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*Research Article*

# Hybrid Steepest Descent Viscosity Method for Triple Hierarchical Variational Inequalities

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We consider a triple hierarchical variational inequality problem (in short, THVIP). By combining hybrid steepest descent method, viscosity method, and projection method, we propose an approximation method to compute the approximate solution of THVIP. We also study the strong convergence of the sequences generated by the proposed method to a solution of THVIP.

## 1. Introduction and Formulations

Let $H$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $C$ be a nonempty closed convex subset of $H$ and let $\Gamma : C \to H$ be a nonlinear mapping. The variational inequality problem (for short, VIP) is to find $x^* \in C$ such that

$$\langle \Gamma x^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The inequality (1.1) is called a variational inequality (in short, VI). If the mapping $\Gamma$ is a monotone operator, then the inequality (1.1) is called a monotone variational inequality. The theory of variational inequalities is well established in the literature because of its applications in science, engineering, social sciences, and so forth. For further detail on variational inequalities and their applications, we refer to [1–10] and the references therein.

It is well known that the VI (1.1) is equivalent to the fixed point equation

$$x^* = P_C(I - \lambda\Gamma)x^*, \tag{1.2}$$

where $\lambda > 0$ and $P_C$ is the metric projection of $H$ onto $C$ which assigns to each $x \in H$ the only point in $C$, denoted by $P_C x$, such that

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\|. \tag{1.3}$$

It is known that the fixed point methods can be implemented to find a solution of the VI (1.1) provided $\Gamma$ satisfies some conditions and $\lambda > 0$ is chosen appropriately. For instance, if $\Gamma$ is Lipschitzian and strongly monotone (i.e., $\langle \Gamma x - \Gamma y,\ x - y\rangle \geq \eta\|x - y\|^2,$ for all $x, y \in C$ for some $\eta > 0$) and $\lambda > 0$ is small enough, then the mapping determined by the right-hand side of (1.2) is a contraction. Hence, the Banach contraction principle guarantees that the sequence $\{x_n\}$ of Picard iterates, given by $x_n = P_C(I - \lambda\Gamma)x_{n-1}$ $(n \geq 1)$, converges strongly to a unique solution of the VI (1.1).

Furthermore, it is also known that if $\Gamma$ is inverse strongly monotone (i.e., there is a constant $\alpha > 0$ such that $\langle \Gamma x - \Gamma y,\ x - y\rangle \geq \alpha\|\Gamma x - \Gamma y\|^2,$ for all $x, y \in C$), then the mapping $P_C(I - \lambda\Gamma)$ is an averaged mapping (namely, there are $\beta \in (0,1)$ and a nonexpansive mapping $T$ such that $P_C(I - \lambda\Gamma) = (1-\beta)I + \beta T$), then the sequence of Picard iterates, $\{(P_C(I - \lambda\Gamma))^n x_0\}$, converges weakly to a solution of the VI (1.1) (if such a solution exists).

In the last decade, the variational inequality problem is considered over the set of fixed points of a nonexpansive mapping; see, for example, [11–15] and the reference therein. In particular, Moudafi and Maingé [12] and Xu [14] considered the following VIP over the set $\mathrm{Fix}(T)$ of fixed points of a nonexpansive mapping $T : C \to C$ (i.e., $C = \mathrm{Fix}(T)$) with $\Gamma = I - V$, where $V$ is another nonexpansive self-mapping on $C$: find $x^* \in \mathrm{Fix}(T)$ such that

$$\langle (I - V)x^*,\ x - x^*\rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T), \tag{1.4}$$

where we assume that $\mathrm{Fix}(T) \neq \emptyset$. It is called hierarchical variational inequality problem (in short, HVIP). The HVIP (1.4) is equivalent to the following fixed point problem:

$$\text{find } x^* \in C \quad \text{such that } x^* = P_{\mathrm{Fix}(T)} \circ V(x^*). \tag{1.5}$$

Let $S$ denote the solution set of the HVIP (1.4). It has been shown in [12] that the HVIP (1.4) contains the HVIP considered in [15], monotone inclusion problem, convex programming problem, minimization problem over a set of fixed points, and so forth, as special cases; see, for example, [12, 14] and the references therein. In the recent past, several kinds of approximation methods for computing the approximate solutions of HVIP are proposed; see, for example, [11–15] and the reference therein. Yamada [15] considered the so-called hybrid steepest descent method for solving the VIP over the set of fixed points of a nonexpansive mapping. Moudafi [11] proposed the viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping which is also a solution of a variational inequality problem. Subsequently, this method was developed by Xu [13]. Moudafi and Maingé [12] and Xu [14] further studied the viscosity method for HVIP.

Very recently, Iiduka [16, 17] considered a variational inequality problem with variational inequality constraint over the set of fixed points of a nonexpansive mapping. Since this problem has a triple structure in contrast with hierarchical constrained optimization problems or hierarchical fixed point problem, it is referred as triple hierarchical-constrained optimization problem (THCOP). He presented some examples of THCOP and developed iterative algorithms to find the solution of such a problem. The convergence analysis of the proposed algorithms is also studied in [16, 17]. Since the original problem is a variational inequality problem, in this paper, we call it the triple hierarchical variational inequality problem (THVIP).

Let $F : C \to H$ be a $\kappa$-Lipschitzian and $\eta$-strongly monotone operator with constants $\kappa$ and $\eta > 0$, respectively. Let $f : C \to H$ be $L$-Lipschitzian with constant $L \geq 0$ and let $T, V : C \to C$ be nonexpansive mappings with $\mathrm{Fix}(T) \neq \emptyset$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma L < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. We consider the following triple hierarchical variational inequality problem (for short, THVIP): find $x^* \in S$ such that

$$\langle (\mu F - \gamma f)x^*,\ x - x^* \rangle \geq 0, \quad \forall x \in S, \tag{1.6}$$

where $S$ denotes the solution set of the hierarchical variational inequality problem (1.4) which is assumed to be nonempty.

Recall the function $g : C \to \mathbb{R}$ is said to be convex if for all $x, y \in C$ and for all $\lambda \in [0,1]$, $g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y)$. It is said to be $\alpha$-strongly convex if there exists $\alpha > 0$ such that for all $x, y \in K$ and for all $\lambda \in [0,1]$, $g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y) - (1/2)\alpha\lambda(1-\lambda)\|x-y\|^2$. It is easy to see that if $g$ is Fréchet differential and $\alpha$-strongly convex, then the gradient $\nabla g$ is $\alpha$-strongly monotone.

Now, we illustrate the triple hierarchical variational inequality problem (for short, THVIP) by an example which is closely related to [17, Example 3.1].

*Example 1.1.* Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and let $f : C \to H$ be $L$-Lipschitz continuous with constant $L > 0$. Suppose that $g_0 : H \to \mathbb{R}$ is a convex function with a $1/\alpha_0$-Lipschitz continuous gradient, $g_1 : H \to \mathbb{R}$ is a convex function with a $1/\alpha_1$-Lipschitz continuous gradient, and $g_2 : H \to \mathbb{R}$ is an $\alpha$-strongly convex function with an $\alpha_2$-Lipschitz continuous gradient. Define $T := P_C(I - \lambda\nabla g_0)$ ($\lambda \in (0, 2\alpha_0]$), $V := P_C(I - \tilde{\lambda}\nabla g_1)$ ($\tilde{\lambda} \in (0, 2\alpha_1]$) and $F := \nabla g_2$. Then $T, V : C \to C$ are nonexpansive mappings with $\mathrm{Fix}(T) = \mathrm{Argmin}_{z\in C} g_0(z)$ and $\mathrm{Fix}(V) = \mathrm{Argmin}_{z\in C} g_1(z)$, and $F$ is $\kappa$-Lipschitzian and $\eta$-strongly monotone with $\kappa = 1/\alpha_2$ and $\eta = \alpha$. Assume that $\mathrm{Argmin}_{z\in C} g_0(z) \cap \mathrm{Argmin}_{z\in C} g_1(z) \neq \emptyset$. Then for the solution set $S$ of the hierarchical variational inequality problem (for short, HVIP), we have

$$\begin{aligned}
\emptyset \neq\ & \operatorname*{Argmin}_{z\in C} g_0(z) \cap \operatorname*{Argmin}_{z\in C} g_1(z) \\
=\ & \mathrm{Fix}(T) \cap \mathrm{Fix}(V) \\
& \subset \{z^* \in \mathrm{Fix}(T) : \langle (I - V)z^*,\ z - z^* \rangle \geq 0,\ \forall z \in \mathrm{Fix}(T)\} \\
=\ & S.
\end{aligned} \tag{1.7}$$

When $0 < \mu < 2\alpha\alpha_2^2$ and $0 \le \gamma L < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\alpha - \mu/\alpha_2^2)}$, we have

$$0 < \mu < \frac{2\eta}{\kappa^2}, \quad 0 \le \gamma L < \tau, \tag{1.8}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. In particular, when $\mu = \eta/\kappa^2 = \alpha\alpha_2^2$, we have

$$0 \le \gamma L < \tau = 1 - \sqrt{1 - \mu\left(2\alpha - \frac{\mu}{\alpha_2^2}\right)} = 1 - \sqrt{1 - \alpha^2\alpha_2^2}. \tag{1.9}$$

In this case, when $\gamma = (1/2L)\alpha^2\alpha_2^2$ (obviously, $\sqrt{1 - \alpha^2\alpha_2^2} < 1 - (1/2)\alpha^2\alpha_2^2$), the following triple hierarchical variational inequality problem (for short, THVIP): find $x^* \in S$ such that

$$\langle (\mu F - \gamma f)x^*,\ x - x^* \rangle \ge 0, \quad \forall x \in S \tag{1.10}$$

reduces to the following THVIP: find $x^* \in S$ such that

$$\left\langle \left(\nabla g_2 - \frac{\alpha}{2L} f\right)x^*,\ x - x^* \right\rangle \ge 0, \quad \forall x \in S. \tag{1.11}$$

In this paper, by combining hybrid steepest descent method, viscosity method, and projection method, we propose an approximation method to compute the approximate solution of THVIP. We also study the strong convergence of the sequences generated by the proposed method to a solution of THVIP. The results of this paper extend and generalize the results given in [12, 14] and several others given in the literature.

## 2. Preliminaries

Throughout the paper, unless other specified, we assume that $C$ is a nonempty closed convex subset of a real Hilbert space $H$. We use $x_n \to x$ and $x_n \rightharpoonup x$ to denote strong and weak convergence to $x$ of the sequence $\{x_n\}$, respectively.

Recall that a mapping $f : C \to H$ is called $L$-Lipschitzian on $C$ if there exists $L \in [0, \infty)$ such that $\|f(x) - f(y)\| \le L\|x - y\|$, for all $x, y \in C$. In particular, if $L \in [0, 1)$ then $f$ is called a contraction on $C$; if $L = 1$ then $f$ is called a nonexpansive mapping on $C$.

We present some basic facts and results which will be used in the sequel.

**Lemma 2.1** (see [18]). *Let $H$ be a real Hilbert space. Then,*

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2 \tag{2.1}$$

*for all $x, y \in H$ and $\lambda \in [0, 1]$.*

The following lemma can be easily proved, and therefore, we omit the proof.

**Lemma 2.2.** *Let $f : C \to H$ be an $L$-Lipschitzian mapping with constant $L \in [0,\infty)$, and let $F : C \to H$ be a $\kappa$-Lipschitzian and $\eta$-strongly monotone operator with constants $\kappa$ and $\eta > 0$, respectively. Then, for $0 \le \gamma L < \mu\eta$,*

$$\langle x - y, (\mu F - \gamma f)x - (\mu F - \gamma f)y \rangle \ge (\mu\eta - \gamma L)\|x - y\|^2, \quad \forall x, y \in C. \tag{2.2}$$

*That is, $\mu F - \gamma f$ is strongly monotone with constant $\mu\eta - \gamma L > 0$.*

**Lemma 2.3** (see [18, Demiclosedness Principle]). *Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. If $\{x_n\}$ weakly converges to $x$ in $C$ and if $\{(I - T)x_n\}$ strongly converges to $y$, then $(I - T)x = y$; in particular, if $y = 0$, then $x \in \mathrm{Fix}(T)$.*

In the following lemma, we present some properties of the projection.

**Lemma 2.4.** *Given $x \in H$ and $z \in C$. Then*

(a) *$z = P_C x$ if and only if there holds the relation:*

$$\langle x - z,\ y - z \rangle \le 0, \quad \forall y \in C. \tag{2.3}$$

(b) *$z = P_C x$ if and only if there holds the relation:*

$$\|x - z\|^2 \le \|x - y\|^2 - \|y - z\|^2, \quad \forall y \in C. \tag{2.4}$$

(c) *$P_C$ is nonexpansive and monotone, that is,*

$$\langle P_C x - P_C y,\ x - y \rangle \ge \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.5}$$

**Lemma 2.5.** *Let $H$ be a real Hilbert space. Then, for all $x, y \in H$,*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y \rangle. \tag{2.6}$$

The following lemma plays a key role in proving the main results of this paper.

**Lemma 2.6** (see [19, Lemma 3.1]). *Let $\lambda \in (0,1]$ and $\mu > 0$. Let $F : C \to H$ be an operator on $C$ such that, for some constants $\kappa, \eta > 0$, $F$ is $\kappa$-Lipschitzian and $\eta$-strongly monotone. Associating with a nonexpansive mapping $T : C \to C$, define the mapping $T^\lambda : C \to H$ by*

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in C. \tag{2.7}$$

*Then $T^\lambda$ is a contraction provided $\mu < 2\eta/\kappa^2$, that is,*

$$\left\| T^\lambda x - T^\lambda y \right\| \le (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in C, \tag{2.8}$$

*where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0,1]$.*

*Remark 2.7.* If $F = I$, where $I$ is the identity operator of $H$. Then $\kappa = \eta = 1$ and hence $\mu < 2\eta/\kappa^2 = 2$. Also, if $\mu = 1$, then it is easy to see that

$$\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} = 1. \tag{2.9}$$

In particular, whenever $\lambda > 0$, we have $T^\lambda x := Tx - \lambda\mu F(Tx) = (1 - \lambda)Tx$.

## 3. Approximation Methods and Convergence Results

Let $F : C \to H$ be a $\kappa$-Lipschitzian and $\eta$-strongly monotone operator with constants $\kappa$ and $\eta > 0$, respectively. Let $f : C \to H$ be a $L$-Lipschitzian mapping with constant $L \geq 0$ and let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma L \leq \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. We consider the hierarchical variational inequality problem (in short, HVIP) of finding $z^* \in \mathrm{Fix}(T)$ such that

$$\langle (\mu F - \gamma f)z^*,\ z - z^* \rangle \geq 0, \quad \forall z \in \mathrm{Fix}(T). \tag{3.1}$$

We denote by $\Omega$ the solution set of the HVIP (3.1).

When $\mu = 1$, $F = I$, $\gamma = \tau = 1$ and $f = V$ are a nonexpansive self-mapping on $C$, the HVIP (3.1) reduces to the following hierarchical variational inequality problem of finding $z^* \in \mathrm{Fix}(T)$ such that

$$\langle (I - V)z^*,\ z - z^* \rangle \geq 0, \quad \forall z \in \mathrm{Fix}(T). \tag{3.2}$$

It is considered and studied in [12, 14].

We consider a mapping $\Theta_t$ on $C$ defined by

$$\Theta_t x = P_C[t\gamma f(x) + (I - t\mu F)Tx], \quad \forall x \in C. \tag{3.3}$$

It is easy to see that $\Theta_t$ is a nonexpansive mapping. Indeed, we have

$$\begin{aligned} \|\Theta_t x - \Theta_t y\| &= \|P_C[t\gamma f(x) + (I - t\mu F)Tx] - P_C[t\gamma f(y) + (I - t\mu F)Ty]\| \\ &\leq t\gamma\|f(x) - f(y)\| + \|(I - t\mu F)Tx - (I - t\mu F)Ty\| \\ &\leq t\gamma L\|x - y\| + (1 - t\tau)\|x - y\| \\ &= (1 - (\tau - \gamma L)t)\|x - y\|. \end{aligned} \tag{3.4}$$

Since $0 \leq \gamma L \leq \tau$, it is known that $\Theta_t$ is nonexpansive on $C$.

**Proposition 3.1.** *Let $F : C \to H$ be a $\kappa$-Lipschitzian and $\eta$-strongly monotone operator with constants $\kappa$ and $\eta > 0$, respectively. Let $f : C \to H$ be $L$-Lipschitzian with constant $L \geq 0$ and let $T : C \to C$ be nonexpansive with $\mathrm{Fix}(T) \neq \emptyset$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma L \leq \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Let $t \in (0, 1)$ and $z_t$ be a fixed point of the mapping $\Theta_t = P_C[t\gamma f + (I - t\mu F)T]$,*

*that is,* $z_t = P_C[t\gamma f(z_t) + (I - t\mu F)Tz_t]$. *Assume* $\{z_t\}$ *remains bounded as* $t \to 0$, *then the following conclusions hold.*

(a) *The solution set* $\Omega$ *of the HVIP* (3.1) *is nonempty and each weak limit point (as* $t \to 0$*) of* $\{z_t\}$ *solves the HVIP* (3.1).

(b) *If* $\mu F - \gamma f$ *is strictly monotone, then the net* $\{z_t\}$ *converges weakly to the (unique) solution of the HVIP* (3.1).

(c) *If* $\mu F - \gamma f$ *is strongly monotone (e.g.,* $\mu\eta > \gamma L$*), then the net* $\{z_t\}$ *converges strongly to a solution of the HVIP* (3.1).

*Proof.* Let $W$ be the set of all weak accumulation points of $\{z_t\}$ as $t \to 0$; that is,

$$W = \{z : z_{t_n} \rightharpoonup z \text{ for some sequence } \{t_n\} \text{ in } (0,1) \text{ such that } t_n \longrightarrow 0\}. \tag{3.5}$$

Then, $W \neq \emptyset$ because $\{z_t\}$ is bounded.

To prove (a), we notice that the boundedness of $\{z_t\}$ implies that $W \neq \emptyset$ and

$$\begin{aligned} \|z_t - Tz_t\| &= \left\|P_C\left[t\gamma f(z_t) + (I - t\mu F)Tz_t\right] - P_C Tz_t\right\| \\ &\leq \left\|t\gamma f(z_t) + (I - t\mu F)Tz_t - Tz_t\right\| \\ &= t\left\|\gamma f(z_t) - \mu F(Tz_t)\right\| \longrightarrow 0 \quad \text{as } t \longrightarrow 0. \end{aligned} \tag{3.6}$$

It thus follows from Lemma 2.3 that $W \subset \mathrm{Fix}(T)$. Take a fixed $\hat{x} \in \mathrm{Fix}(T)$ arbitrarily and set

$$w_t = t\gamma f(z_t) + (I - t\mu F)Tz_t, \quad \forall t \in (0,1). \tag{3.7}$$

Then $z_t = P_C w_t$ and

$$\begin{aligned} z_t - \hat{x} &= P_C w_t - w_t + t\gamma f(z_t) + (I - t\mu F)Tz_t - \hat{x} \\ &= P_C w_t - w_t + t(\gamma f(z_t) - \mu F\hat{x}) + (I - t\mu F)Tz_t - (I - t\mu F)\hat{x} \\ &= P_C w_t - w_t + t\gamma(f(z_t) - f(\hat{x})) + t(\gamma f(\hat{x}) - \mu F\hat{x}) \\ &\quad + (I - t\mu F)Tz_t - (I - t\mu F)T\hat{x}. \end{aligned} \tag{3.8}$$

Since $P_C$ is the metric projection from $H$ onto $C$, utilizing Lemma 2.4, we have

$$\langle P_C w_t - w_t,\ P_C w_t - \hat{x}\rangle \leq 0. \tag{3.9}$$

Hence, utilizing $\kappa$-Lipschitzian property of $F$, we get

$$\begin{aligned}
\|z_t-\hat{x}\|^2 &= \langle P_C w_t - w_t,\ P_C w_t - \hat{x}\rangle + \langle w_t-\hat{x},\ z_t-\hat{x}\rangle \\
&\le \langle w_t-\hat{x},\ z_t-\hat{x}\rangle \\
&= t\langle \gamma f(z_t)-\mu F\hat{x},\ z_t-\hat{x}\rangle + \langle (I-t\mu F)Tz_t-(I-t\mu F)\hat{x},\ z_t-\hat{x}\rangle \\
&\le t(\langle (\gamma f-\mu F)z_t,\ z_t-\hat{x}\rangle + \mu\langle Fz_t-F\hat{x},\ z_t-\hat{x}\rangle) \\
&\quad + \frac{1}{2}\left(\|(I-t\mu F)Tz_t-(I-t\mu F)\hat{x}\|^2+\|z_t-\hat{x}\|^2\right) \\
&= t(\langle (\gamma f-\mu F)z_t,\ z_t-\hat{x}\rangle + \mu\langle Fz_t-F\hat{x},\ Tz_t-\hat{x}\rangle + \mu\langle Fz_t-F\hat{x},\ z_t-Tz_t\rangle) \\
&\quad + \frac{1}{2}\left[\|Tz_t-\hat{x}\|^2-2t\mu\langle FTz_t-F\hat{x},\ Tz_t-\hat{x}\rangle + t^2\mu^2\|FTz_t-F\hat{x}\|^2+\|z_t-\hat{x}\|^2\right] \\
&= t(\langle (\gamma f-\mu F)z_t,\ z_t-\hat{x}\rangle + \mu\langle FTz_t-F\hat{x},\ Tz_t-\hat{x}\rangle \\
&\quad + \mu\langle Fz_t-FTz_t,\ Tz_t-\hat{x}\rangle + \mu\langle Fz_t-F\hat{x},\ z_t-Tz_t\rangle) \\
&\quad + \frac{1}{2}\left[\|Tz_t-\hat{x}\|^2-2t\mu\langle FTz_t-F\hat{x},\ Tz_t-\hat{x}\rangle + t^2\mu^2\|FTz_t-F\hat{x}\|^2+\|z_t-\hat{x}\|^2\right] \\
&= t(\langle (\gamma f-\mu F)z_t,\ z_t-\hat{x}\rangle + \mu\langle Fz_t-FTz_t,\ Tz_t-\hat{x}\rangle + \mu\langle Fz_t-F\hat{x},\ z_t-Tz_t\rangle) \\
&\quad + \frac{1}{2}\left[\|Tz_t-\hat{x}\|^2+t^2\mu^2\|FTz_t-F\hat{x}\|^2+\|z_t-\hat{x}\|^2\right] \\
&\le t\langle (\gamma f-\mu F)z_t,\ z_t-\hat{x}\rangle + 2t\mu\kappa\|z_t-Tz_t\|\|z_t-\hat{x}\| + \left(1+\frac{t^2\mu^2\kappa^2}{2}\right)\|z_t-\hat{x}\|^2.
\end{aligned} \tag{3.10}$$

It follows that

$$\langle (\mu F-\gamma f)z_t,\ z_t-\hat{x}\rangle \le \frac{t\mu^2\kappa^2}{2}\|z_t-\hat{x}\|^2 + 2\mu\kappa\|z_t-Tz_t\|\|z_t-\hat{x}\|. \tag{3.11}$$

Note that $0\le\gamma L\le\tau$ and

$$\begin{aligned}
\mu\eta\ge\tau &\Longleftrightarrow \mu\eta\ge 1-\sqrt{1-\mu(2\eta-\mu\kappa^2)} \\
&\Longleftrightarrow \sqrt{1-\mu(2\eta-\mu\kappa^2)}\ge 1-\mu\eta \\
&\Longleftrightarrow 1-2\mu\eta+\mu^2\kappa^2\ge 1-2\mu\eta+\mu^2\eta^2 \\
&\Longleftrightarrow \kappa^2\ge\eta^2 \\
&\Longleftrightarrow \kappa\ge\eta.
\end{aligned} \tag{3.12}$$

Since $0 \le \gamma L \le \tau \le \mu\eta$, we have $\mu\eta - \gamma L \ge 0$. Thus, utilizing the $\eta$-strong monotonicity of $F$ and $L$-Lipschitzian property of $f$, we know that $\mu F - \gamma f$ is monotone because the following inequality holds:

$$\langle (\mu F - \gamma f)x - (\mu F - \gamma f)y,\ x - y\rangle \ge (\mu\eta - \gamma L)\|x - y\|^2, \quad \forall x, y \in C. \tag{3.13}$$

Consequently, we have

$$\langle (\mu F - \gamma f)z_t,\ z_t - \hat{x}\rangle \ge \langle (\mu F - \gamma f)\hat{x},\ z_t - \hat{x}\rangle. \tag{3.14}$$

This together with (3.11) implies that

$$\langle (\mu F - \gamma f)\hat{x}, z_t - \hat{x}\rangle \le \frac{t\mu^2\kappa^2}{2}\|z_t - \hat{x}\|^2 + 2\mu\kappa\|z_t - Tz_t\|\|z_t - \hat{x}\|. \tag{3.15}$$

Now, if $\tilde{x} \in W \subset \mathrm{Fix}(T)$ and if $t_n \to 0$ is such that $z_{t_n} \rightharpoonup \tilde{x}$, then we obtain from (3.15) and $\|z_t - Tz_t\| \to 0$ that

$$\langle (\mu F - \gamma f)\hat{x},\ \tilde{x} - \hat{x}\rangle \le 0, \quad \forall \hat{x} \in \mathrm{Fix}(T). \tag{3.16}$$

Replacing $\hat{x}$ by $\tilde{x} + \lambda(x - \tilde{x}) \in \mathrm{Fix}(T)$ in (3.16), where $\lambda \in (0,1)$ and $x \in \mathrm{Fix}(T)$, we get

$$\langle (\mu F - \gamma f)(\tilde{x} + \lambda(x - \tilde{x})),\ \tilde{x} - x\rangle \le 0. \tag{3.17}$$

Letting $\lambda \to 0$ yields

$$\langle (\mu F - \gamma f)\tilde{x},\ \tilde{x} - x\rangle \le 0, \quad \forall x \in \mathrm{Fix}(T). \tag{3.18}$$

Consequently, $\tilde{x} \in \Omega$.

To see (b), we assume that $\{t'_n\}$ is another null sequence in $(0,1)$ such that $x_{t'_n}\hat{x}$. Then $\hat{x} \in \mathrm{Fix}(T)$ and by replacing $x$ by $\hat{x}$ in (3.18), we get

$$\langle (\mu F - \gamma f)\tilde{x},\ \tilde{x} - \hat{x}\rangle \le 0. \tag{3.19}$$

By interchanging $\tilde{x}$ and $\hat{x}$, we get

$$\langle (\mu F - \gamma f)\hat{x},\ \hat{x} - \tilde{x}\rangle \le 0. \tag{3.20}$$

Adding up (3.19) and (3.20) yields

$$\langle (\mu F - \gamma f)\tilde{x} - (\mu F - \gamma f)\hat{x},\ \tilde{x} - \hat{x}\rangle \le 0. \tag{3.21}$$

So the strict monotonicity of $\mu F - \gamma f$ implies that $\tilde{x} = \hat{x}$ and $\{z_t\}$ converges weakly.

Finally, to prove (c), we observe that the strong monotonicity of $\mu F - \gamma f$ and (3.11) implies that

$$\alpha\|z_t - \hat{x}\|^2 + \langle(\mu F - \gamma f)\hat{x},\ z_t - \hat{x}\rangle \le \frac{t\mu^2\kappa^2}{2}\|z_t - \hat{x}\|^2 + 2\mu\kappa\|z_t - Tz_t\|\|z_t - \hat{x}\|, \tag{3.22}$$

where $\alpha > 0$ is the strong monotonicity constant of $\mu F - \gamma f$; that is,

$$\langle(\mu F - \gamma f)x - (\mu F - \gamma f)y,\ x - y\rangle \ge \alpha\|x - y\|^2, \quad \forall x, y \in C. \tag{3.23}$$

A straightforward consequence of (3.22) is that if $\hat{x} \in W$ and if $z_{t_n} \rightharpoonup \hat{x}$ for some null sequence $\{t_n\}$ in $(0,1)$, then we must have $z_{t_n} \to \hat{x}$. This shows that $\{z_t\}$ is relatively compact in the norm topology, and each of its limit points solves the HVIP (3.1). Finally repeating the argument in the weak convergence case of (b), we see that $\{z_t\}$ can have exactly one limit point; hence, $\{z_t\}$ converges in norm. □

**Corollary 3.2** (see [14, Proposition 3.1]). *Let $V, T : C \to C$ be nonexpansive mappings with* $\mathrm{Fix}(T) \neq \emptyset$. *Let $t \in (0,1)$ and $z_t$ be a fixed point of the mapping $W_t = tV + (1-t)T$, that is, $z_t = tVz_t + (1-t)Tz_t$. Assume $\{z_t\}$ remains bounded as $t \to 0$, then the following conclusions hold.*

(a) *The solution set $S$ of the HVIP* (1.4) *is nonempty and each weak limit point (as $t \to 0$) of $\{z_t\}$ solves the HVIP* (1.4).

(b) *If $I - V$ is strictly monotone, then the net $\{z_t\}$ converges weakly to the solution of the HVIP* (1.4).

(c) *If $I - V$ is strongly monotone (e.g., $V$ is a contraction), then the net $\{z_t\}$ converges strongly to a solution of the HVIP* (1.4).

Now by combining hybrid steepest descent method, viscosity method, and projection method, we define, for each $s, t \in (0,1)$, two mappings $W_t$ and $f_{s,t}$ by

$$W_t = tV + (1-t)T, \qquad f_{s,t} = P_C\left[s\gamma f + (I - s\mu F)W_t\right]. \tag{3.24}$$

It is easy to see that $W_t$ is a nonexpansive self-mapping on $C$. Moreover, utilizing Lemma 2.6, we can see that $f_{s,t}$ is a $(1-(\tau-\gamma L)s)$-contraction. Indeed, observe that

$$\begin{aligned}
\|f_{s,t}(x) - f_{s,t}(y)\| &= \|P_C[s\gamma f(x) + (I - s\mu F)W_t x] - P_C[s\gamma f(y) + (I - s\mu F)W_t y]\| \\
&\le \|[s\gamma f(x) + (I - s\mu F)W_t x] - [s\gamma f(y) + (I - s\mu F)W_t y]\| \\
&\le s\gamma\|f(x) - f(y)\| + \|(I - s\mu F)W_t x - (I - s\mu F)W_t y\| \\
&\le s\gamma L\|x - y\| + (1 - s\tau)\|x - y\| \\
&= (1 - s(\tau - \gamma L))\|x - y\|.
\end{aligned} \tag{3.25}$$

Let $x_{s,t}$ be the unique fixed point of $f_{s,t}$. Namely, $x_{s,t}$ is the unique solution in $C$ to the following:

$$\begin{aligned} x_{s,t} &= P_C\left[s\gamma f(x_{s,t}) + (I - s\mu F)W_t x_{s,t}\right] \\ &= P_C\left[s\gamma f(x_{s,t}) + (I - s\mu F)(tVx_{s,t} + (1-t)Tx_{s,t})\right]. \end{aligned} \tag{3.26}$$

**Theorem 3.3.** *Let $F : C \to H$ be a $\kappa$-Lipschitzian and $\eta$-strongly monotone operator with constants $\kappa$ and $\eta > 0$, respectively. Let $f : C \to H$ be $L$-Lipschitzian with constant $L \geq 0$ and let $V, T : C \to C$ be nonexpansive with $\mathrm{Fix}(T) \neq \emptyset$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma L < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. For each $s, t \in (0,1)$, let $x_{s,t}$ be the unique solution to* (3.26). *Assume also that, for each $t \in (0,1)$, $\mathrm{Fix}(W_t)$ is nonempty (but not necessarily bounded), and the following assumption holds:*

$$\emptyset \neq S \subset \|\cdot\| - \liminf_{t\to 0} \mathrm{Fix}(W_t) := \{z : \exists z_t \in \mathrm{Fix}(W_t) \text{ such that } z_t \longrightarrow z\}. \tag{A}$$

*Then the strong $\lim_{s\to 0} x_{s,t} =: x_t$ exists for each $t \in (0,1)$. Moreover, the strong $\lim_{t\to 0} x_t =: x_\infty$ exists and solves the THVIP* (1.6). *Hence, for any null sequence $\{s_n\}$ in $(0,1)$, there is another null sequence $\{t_n\}$ in $(0,1)$ such that $x_{s_n,t_n} \to x_\infty$ in norm, as $n \to \infty$.*

*Proof.* Observe that the condition $0 \leq \gamma L < \tau$ and the fact $\tau \leq \mu\eta$ imply that

$$0 \leq \gamma L < \tau \leq \mu\eta. \tag{3.27}$$

Therefore, $\mu F - \gamma f$ is a strongly monotone operator with constant $\mu\eta - \gamma L > 0$. Since, for each fixed $t \in (0,1)$, the fixed point set $\mathrm{Fix}(W_t)$ of $W_t$ is nonempty, we can apply Proposition 3.1 (c) to get that

$$x_t := \|\cdot\| - \lim_{s\to 0} x_{s,t} \tag{3.28}$$

exists in $\mathrm{Fix}(W_t)$ and solves the following hierarchical variational inequality problem of finding $x_t \in \mathrm{Fix}(W_t)$ such that

$$\langle (\mu F - \gamma f)x_t,\ x - x_t\rangle \geq 0, \quad \forall x \in \mathrm{Fix}(W_t). \tag{3.29}$$

Equivalently, $x_t = P_{\mathrm{Fix}(W_t)}(I - \mu F + \gamma f)x_t$, where $P_{\mathrm{Fix}(W_t)}$ is the metric projection from $H$ onto $\mathrm{Fix}(W_t)$.

Utilizing the strong monotonicity of $\mu F - \gamma f$, we conclude from (3.29) that for each $z \in \mathrm{Fix}(W_t)$,

$$\begin{aligned} (\mu\eta - \gamma L)\|x_t - z\|^2 &\leq \langle (\mu F - \gamma f)x_t - (\mu F - \gamma f)z,\ x_t - z\rangle \\ &= -\langle (\mu F - \gamma f)x_t,\ z - x_t\rangle + \langle (\gamma f - \mu F)z,\ x_t - z\rangle \\ &\leq \langle (\gamma f - \mu F)z,\ x_t - z\rangle. \end{aligned} \tag{3.30}$$

Hence,

$$\|x_t - z\|^2 \leq \frac{1}{\mu\eta - \gamma L}\langle(\gamma f - \mu F)z,\ x_t - z\rangle, \quad \forall z \in \mathrm{Fix}(W_t). \tag{3.31}$$

This implies that

$$\|x_t - z\| \leq \frac{1}{\mu\eta - \gamma L}\|(\gamma f - \mu F)z\|, \quad \forall z \in \mathrm{Fix}(W_t). \tag{3.32}$$

The inequality (3.32) is yet to imply the boundedness of $\{x_t\}$ since $z$ may depend on $t$. However, since the solution set $S$ of the HVIP (1.4) is nonempty, we can take (an arbitrary) $v \in S$ and use assumption (A) to find $z_t \in \mathrm{Fix}(W_t)$ such that $z_t \to v$ in norm as $t \to 0$. Hence, $\{z_t\}$ must be bounded (as $t \to 0$). The inequality (3.32) implies

$$\begin{aligned}\|x_t\| &\leq \|x_t - z_t\| + \|z_t - v\| + \|v\| \\ &\leq \frac{1}{\mu\eta - \gamma L}\|(\gamma f - \mu F)z_t\| + \|z_t - v\| + \|v\|,\end{aligned} \tag{3.33}$$

and this is sufficient to ensure that $\{x_t\}$ is bounded (as $t$ closes 0).

Now, the boundedness of $\{x_t\}$ allows us to apply Corollary 3.2 (a) to conclude that every weak limit point $\tilde{x}$ of $\{x_t\}$ belongs to the solution set $S$ of the HVIP (1.4). Then (3.31) guarantees that every such weak limit point $\tilde{x}$ of $\{x_t\}$ is also a strong limit point of $\{x_t\}$. Indeed, if $\{t_n\}$ is a null sequence in $(0,1)$ and if $x_{t_n} \rightharpoonup \tilde{x}$, then $\tilde{x} \in S$. By assumption (A), we get a sequence $\{z_n\}$ such that $z_n \in \mathrm{Fix}(W_{t_n})$ for all $n$ and $z_n \to \tilde{x}$ in norm. From (3.31) we derive

$$\begin{aligned}\|x_{t_n} - \tilde{x}\|^2 &= \|(x_{t_n} - z_n) + (z_n - \tilde{x})\|^2 \\ &\leq 2\left(\|x_{t_n} - z_n\|^2 + \|z_n - \tilde{x}\|^2\right) \\ &\leq \frac{2}{\mu\eta - \gamma L}\langle(\gamma f - \mu F)z_n,\ x_{t_n} - z_n\rangle + 2\|z_n - \tilde{x}\|^2.\end{aligned} \tag{3.34}$$

However, $\langle(\gamma f - \mu F)z_n,\ x_{t_n} - z_n\rangle \to 0$ since $(\gamma f - \mu F)z_n \to (\gamma f - \mu F)\tilde{x}$ in norm and $x_{t_n} - z_n \to 0$ weakly, and we find that the right-hand side of (3.34) tends to zero. Hence, $x_{t_n} \to \tilde{x}$ in norm.

So to prove the strong convergence of the entire net $\{x_t\}$, it remains to prove that $\{x_t\}$ can have only one strong limit point. Let $\tilde{x}$ and $\tilde{x}'$ be two strong limit points of $\{x_t\}$ and assume that $x_{t_n} \to \tilde{x}$ and $x_{t'_n} \to \tilde{x}'$ both in norm, where $\{t_n\}$ and $\{t'_n\}$ are null sequences in $(0,1)$. It remains to verify that $\tilde{x} = \tilde{x}'$.

Since $\tilde{x}' \in S$, by assumption (A), we can find $z_t \in \mathrm{Fix}(W_t)$ such that $z_t \to \tilde{x}'$ in norm as $t \to 0$. The HVIP (3.29) implies

$$\langle(\mu F - \gamma f)x_{t_n},\ z_{t_n} - x_{t_n}\rangle \geq 0. \tag{3.35}$$

Taking the limit as $n \to \infty$ yields

$$\langle (\mu F - \gamma f)\tilde{x},\ \tilde{x}' - \tilde{x} \rangle \geq 0. \tag{3.36}$$

Similarly, we have

$$\langle (\mu F - \gamma f)\tilde{x}',\ \tilde{x} - \tilde{x}' \rangle \geq 0. \tag{3.37}$$

Adding up (3.36) and (3.37) gives

$$\langle (\mu F - \gamma f)\tilde{x} - (\mu F - \gamma f)\tilde{x}',\ \tilde{x} - \tilde{x}' \rangle \leq 0. \tag{3.38}$$

Utilizing Lemma 2.2, we know that $\mu F - \gamma f$ is strongly monotone with constant $\mu\eta - \gamma L > 0$. Hence, from (3.38) it follows that $\tilde{x} = \tilde{x}'$ and so $\{x_t\}$ converges in norm to (say) $x_\infty$.

Now, for any $v \in S$, since by assumption (A), we can find $z_t \in \mathrm{Fix}(W_t)$ such that $z_t \to v$ in norm, (3.29) then implies

$$\langle (\mu F - \gamma f)x_t,\ v - x_t \rangle \geq \langle (\mu F - \gamma f)x_t,\ v - z_t \rangle \longrightarrow 0 \tag{3.39}$$

which in turns implies

$$\langle (\mu F - \gamma f)x_\infty,\ v - x_\infty \rangle \geq 0, \quad \forall v \in S, \tag{3.40}$$

that is, $x_\infty = P_S(I - \mu F + \gamma f)x_\infty$, the unique fixed point of the contraction $P_S(I - \mu F + \gamma f)$. Finally, for any null sequence $\{s_n\}$ in $(0,1)$, using a diagonalization argument (cf. [1]), we can find another null sequence $\{t_n\}$ in $(0,1)$ such that $x_{s_n,t_n} \to x_\infty$ in norm, as $n \to \infty$. □

*Remark 3.4.* Theorem 3.3 shows that for any null sequence $\{s_n\}$ in $(0,1)$, there is another null sequence $\{t_n\}$ in $(0,1)$ such that $x_{s_n,t_n} \to x_\infty$ in norm, as $n \to \infty$, and $x_\infty$ is a solution to the HVIP (3.40). Theorem 3.3 is the main result of the present paper in which we improve the result of Moudafi and Maingé [12] by proving that $\{x_t\}$ actually converges strongly and also by removing the boundedness of the set $\{\mathrm{Fix}(W_t) : 0 < t < 1\}$. Our proof is different from that of [12]. In the meantime, Theorem 3.3 covers [14, Theorem 3.2] as a special case. For instance, whenever we put $\mu = 1$, $F = I$, $\gamma = \tau = 1$, and let the $L$-Lipschitzian mapping $f : C \to H$ be a (self-) contraction with coefficient $\rho \in [0,1)$, our Theorem 3.3 reduces to [14, Theorem 3.2].

Now, we present a general result. We show that as long as $t_s$ is taken so that $t_s = o(s)$ (i.e., $\lim_{s\to 0} t_s/s = 0$), then $x_{s,t_s} \to z_\infty$ in norm, and moreover, $z_\infty$ solves the HVIP (3.40) on the larger set $\mathrm{Fix}(T)$ (i.e., $z_\infty$ is the unique fixed point in $\mathrm{Fix}(T)$ of the contraction $P_{\mathrm{Fix}(T)}(I - \mu F + \gamma f)$), without the assumption (A). However, for such a general choice of $\{t_s\}$, this solution $z_\infty$ may differ from the solution $x_\infty$ of the HVIP (3.40) on the smaller set $S$ (i.e., $x_\infty$ is the unique fixed point in $S$ of the contraction $P_S(I - \mu F + \gamma f)$). We will verify this by taking $t_s = s^2$ for simplicity (the argument, however, works for any net $\{t_s\}$ in $(0,1)$ such that $\lim_{s\to 0} t_s/s = 0$).

**Theorem 3.5.** *Let $F : C \to H$ be a $\kappa$-Lipschitzian and $\eta$-strongly monotone operator with constants $\kappa$ and $\eta > 0$, respectively. Let $f : C \to H$ be $L$-Lipschitzian with constant $L \geq 0$ and let $V, T$ :*

*$C \to C$ be nonexpansive with* $\mathrm{Fix}(T) \neq \emptyset$. *Let* $0 < \mu < 2\eta/\kappa^2$ *and* $0 \leq \gamma L < \tau$, *where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. *For each* $s \in (0,1)$, *let* $x_s$ *be the unique solution in* $C$ *to the following:*

$$x_s = P_C\left[s\gamma f(x_s) + (I - s\mu F)\left(s^2 V x_s + \left(1 - s^2\right)T x_s\right)\right]. \tag{3.41}$$

*Then, as* $s \to 0$, $x_s$ *converges in norm to the solution of the HVIP of finding* $z_\infty \in \mathrm{Fix}(T)$ *such that*

$$\left\langle (\mu F - \gamma f) z_\infty,\ z - z_\infty \right\rangle \geq 0, \quad \forall z \in \mathrm{Fix}(T); \tag{3.42}$$

*equivalently,* $z_\infty = P_{\mathrm{Fix}(T)}(I - \mu F + \gamma f) z_\infty$.

*Proof.* Write $W_s$ (instead of $W_{s^2}$) for $s^2 V + (1 - s^2)T$; then

$$x_s = P_C\left[s\gamma f(x_s) + (I - s\mu F) W_s x_s\right]. \tag{3.43}$$

Take a fixed $z \in \mathrm{Fix}(T)$ arbitrarily and put

$$y_s = s\gamma f(x_s) + (I - s\mu F) W_s x_s, \quad \forall s \in (0,1). \tag{3.44}$$

Then from (3.41) we get $x_s = P_C y_s$. Since $P_C$ is the metric projection from $H$ onto $C$, we have

$$\left\langle P_C y_s - y_s,\ P_C y_s - z \right\rangle \leq 0. \tag{3.45}$$

Also, observe that

$$\begin{aligned} y_s - z &= s\gamma f(x_s) + (I - s\mu F) W_s x_s - z \\ &= s(\gamma f(x_s) - \mu F W_s z) + (I - s\mu F) W_s x_s - (I - s\mu F) W_s z + W_s z - z \\ &= s\gamma(f(x_s) - f(z)) + s(\gamma f(z) - \mu F W_s z) + (I - s\mu F) W_s x_s - (I - s\mu F) W_s z \\ &\quad + s^2 (V - I) z. \end{aligned} \tag{3.46}$$

Utilizing Lemma 2.6, we deduce from (3.41) that

$$\begin{aligned}
\|x_s - z\|^2 &= \langle P_C y_s - y_s,\ P_C y_s - z\rangle + \langle y_s - z,\ x_s - z\rangle \\
&\le \langle y_s - z,\ x_s - z\rangle \\
&= s\gamma\langle f(x_s) - f(z),\ x_s - z\rangle + s\langle \gamma f(z) - \mu F W_s z,\ x_s - z\rangle \\
&\quad + \langle (I - s\mu F)W_s x_s - (I - s\mu F)W_s z,\ x_s - z\rangle + s^2\langle (V - I)z,\ x_s - z\rangle \\
&\le s\gamma\|f(x_s) - f(z)\|\|x_s - z\| + s\langle \gamma f(z) - \mu F W_s z,\ \ x_s - z\rangle \\
&\quad + \|(I - s\mu F)W_s x_s - (I - s\mu F)W_s z\|\|x_s - z\| + s^2\langle (V - I)z,\ \ x_s - z\rangle \\
&\le s\gamma L\|x_s - z\|^2 + s\langle \gamma f(z) - \mu F W_s z,\ x_s - z\rangle + (1 - s\tau)\|x_s - z\|^2 \\
&\quad + s^2\langle (V - I)z,\ x_s - z\rangle \\
&= (1 - s(\tau - \gamma L))\|x_s - z\|^2 + s\langle \gamma f(z) - \mu F W_s z,\ x_s - z\rangle \\
&\quad + s^2\langle (V - I)z,\ x_s - z\rangle.
\end{aligned} \tag{3.47}$$

It follows that, for any fixed $z \in \mathrm{Fix}(T)$,

$$\|x_s - z\|^2 \le \frac{1}{\tau - \gamma L}\left(\langle \gamma f(z) - \mu F W_s z,\ x_s - z\rangle + s\langle (V - I)z,\ x_s - z\rangle\right). \tag{3.48}$$

This implies that

$$\|x_s - z\| \le \frac{1}{\tau - \gamma L}\left(\|\gamma f(z) - \mu F W_s z\| + \|(V - I)z\|\right). \tag{3.49}$$

In particular, $\{x_s\}$ is bounded, and from (3.41), we further get

$$\begin{aligned}
\|x_s - Tx_s\| &= \left\|P_C\left[s\gamma f(x_s) + (I - s\mu F)\left(s^2 V x_s + \left(1 - s^2\right)Tx_s\right)\right] - P_C T x_s\right\| \\
&\le \left\|s\gamma f(x_s) + (I - s\mu F)\left(s^2 V x_s + \left(1 - s^2\right)Tx_s\right) - Tx_s\right\| \\
&\le s\left\|\gamma f(x_s) - \mu F\left(s^2 V x_s + \left(1 - s^2\right)Tx_s\right)\right\| + s^2\|Vx_s - Tx_s\| \longrightarrow 0,
\end{aligned} \tag{3.50}$$

as $s \to 0$. Lemma 2.3 ensures that every weak limit point, as $s \to 0$, of $\{x_s\}$ is a fixed point of $T$. Going back to (3.48), we find that each weak limit point of $\{x_s\}$ is actually a strong limit point of $\{x_s\}$ because

$$\begin{aligned}
&\left|\langle \gamma f(z) - \mu F W_s z,\ x_s - z\rangle\right| \\
&\quad = \left|\langle \gamma f(z) - \mu F z,\ x_s - z\rangle + \langle \mu F z - \mu F W_s z,\ x_s - z\rangle\right| \\
&\quad \le \left|\langle \gamma f(z) - \mu F z,\ x_s - z\rangle\right| + \mu \|F z - F W_s z\| \|x_s - z\| \\
&\quad \le \left|\langle \gamma f(z) - \mu F z,\ x_s - z\rangle\right| + \mu \kappa s^2 \|(V - I) z\| \|x_s - z\|.
\end{aligned} \tag{3.51}$$

So to prove the strong convergence of $\{x_s\}$, we need only to show the uniqueness of strong limit points of $\{x_s\}$. Assuming $\{s_n\}$ and $\{s_n'\}$ are null sequences in $(0,1)$ such that $x_{s_n} \to v$ and $x_{s_n'} \to v'$, both in norm. Observing that (3.41) implies

$$(\mu F - \gamma f) x_s = \frac{1}{s}(P_C y_s - y_s) - \frac{1}{s}(I - W_s) x_s + \mu (F x_s - F W_s x_s), \tag{3.52}$$

where $y_s = s\gamma f(x_s) + (I - s\mu F) W_s x_s$ and $x_s = P_C y_s$. Utilizing Lemmas 2.4 and 2.6, we deduce from the monotonicity of $I - W_s$ that for any fixed $z \in \mathrm{Fix}(T)$,

$$\begin{aligned}
&\langle (\mu F - \gamma f) x_s,\ x_s - z\rangle \\
&\quad = \frac{1}{s}\langle P_C y_s - y_s,\ P_C y_s - z\rangle - \frac{1}{s}\langle (I - W_s) x_s,\ x_s - z\rangle + \mu \langle F x_s - F W_s x_s,\ x_s - z\rangle \\
&\quad \le \frac{1}{s}\langle P_C y_s - y_s,\ P_C y_s - z\rangle - \frac{1}{s}[\langle (I - W_s) x_s - (I - W_s) z,\ x_s - z\rangle \\
&\qquad + \langle (I - W_s) z,\ x_s - z\rangle] + \mu \langle F x_s - F W_s x_s,\ x_s - z\rangle \\
&\quad \le -s\langle (I - V) z,\ x_s - z\rangle + \mu \kappa \|x_s - W_s x_s\| \|x_s - z\| \\
&\quad = -s\langle (I - V) z,\ x_s - z\rangle + \mu \kappa \left\|(x_s - T x_s) + s^2 (T - V) x_s\right\| \|x_s - z\|.
\end{aligned} \tag{3.53}$$

In particular, we have from $I - W_s$ that for any fixed $z \in \mathrm{Fix}(T)$,

$$\begin{aligned}
\langle (\mu F - \gamma f) x_{s_n},\ x_{s_n} - v'\rangle \le\ & -s_n \langle (I - V) v',\ x_{s_n} - v'\rangle \\
& + \mu \kappa \left\|(x_{s_n} - T x_{s_n}) + s_n^2 (T - V) x_{s_n}\right\| \|x_{s_n} - v'\|.
\end{aligned} \tag{3.54}$$

So letting $n \to \infty$ yields

$$\langle (\mu F - \gamma f) v,\ v - v'\rangle \le 0. \tag{3.55}$$

Repeating the above argument obtains

$$\langle (\mu F - \gamma f) v',\ v' - v\rangle \le 0. \tag{3.56}$$

Adding up (3.55) and (3.56) gives us that

$$\langle (\mu F - \gamma f)v - (\mu F - \gamma f)v', \ v - v' \rangle \leq 0. \tag{3.57}$$

The strong monotonicity of $\mu F - \gamma f$ (Lemma 2.2) then implies $v = v'$. Finally, taking the limit as $s \to 0$ in (3.53) and letting $z_\infty = \|\cdot\| - \lim_{s\to 0} x_s$, we conclude immediately that $z_\infty$ solves the variational inequality of finding $z_\infty \in \mathrm{Fix}(T)$ such that

$$\langle (\mu F - \gamma f)z_\infty, \ z_\infty - z \rangle \leq 0, \quad \forall z \in \mathrm{Fix}(T). \tag{3.58}$$

Equivalently, $z_\infty = P_{\mathrm{Fix}(T)}(I - \mu F + \gamma f)z_\infty$. The proof is therefore complete. □

*Remark 3.6.* If $T$ and $V$ have a common fixed point, then it is not hard to see that $\mathrm{Fix}(W_t) = \mathrm{Fix}(T) \cap \mathrm{Fix}(V)$ for all $t \in (0,1)$. Indeed, it suffices to show the inclusion $\mathrm{Fix}(W_t) \subset \mathrm{Fix}(T) \cap \mathrm{Fix}(V)$. Let $z \in \mathrm{Fix}(W_t)$. Then for any fixed $p \in \mathrm{Fix}(T) \cap \mathrm{Fix}(V)$ we have

$$\begin{aligned} \|z - p\|^2 &= t\|Vz - p\|^2 + (1-t)\|Tz - p\|^2 - t(1-t)\|Vz - Tz\|^2 \\ &\leq \|z - p\|^2 - t(1-t)\|Vz - Tz\|^2. \end{aligned} \tag{3.59}$$

This implies $Vz = Tz = z$; that is $z \in \mathrm{Fix}(T) \cap \mathrm{Fix}(V)$. Furthermore, it is clear that $\mathrm{Fix}(T) \cap \mathrm{Fix}(V) \subset S$. In this case, assumption (A) is reduced to the assumption $S \subset \mathrm{Fix}(T) \cap \mathrm{Fix}(V)$. Therefore, assumption (A) is equivalent to the assumption $S = \mathrm{Fix}(T) \cap \mathrm{Fix}(V)$.

**Corollary 3.7.** *Let $F : C \to H$ be a $\kappa$-Lipschitzian and $\eta$-strongly monotone operator with constants $\kappa$ and $\eta > 0$, respectively. Let $f : C \to H$ be $L$-Lipschitzian with constant $L \geq 0$ and let $V, T : C \to C$ be nonexpansive with $\mathrm{Fix}(T) \cap \mathrm{Fix}(V) \neq \emptyset$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma L < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. For each $s, t \in (0,1)$, let $x_{s,t}$ be the unique solution to* (3.26). *Then the conclusion of Theorem 3.3 holds. Namely, the strong $\lim_{s\to 0} x_{s,t} =: x_t$ exists for each fixed $t \in (0,1)$, and moreover, the strong $\lim_{t\to 0} x_t =: x_\infty$ exists and solves the THVIP* (1.6).

*Proof.* Since $\mathrm{Fix}(W_t) = \mathrm{Fix}(T) \cap \mathrm{Fix}(V)$ is independent of $t$; the $z$ in both (3.31) and (3.32) does not depend on $t$. Hence, it is immediately clear that $\{x_t\}$ is bounded, which then implies via (3.31) that every weak accumulation point of $\{x_t\}$ is also a strong accumulation point of $\{x_t\}$. Eventually, $\{x_t\}$ converges in norm as shown in the final part of the proof of Theorem 3.5. □

*Remark 3.8.* Theorems 3.3 and 3.5 improve and extend [14, Theorems 3.2 and 3.4], respectively, in the following ways.

(a) The contraction mapping $f : C \to C$ in [14, Theorems 3.2 and 3.4] is extended to the case of (possibly nonself) $L$-Lipschitzian mapping $f : C \to H$ from a nonempty closed convex subset $C$ to $H$.

(b) The convex combination of (self) contraction mapping $f$ and nonexpansive mapping $W_t$ in the implicit scheme in [14, Theorem 3.2] is extended to the linear combination of (possibly nonself) $L$-Lipschitzian mapping $f$ and hybrid steepest descent method involving $W_t$. In particular, if $t = s^2$, Theorem 3.5 is an extension of [14, Theorem 3.4].

(c) In order to guarantee that the net $\{x_{s,t}\}$ generated by the implicit scheme still lies in $C$, the implicit scheme in [14, Theorem 3.2] is extended to develop our new implicit scheme (3.26) by virtue of the projection method. In particular, if $t = s^2$, [14, Theorem 3.4] is extended to the corresponding case in our Theorem 3.5.

(d) The new technique of argument is applied to derive our Theorems 3.3 and 3.5. For instance, the characteristic properties (Lemma 2.4) of the metric projection play a key role in proving the strong convergence of the nets $\{x_{s,t}\}_{s,t\in(0,1)}$ and $\{x_s\}_{s\in(0,1)}$ in our Theorems 3.3 and 3.5, respectively.

(e) If we put $\mu = 1, F = I$ and $\gamma = \tau = 1$ and let $f$ be a contractive self-mapping on $C$ with coefficient $\rho \in [0,1)$, then our Theorems 3.3 and 3.5 reduce to [14, Theorems 3.2 and 3.4], respectively. Thus, our Theorems 3.3 and 3.5 cover [14, Theorems 3.2 and 3.4] as special cases, respectively.

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