\right) + \beta u_x^n, \\ u(0, x) &= u_0(x), \end{aligned} \tag{3.7}$$

which is equivalent to

$$\begin{aligned} u_t + buu_x &= -\Lambda^{-2}\left[\left(2ku + \frac{m}{2}u^2\right)_x + buu_x - \frac{3b-a}{2}\partial_x\left(u_x^2\right) + \beta u_x^n\right], \\ u(0,x) &= u_0(x). \end{aligned} \tag{3.8}$$

We set $A(u) = bu\partial_x$ with constant $b > 0$, $Y = H^s(R)$, $X = H^{s-1}(R)$, $\Lambda = (1-\partial_x^2)^{1/2}$, $f(u) = -\Lambda^{-2}(2ku + (m/2)u^2)_x + b\Lambda^{-2}(uu_x) - ((3b-a)/2)\Lambda^{-2}\partial_x(u_x^2) + \beta\Lambda^{-2}u_x^n$, and $Q = \Lambda^s$. We know that Q is an isomorphism of H^s onto H^{s-1}. In order to prove Theorem 2.1, we only need to check that $A(u)$ and $f(u)$ satisfy assumptions (I)–(III).

Lemma 3.1. *The operator $A(u) = u\partial_x$ with $u \in H^s(R)$, $s > 3/2$ belongs to $G(H^{s-1}, 1, \beta)$.*

Lemma 3.2. *Let $A(u) = bu\partial_x$ with $u \in H^s$ and $s > 3/2$, then $A(u) \in L(H^s, H^{s-1})$ for all $u \in H^s$. Moreover,*

$$\|(A(u) - A(z))w\|_{H^{s-1}} \le \rho_1 \|u - z\|_{H^{s-1}} \|w\|_{H^s}, \quad u, z, w \in H^s(R). \tag{3.9}$$

Lemma 3.3. *For $s > 3/2$, $u, z \in H^s$, and $w \in H^{s-1}$, it holds that $B(u) = [\Lambda, u\partial_x]\Lambda^{-1} \in L(H^{s-1})$ for $u \in H^s$ and*

$$\|(B(u) - B(z))w\|_{H^{s-1}} \le \rho_2 \|u - z\|_{H^s} \|w\|_{H^{s-1}}. \tag{3.10}$$

Proofs of the above Lemmas 3.1–3.3 can be found in [24] or [25].

Lemma 3.4 (see [23]). *Let r and q be real numbers such that $-r < q \le r$, then*

$$\begin{aligned} \|uv\|_{H^q} &\le c\|u\|_{H^r}\|v\|_{H^q}, \quad \text{if } r > \frac{1}{2}, \\ \|uv\|_{H^{r+q-1/2}} &\le c\|u\|_{H^r}\|v\|_{H^q}, \quad \text{if } r < \frac{1}{2}. \end{aligned} \tag{3.11}$$

Lemma 3.5. *Let $u, z \in H^s$ with $s > 3/2$ and $f(u) = -\Lambda^{-2}(2ku + (m/2)u^2)_x + b\Lambda^{-2}(uu_x) - ((3b - a)/2)\Lambda^{-2}\partial_x(u_x^2) + \beta\Lambda^{-2}(u_x^n)$, then f is bounded in H^s and satisfies*

$$\begin{aligned} \|f(u) - f(z)\|_{H^s} &\le \rho_3 \|u - z\|_{H^s}, \\ \|f(u) - f(z)\|_{H^{s-1}} &\le \rho_4 \|u - z\|_{H^{s-1}}. \end{aligned} \tag{3.12}$$

Proof. Using the algebra property of the space H^{s_0} with $s_0 > 1/2$ and $s-1 > 1/2$, we have

$$\begin{aligned}
&\|f(u)-f(z)\|_{H^s}\\
&\quad\le \left\|\Lambda^{-2}\left(\left(2ku+\frac{m}{2}u^2\right)_x-\left(2kz+\frac{m}{2}z^2\right)_x\right)\right\|_{H^s}+\left\|b\Lambda^{-2}(uu_x-zz_x)\right\|_{H^s}\\
&\qquad+\left\|\frac{3b-a}{2}\Lambda^{-2}\partial_x\left(u_x^2-z_x^2\right)\right\|_{H^s}+\left\|\beta\Lambda^{-2}(u_x^n-z_x^n)\right\|_{H^s}\\
&\quad\le c\Bigg(\|u-z\|_{H^{s-1}}(1+\|u\|_{H^{s-1}}+\|z\|_{H^{s-1}})+\left\|\Lambda^{-2}\left(u^2-z^2\right)_x\right\|_{H^s}\\
&\qquad\qquad+\left\|u_x^2-z_x^2\right\|_{H^{s-1}}+\|u_x-z_x\|_{H^{s-1}}\sum_{j=0}^{n-1}\|u_x\|_{H^{s-1}}^{n-j}\|z_x\|_{H^{s-1}}^{j}\Bigg)\\
&\quad\le c\Bigg(\|u-z\|_{H^s}(1+\|u\|_{H^s}+\|z\|_{H^s})+\|(u-z)(u+z)\|_{H^{s-1}}\\
&\qquad\qquad+\left\|u_x^2-z_x^2\right\|_{H^{s-1}}+\|u_x-z_x\|_{H^{s-1}}\sum_{j=0}^{n-1}\|u\|_{H^s}^{n-j}\|z\|_{H^s}^{j}\Bigg)\\
&\quad\le c\|u-z\|_{H^s}\left(1+\|u\|_{H^s}+\|z\|_{H^s}+\sum_{j=0}^{n-1}\|u\|_{H^s}^{n-j}\|z\|_{H^s}^{j}\right)\\
&\quad\le c\rho_3\|u-z\|_{H^s},
\end{aligned}\tag{3.13}$$

from which we obtain (3.12).

Applying Lemma 3.4, $uu_x = 1/2(u^2)_x$, $s > 3/2$, we get

$$\begin{aligned}
&\|f(u)-f(z)\|_{H^{s-1}}\\
&\quad\le c\Bigg(\left\|2ku+\frac{m}{2}u^2-\left(2kz+\frac{m}{2}z^2\right)\right\|_{H^{s-2}}+\left\|u^2-z^2\right\|_{H^{s-2}}\\
&\qquad\qquad+\|(u_x-z_x)(u_x+z_x)\|_{H^{s-2}}+\|u_x^n-z_x^n\|_{H^{s-1}}\Bigg)\\
&\quad\le c\|u-z\|_{H^{s-2}}(1+\|u\|_{H^{s-1}}+\|z\|_{H^{s-1}})\\
&\qquad+c\|u_x-z_x\|_{H^{s-2}}(\|u_x\|_{H^{s-1}}+\|z_x\|_{H^{s-1}})\\
&\qquad+c\|u_x-z_x\|_{H^{s-2}}\sum_{j=0}^{n-1}\|u_x\|_{H^{s-1}}^{n-j}\|z_x\|_{H^{s-1}}^{j}\\
&\quad\le c\|u-z\|_{H^{s-1}}\left(1+\|u\|_{H^s}+\|z\|_{H^s}+\sum_{j=0}^{n-1}\|u\|_{H^s}^{n-j}\|z\|_{H^s}^{j}\right),
\end{aligned}\tag{3.14}$$

which completes the proof of (3.12). □

Proof of Theorem 2.1. Using the Kato theorem, Lemmas 3.1, 3.2, 3.3, and 3.5, we know that system (3.11) or problem (2.2) has a unique solution

$$u(t,x) \in C([0,T);H^s(R)) \bigcap C^1\left([0,T);H^{s-1}\right). \tag{3.15}$$

□

4. Proofs of Theorems 2.2 and 2.3

Before establishing the proofs of Theorems 2.2 and 2.3, we give several lemmas.

Lemma 4.1 (Kato and Ponce [26]). *If $r \geq 0$, then $H^r \bigcap L^\infty$ is an algebra. Moreover,*

$$\|uv\|_{H^r} \leq c(\|u\|_{L^\infty}\|v\|_{H^r} + \|u\|_{H^r}\|v\|_{L^\infty}), \tag{4.1}$$

where c is a constant depending only on r.

Lemma 4.2 (Kato and Ponce [26]). *Let $r > 0$. If $u \in H^r \bigcap W^{1,\infty}$ and $v \in H^{r-1} \bigcap L^\infty$, then*

$$\|[\Lambda^r,u]v\|_{L^2} \leq c\left(\|\partial_x u\|_{L^\infty}\left\|\Lambda^{r-1}v\right\|_{L^2} + \|\Lambda^r u\|_{L^2}\|v\|_{L^\infty}\right), \tag{4.2}$$

where $[\Lambda^r,u]v = \Lambda^r(uv) - u\Lambda^r v$.

Using the first equation of problem (2.2) *gives rise to*

$$\frac{d}{dt}\left[\int_R \left(u^2+u_x^2\right)dx\right] + (a-2b)\int_R (u_x)^3 dx = 2\beta\int_R uu_x^n\,dx, \tag{4.3}$$

from which one has

$$\int_R \left(u^2+u_x^2\right)dx + \int_0^t\left[\int_R\left((a-2b)u_x^3 - 2\beta uu_x^n\right)dx\right]d\tau = \int_R\left(u_0^2+u_{0x}^2\right)dx. \tag{4.4}$$

Lemma 4.3. *Let $s \geq 3/2$, and the function $u(t,x)$ is a solution of the problem* (2.2) *and the initial data $u_0(x) \in H^s$, then it holds that*

$$\|u\|_{L^\infty} \leq c\|u\|_{H^1} \leq c\|u_0\|_{H^1} e^{c_0\int_0^t(\|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^{n-1})d\tau}, \tag{4.5}$$

where $c_0 = 1/2\max(|a-2b|,|\beta|)$.

For $q \in (0,s-1]$, there is a constant c depending only on q such that

$$\int_R \left(\Lambda^{q+1}u\right)^2 dx \leq \int_R\left(\Lambda^{q+1}u_0\right)^2 dx + c\int_0^t\left(\|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^{n-1}\right)\|u\|_{H^{q+1}}^2\,d\tau. \tag{4.6}$$

If $q \in [0,s-1]$, there is a constant c depending only on q such that

$$\|u_t\|_{H^q} \leq c\|u\|_{H^{q+1}}\left(1 + \|u\|_{H^1} + \|u_x\|_{L^\infty}^{n-1}\right). \tag{4.7}$$

Proof. Using $\|u\|_{H^1}^2 \le c\int_R (u^2+u_x^2)dx$ and (4.4) derives (4.5).

We write (1.5) in the equivalent form

$$u_t - u_{txx} = -\left[2ku + \frac{m}{2}u^2\right]_x + \frac{b}{2}\partial_x^3 u^2 - \frac{3b-a}{2}\partial_x\left(u_x^2\right) + \beta u_x^n. \tag{4.8}$$

Applying $\partial_x^2 = -\Lambda^2 + 1$ and the Parseval's equality gives rise to

$$\int_R \Lambda^q u \Lambda^q \partial_x^3\left(u^2\right)dx = -2\int_R \left(\Lambda^{q+1}u\right)\Lambda^{q+1}(uu_x)dx + 2\int_R (\Lambda^q u)\Lambda^q(uu_x)dx. \tag{4.9}$$

For $q \in (0, s-1]$, applying $(\Lambda^q u)\Lambda^q$ on both sides of (4.8), noting the above equality, and integrating the new equation with respect to x by parts, we obtain the equation

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\int_R \left((\Lambda^q u)^2 + (\Lambda^q u_x)^2\right)dx\right] \\
&\quad = -\int_R (\Lambda^q u)\Lambda^q\left[2ku + \frac{m}{2}u^2\right]_x dx - b\int_R \left(\Lambda^{q+1}u\right)\Lambda^{q+1}(uu_x)dx \\
&\qquad + \frac{3b-a}{2}\int_R (\Lambda^q u_x)\Lambda^q\left(u_x^2\right)dx + b\int_R (\Lambda^q u)\Lambda^q(uu_x)dx + \beta\int_R \Lambda^q u \Lambda^q u_x^n\, dx.
\end{aligned} \tag{4.10}$$

We will estimate each of the terms on the right-hand side of (4.10). For the first and the fourth terms, using integration by parts, the Cauchy-Schwartz inequality, and Lemmas 4.1-4.2, we have

$$\begin{aligned}
\int_R (\Lambda^q u)\Lambda^q(uu_x)dx &= \int_R (\Lambda^q u)[\Lambda^q(uu_x) - u\Lambda^q u_x]dx + \int_R (\Lambda^q u)u\Lambda^q u_x\, dx \\
&\le c\|u\|_{H^q}(\|u_x\|_{L^\infty}\|u\|_{H^q} + \|u_x\|_{L^\infty}\|u\|_{H^q}) \\
&\quad + \frac{1}{2}\|u_x\|_{L^\infty}\|\Lambda^q u\|_{L^2}^2 \\
&\le c\|u\|_{H^q}^2\|u_x\|_{L^\infty},
\end{aligned} \tag{4.11}$$

where c only depends on q. Using the above estimate to the second term yields

$$\int_R \left(\Lambda^{q+1}u\right)\Lambda^{q+1}(uu_x)dx \le c\|u\|_{H^{q+1}}^2\|u_x\|_{L^\infty}. \tag{4.12}$$

For the third term, using Lemma 4.1 gives rise to

$$\begin{aligned}\int_R (\Lambda^q u_x)\Lambda^q\left(u_x^2\right)dx &\le \|\Lambda^q u_x\|_{L^2}\left\|\Lambda^q u_x^2\right\|_{L^2}\\ &\le c\|u\|_{H^{q+1}}\|u_x\|_{L^\infty}\|u_x\|_{H^q}\\ &\le c\|u\|^2_{H^{q+1}}\|u_x\|_{L^\infty}.\end{aligned} \tag{4.13}$$

For the last term, using Lemma 4.1 repeatedly, we get

$$\left|\int_R (\Lambda^q u)\Lambda^q (u_x)^n dx\right| \le \|u\|_{H^q}\|u_x^n\|_{H^q} \le c\|u\|^2_{H^{q+1}}\|u_x\|^{n-1}_{L^\infty}. \tag{4.14}$$

It follows from (4.10)–(4.14) that

$$\begin{aligned}&\frac{1}{2}\int_R\left[(\Lambda^q u)^2 + (\Lambda^q u_x)^2\right]dx - \frac{1}{2}\int_R\left[(\Lambda^q u_0)^2 + (\Lambda^q u_{0x})^2\right]dx\\ &\quad\le c\int_0^t\left(\|u_x\|_{L^\infty} + \|u_x\|^{n-1}_{L^\infty}\right)\|u\|^2_{H^{q+1}}d\tau,\end{aligned} \tag{4.15}$$

which results in (4.6). Applying the operator $(1-\partial_x^2)^{-1}$ on both sides of (4.8) yields the equation

$$u_t = \left(1-\partial_x^2\right)^{-1}\left[-\left[2ku + \frac{m}{2}u^2\right]_x + \frac{b}{2}\partial_x^3 u^2 - \frac{3b-a}{2}\partial_x\left(u_x^2\right) + \beta u_x^n\right]. \tag{4.16}$$

Multiplying both sides of (4.16) by $(\Lambda^q u_t)\Lambda^q$ for $q \in [0, s-1]$ and integrating the resultant equation by parts give rise to

$$\begin{aligned}&\int_R (\Lambda^q u_t)^2 dx\\ &\quad= \int_R (\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\left[-\left[2ku + \frac{m}{2}u^2\right]_x + \frac{b}{2}\partial_x^3 u^2 - \frac{3b-a}{2}\partial_x\left(u_x^2\right) + \beta u_x^n\right]dx.\end{aligned} \tag{4.17}$$

On the right-hand side of (4.17), we have

$$\left|\int_R (\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q(-2ku_x)dx\right| \le |2k|\|u_t\|_{H^q}\|u\|_{H^q}, \tag{4.18}$$

$$\begin{aligned}
&\left|\int_R (\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\partial_x\left(-\frac{m}{2}u^2-\frac{3b-a}{2}u_x^2\right)dx\right| \\
&\quad\le \|u_t\|_{H^q}\times\left(\int_R\left(1+\xi^2\right)^{q-1}d\xi\left(\int_R\left[\frac{m}{2}\widehat{u}(\xi-\eta)\widehat{u}(\eta)+\frac{3b-a}{2}\widehat{u_x}(\xi-\eta)\widehat{u_x}(\eta)\right]d\eta\right)^2\right)^{1/2} \\
&\quad\le c\|u_t\|_{H^q}\left(\int_R\frac{c(\|u\|_{H^q}\|u\|_{L^2}+\|u_x\|_{L^2}\|u_x\|_{H^q})}{1+\xi^2}d\xi\right)^{1/2} \\
&\quad\le c\|u_t\|_{H^q}\|u\|_{H^1}\|u\|_{H^{q+1}},
\end{aligned} \tag{4.19}$$

in which we have used Lemma 4.1. As

$$\begin{aligned}
\int_R (\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\partial_x^2(uu_x)dx = &-\int_R(\Lambda^q u_t)\Lambda^q(uu_x)dx \\
&+\int_R(\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q(uu_x)dx,
\end{aligned} \tag{4.20}$$

by using $\|uu_x\|_{H^q}\le c\|u^2\|_{H^{q+1}}\le c\|u\|_{L^\infty}\|u\|_{H^{q+1}}\le c\|u\|_{H^1}\|u\|_{H^{q+1}}$, we have

$$\left|\int_R(\Lambda^q u_t)\Lambda^q(uu_x)dx\right|\le c\|u_t\|_{H^q}\|uu_x\|_{H^q}\le c\|u_t\|_{H^q}\|u\|_{H^1}\|u\|_{H^{q+1}}, \tag{4.21}$$

$$\left|\int_R(\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q(uu_x)dx\right|\le c\|u_t\|_{H^q}\|u\|_{H^1}\|u\|_{H^{q+1}}. \tag{4.22}$$

Using the Cauchy-Schwartz inequality and Lemma 4.1 yields

$$\left|\int_R(\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q(u_x^n)dx\right|\le c\|u_t\|_{H^q}\|u_x\|_{L^\infty}^{n-1}\|u\|_{H^{q+1}}. \tag{4.23}$$

Applying (4.18)–(4.23) to (4.17) yields the inequality

$$\|u_t\|_{H^q}\le c\|u\|_{H^{q+1}}\left(1+\|u\|_{H^1}+\|u_x\|^{n-1}\right), \tag{4.24}$$

for a constant $c>0$. □

Lemma 4.4. *For $s > 0$, $u_0 \in H^s(R)$, and $u_{\varepsilon 0} = \phi_\varepsilon \star u_0$, the following estimates hold for any ε with $\varepsilon \in (0, 1/4)$:*

$$\begin{gathered}
\|u_{\varepsilon 0x}\|_{L^\infty} \le c\|u_{0x}\|_{L^\infty}, \quad \|u_{\varepsilon 0}\|_{H^q} \le c \quad \text{if } q \le s, \\
\|u_{\varepsilon 0}\|_{H^q} \le c\varepsilon^{(s-q)/4} \quad \text{if } q > s, \\
\|u_{\varepsilon 0} - u_0\|_{H^q} \le c\varepsilon^{(s-q)/4} \quad \text{if } q \le s, \\
\|u_{\varepsilon 0} - u_0\|_{H^s} = o(1),
\end{gathered} \tag{4.25}$$

where c is a constant independent of ε.

The proof of this lemma can be found in [21].

Applying Lemmas 4.3 and 4.4, we can now state the following lemma, which plays an important role in proving existence of weak solutions.

Lemma 4.5. *For $s \ge 1$ and $u_0 \in H^s(R)$, there exists a constant c independent of ε, such that the solution u_ε of problem (2.3) satisfies*

$$\|u_\varepsilon\|_{H^1} \le ce^{c_0 \int_0^t (\|u_{\varepsilon x}\|_{L^\infty} + \|u_{\varepsilon x}\|_{L^\infty}^{n-1}) d\tau} \quad \text{for } t \in [0, T_\varepsilon), \tag{4.26}$$

where $c_0 = 1/2 \max(|a - 2b|, |\beta|)$.

Proof. The proof can be directly obtained from Lemma 4.4 and inequality (4.5). □

Proof of Theorem 2.2. Using notation $u = u_\varepsilon$ and differentiating (4.16) with respect to x give rise to

$$\begin{aligned}
u_{xt} + buu_{xx} + \frac{a-b}{2}u_x^2 = 2ku + \frac{m-b}{2}u^2 \\
- \Lambda^{-2}\left[2ku + \frac{m-b}{2}u^2 + \frac{3b-a}{2}u_x^2 - \beta(u_x^n)_x\right].
\end{aligned} \tag{4.27}$$

Using

$$\begin{aligned}
\int_R uu_{xx}(u_x)^{2p+1}dx &= \int_R u(u_x)^{2p+1}du_x \\
&= -\int_R u_x\left[(u_x)^{2p+2} + (2p+1)u(u_x)^{2p}u_{xx}\right]dx,
\end{aligned} \tag{4.28}$$

we get

$$\int_R uu_{xx}(u_x)^{2p+1}dx = -\frac{1}{2p+2}\int_R (u_x)^{2p+3}dx. \tag{4.29}$$

Letting $p > 0$ be an integer and multiplying (4.27) by $(u_x)^{2p+1}$ and then integrating the resulting equation with respect to x yield the equality

$$\begin{aligned}
&\frac{1}{2p+2}\frac{d}{dt}\int_R (u_x)^{2p+2}dx + \frac{(a-b)p+a-2b}{2p+2}\int_R (u_x)^{2p+3}dx \\
&= \int_R (u_x)^{2p+1}\left(2ku + \frac{m-b}{2}u^2\right)dx \\
&\quad - \int_R (u_x)^{2p+1}\Lambda^{-2}\left[2ku + \frac{m-b}{2}u^2 + \frac{3b-a}{2}u_x^2 - \beta(u_x^n)_x\right]dx.
\end{aligned} \tag{4.30}$$

Applying the Hölder's inequality, we get

$$\begin{aligned}
\frac{1}{2p+2}\frac{d}{dt}\int_R (u_x)^{2p+2}dx \le{}& \left\{\left(\int_R \left|2ku + \frac{m-b}{2}u^2\right|^{2p+2}dx\right)^{1/(2p+2)}\right. \\
&\left. + \left(\int_R |G|^{2p+2}dx\right)^{1/(2p+2)}\right\}\left(\int_R |u_x|^{2p+2}dx\right)^{(2p+1)/(2p+2)} \\
&+ \left|\frac{(a-b)p+a-2b}{2p+2}\right|\|u_x\|_{L^\infty}\int_R |u_x|^{2p+2}dx,
\end{aligned} \tag{4.31}$$

or

$$\begin{aligned}
&\frac{d}{dt}\left(\int_R (u_x)^{2p+2}dx\right)^{1/(2p+2)} \\
&\le \left\{\left(\int_R \left|2ku + \frac{m-b}{2}u^2\right|^{2p+2}dx\right)^{1/(2p+2)} + \left(\int_R |G|^{2p+2}dx\right)^{1/(2p+2)}\right\} \\
&\quad + \left|\frac{(a-b)p+a-2b}{2p+2}\right|\|u_x\|_{L^\infty}\left(\int_R |u_x|^{2p+2}dx\right)^{1/(2p+2)},
\end{aligned} \tag{4.32}$$

where

$$G = \Lambda^{-2}\left[2ku + \frac{m-b}{2}u^2 + \frac{3b-a}{2}u_x^2 - \beta(u_x^n)_x\right]. \tag{4.33}$$

Since $\|f\|_{L^p} \to \|f\|_{L^\infty}$ as $p \to \infty$ for any $f \in L^\infty \bigcap L^2$, integrating (4.32) with respect to t and taking the limit as $p \to \infty$ result in the estimate

$$\|u_x\|_{L^\infty} \le \|u_{0x}\|_{L^\infty} + \int_0^t \left[c\left(\left\|2ku + \frac{m-b}{2}u^2\right\|_{L^\infty} + \|G\|_{L^\infty}\right) + \frac{a-b}{2}\|u_x\|_{L^\infty}^2\right]d\tau. \tag{4.34}$$

Using the algebraic property of $H^{s_0}(R)$ with $s_0 > 1/2$ and Lemma 4.5 leads to

$$\begin{aligned}
\left\|2ku + \frac{m-b}{2}u^2\right\|_{L^\infty} &\le c\left\|2ku + \frac{m-b}{2}u^2\right\|_{H^{1/2+}} \\
&\le c\left(\|u\|_{H^1} + \|u\|_{H^1}^2\right) \\
&\le ce^{2c_0\int_0^t(\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^{n-1})d\tau},
\end{aligned} \tag{4.35}$$

$$\begin{aligned}
\|G\|_{L^\infty} &= \left\|\Lambda^{-2}\left[2ku + \frac{m-b}{2}u^2 + \frac{3b-a}{2}u_x^2 - \beta(u_x^n)_x\right]\right\|_{L^\infty} \\
&\le c\left(\left\|\Lambda^{-2}\left(2ku + \frac{m-b}{2}u^2\right)\right\|_{H^{1/2+}} + \left\|\Lambda^{-2}u_x^2\right\|_{H^{1/2+}} + \left\|\Lambda^{-2}(u_x^n)_x\right\|_{H^{1/2+}}\right) \\
&\le c\left(\left\|2ku + \frac{m-b}{2}u^2\right\|_{H^{1/2+}} + \|u\|_{H^1}\|u_x\|_{L^\infty} + \|u_x^n\|_{H^0}\right) \\
&\le c\left(\|u\|_{H^1} + \|u\|_{H^1}^2 + \|u\|_{H^1}\|u_x\|_{L^\infty} + \|u\|_{H^1}\|u_x\|_{L^\infty}^{n-1}\right) \\
&\le c\left(1 + \|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^{n-1}\right)e^{2c_0\int_0^t(\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^{n-1})d\tau},
\end{aligned} \tag{4.36}$$

where c is independent of ε, $c_0 = 1/2\max(|a-2b|, |\beta|)$ and $H^{1/2+}$ means that there exists a sufficiently small $\delta > 0$ such that $\|u_\varepsilon\|_{H^{1/2+}} = \|u_\varepsilon\|_{H^{1/2+\delta}}$. From Lemma 4.5, we have

$$\int_0^t \|G\|_{L^\infty}d\tau \le c\int_0^t\left(1 + \|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^{n-1}\right) \times e^{2c_0\int_0^\tau(\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^{n-1})d\xi}\, d\tau. \tag{4.37}$$

Applying (4.25), (4.34), (4.35), and (4.37) and writing out the subscript ε of u, we obtain

$$\begin{aligned}
\|u_{\varepsilon x}\|_{L^\infty} &\le c\|u_{0x}\|_{L^\infty} \\
&\quad + c\int_0^t\left[\left(1 + \|u_{\varepsilon x}\|_{L^\infty} + \|u_{\varepsilon x}\|_{L^\infty}^{n-1}\right) \times e^{2c_0\int_0^\tau(\|u_{\varepsilon x}\|_{L^\infty}+\|u_{\varepsilon x}\|_{L^\infty}^{n-1})d\xi} + \|u_{\varepsilon x}\|_{L^\infty}^2\right]\, d\tau.
\end{aligned} \tag{4.38}$$

It follows from the contraction mapping principle that there is a $T > 0$ such that the equation

$$\begin{aligned}
\|W\|_{L^\infty} &= c\|u_{0x}\|_{L^\infty} \\
&\quad + c\int_0^t\left[\left(1 + \|W\|_{L^\infty} + \|W\|_{L^\infty}^{n-1}\right) \times e^{2c_0\int_0^\tau(\|W\|_{L^\infty}+\|W\|_{L^\infty}^{n-1})d\xi} + \|W\|_{L^\infty}^2\right]d\tau
\end{aligned} \tag{4.39}$$

has a unique solution $W \in C[0,T]$. From (4.39), we know that the variable T only depends on c and $\|u_{0x}\|_{L^\infty}$. Using the theorem presented on page 51 in [16] or Theorem 2 in Section 1.1

in [27] derives that there are constants $T > 0$ and $c > 0$ independent of ε such that $\|u_{\varepsilon x}\|_{L^\infty} \leq W(t)$ for arbitrary $t \in [0, T]$, which leads to the conclusion of Theorem 2. □

Remark 4.6. Under the assumptions of Theorem 2.2, there exist two constants T and c, both independent of ε, such that the solution u_ε of problem (2.3) satisfies $\|u_{\varepsilon x}\|_{L^\infty} \leq c$ for any $t \in [0, T]$. This states that in Lemma 4.5, there exists a T independent of ε such that (4.26) holds.

Using Theorem 2.2, Lemma 4.5, (4.6), (4.7), notation $u_\varepsilon = u$, and Gronwall's inequality results in the inequalities

$$\begin{gathered}
\|u_\varepsilon\|_{H^q} \leq c \exp\left[c\int_0^t \left(\|u_{\varepsilon x}\|_{L^\infty} + \|u_{\varepsilon x}\|_{L^\infty}^{n-1}\right) d\tau\right] \leq c, \\
\|u_{\varepsilon t}\|_{H^r} \leq \|u_\varepsilon\|_{H^{r+1}} \left(1 + \|u_\varepsilon\|_{H^1} + \|u_{\varepsilon x}\|_{L^\infty}^{n-1}\right) \leq c,
\end{gathered} \tag{4.40}$$

where $q \in (0, s], r \in (0, s-1]$, and $t \in [0, T)$. It follows from Aubin's compactness theo-rem that there is a subsequence of $\{u_\varepsilon\}$, denoted by $\{u_{\varepsilon_{n_1}}\}$, such that $\{u_{\varepsilon_{n_1}}\}$ and their temporal derivatives $\{u_{\varepsilon_{n_1} t}\}$ are weakly convergent to a function $u(t, x)$ and its derivative u_t in $L^2([0, T], H^s)$ and $L^2([0, T], H^{s-1})$, respectively. Moreover, for any real number $R_1 > 0$, $\{u_{\varepsilon_{n_1}}\}$ is convergent to the function u strongly in the space $L^2([0, T], H^q(-R_1, R_1))$ for $q \in (0, s]$, and $\{u_{\varepsilon_{n_1} t}\}$ converges to u_t strongly in the space $L^2([0, T], H^r(-R_1, R_1))$ for $r \in [0, s-1]$.

Proof of Theorem 2.3. From Theorem 2.2, we know that $\{u_{\varepsilon_{n_1} x}\}(\varepsilon_{n_1} \to 0)$ is bounded in the space L^∞. Thus, the sequences$\{u_{\varepsilon_{n_1}}\}$, $\{u_{\varepsilon_{n_1} x}\}$, $\{u^2_{\varepsilon_{n_1} x}\}$, and $\{u^n_{\varepsilon_{n_1} x}\}$ are weakly convergent to u, u_x, u_x^2, and u_x^n in $L^2([0, T], H^r(-R_1, R_1))$ for any $r \in [0, s-1)$, separately. Hence, u satisfies the equation

$$-\int_0^T \int_R u(g_t - g_{xxt}) dx\, dt = \int_0^T \int_R \left[\left(2ku + \frac{m}{2}u^2 + \frac{3b-a}{2}u_x^2\right) g_x - \frac{b}{2}u^2 g_{xxx} - \beta u_x^n g\right] dxdt, \tag{4.41}$$

with $u(0, x) = u_0(x)$ and $g \in C_0^\infty$. Since $X = L^1([0, T] \times R)$ is a separable Banach space and $\{u_{\varepsilon_{n_1} x}\}$ is a bounded sequence in the dual space $X^* = L^\infty([0, T] \times R)$ of X, there exists a subsequence of $\{u_{\varepsilon_{n_1} x}\}$, still denoted by $\{u_{\varepsilon_{n_1} x}\}$, weakly star convergent to a function v in $L^\infty([0, T] \times R)$. As $\{u_{\varepsilon_{n_1} x}\}$ weakly converges to u_x in $L^2([0, T] \times R)$, it results that $u_x = v$ almost everywhere. Thus, we obtain $u_x \in L^\infty([0, T] \times R)$. □

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