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Research Article

Some Fixed and Periodic Points in Abstract Metric Spaces

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We generalize in this paper some results on common fixed points of two, respectively four, contractive-type mappings in abstract metric spaces by removing condition of normality of the cone in their formulations. Further, some results about periodic points of self-maps are extended to the setting of abstract metric spaces.

1. Introduction and Preliminaries

Ordered normed spaces, cones, and topical functions have applications in applied mathematics, for instance, in using Newton's approximation method [1–4] and in optimization theory [5, 6]. K-metric and K-normed spaces were introduced in the mid-20th century ([2], see also [3, 4]) by replacing an ordered Banach space instead of the set of real numbers, as the codomain for a metric. Huang and Zhang [7] reintroduced such spaces under the name of cone metric spaces, but went further, defining convergent and Cauchy sequences in the terms of interior points of the underlying cone. In such a way, nonnormal cones can be used as well (although they used only normal cones), paying attention to the fact that Sandwich Theorem and continuity of the metric may not hold. These and other authors (e.g., [8–14]) proved some fixed point theorems for contractive-type mappings in cone metric spaces, as well as topological vector-space-valued cone metric spaces (e.g., [15, 16]).

The following definitions and results will be needed in the sequel (see, e.g., [2, 3, 5, 17, 18]).

Let E be a real topological vector space. A subset K of E is called a *cone* if (a) K is closed, nonempty and $K \neq \{\theta\}$; (b) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in K$ imply that $ax + by \in K$; (c) $K \cap (-K) = \{\theta\}$.

Given a cone K, we define the partial ordering $\preceq$ with respect to K by $x \preceq y$ if and only if $y - x \in K$. We will write $x \ll y$ for $y - x \in \operatorname{int} K$, where $\operatorname{int} K$ stands for the interior of K and use $x \prec y$ for ($x \preceq y$ and $x \neq y$). If $\operatorname{int} K \neq \emptyset$, then K is called a *solid cone* [3]. Note that the notation $0 \ll c$ for an interior point of a positive cone was first used by Kreĭn and Rutman [19].

The cone K in the topological vector space E is called *normal* if E has a base of neighborhoods of θ consisting of order-convex subsets (see [16]). In the case of a normed space, this is equivalent to the condition that there is a number $k > 0$ such that, for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $\|x\| \leq k\|y\|$. Equivalently, the cone K is normal if

$$(\forall n) \quad x_n \preceq y_n \preceq z_n, \qquad \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \text{ imply } \lim_{n\to\infty} y_n = x. \tag{1.1}$$

For details see [5].

Example 1.1 (see [3]). Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $K = \{x \in E : x(t) \geq 0\}$. This cone is nonnormal. Consider, for example, $x_n(t) = t^n/n$ and $y_n(t) = 1/n$. Then $\theta \preceq x_n \preceq y_n$, and $\lim_{n\to\infty} y_n = \theta$, but $\|x_n\| = \max_{t\in[0,1]}|t^n/n| + \max_{t\in[0,1]}|t^{n-1}| = 1/n + 1 > 1$; hence x_n does not converge to zero. It follows by (1.1) that K is a nonnormal cone.

Definition 1.2 (see [4, 15, 16]). Let X be a nonempty set and E a topological vector space with a cone K. Suppose that a mapping $d : X \times X \to E$ satisfies the following:

(d_1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d_3) $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The function d is called an *abstract metric* and (X, d) is called an *abstract metric space* (or a topological vector-space-valued cone metric space or a K-metric space); we will use the first mentioned term.

The concept of an abstract metric space is obviously more general than that of a metric space. If E is a Banach space then abstract metric space becomes a cone metric space of [7]. For new results in cone metric spaces see [20–26].

Definition 1.3. Let (X, d) be an abstract metric space. We say that a sequence $\{x_n\}$ in X is

(i) a *Cauchy sequence* if, for every c in E with $\theta \ll c$, there is an $n_0 \in \mathbb{N}$ such that for all $m, n > n_0$, $d(x_m, x_n) \ll c$;

(ii) a *convergent sequence* if, for every c in E with $\theta \ll c$, there is an $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \ll c$ for some fixed $x \in X$.

An abstract metric space X is said to be *complete* if every Cauchy sequence in X is convergent in X.

Let (X, d) be an abstract metric space. The following properties are often used, particularly in the case when the underlying cone is nonnormal. The only assumption is that the cone K is solid. For details about these properties see, for example, [11].

(p_1) If $a \preceq ha$ where $a \in K$ and $h \in [0, 1)$, then $a = \theta$.

(p_2) If $\theta \preceq u \ll c$ for each c, $\theta \ll c$, then $u = \theta$.

(p_3) If $u \preceq v$ and $v \ll w$, then $u \ll w$.

(p_4) If $c \in \operatorname{int} K$, $\theta \preceq x_n$, and $x_n \to \theta$, then there exists $k \in \mathbb{N}$ such that, for all $n > k$ we have $x_n \ll c$. (Note that, in general, the converse is not true. Indeed, in Example 1.1, $x_n \nrightarrow \theta$, but $x_n \ll c$ for n sufficiently large.)

In generalizing some theorems of Huang-Zhang [7], Abbas and Rhoades proved the following result in abstract metric spaces over normal cones.

Theorem 1.4 (see [9]). *Let (X, d) be a complete cone metric space over a normal cone. Suppose that $f, g : X \to X$ are two self-maps satisfying*

$$d(fx, gy) \preceq \alpha d(x, y) + \beta[d(x, fx) + d(y, gy)] + \gamma[d(x, gy) + d(y, fx)], \tag{1.2}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$, and $\alpha + 2\beta + 2\gamma < 1$. Then f and g have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and conversely.

Sing et al. extended this result of Abbas-Rhoades to four maps. They proved the following theorem.

Theorem 1.5 (see [27]). *Let (X, d) be a complete cone metric space over a normal cone. Suppose that the mappings f, g, S, and T are four selfmaps on X such that $fX \subset TX$ and $gX \subset SX$ and satisfying*

$$d(fx, gy) \preceq \alpha d(Sx, Ty) + \beta[d(Sx, fx) + d(Ty, gy)] + \gamma[d(Sx, gy) + d(Ty, fx)], \tag{1.3}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. Suppose that the pairs $\{f, S\}$ and $\{g, T\}$ are weakly compatible. Then f, g, S, and T have a unique common fixed point.

In 1977, Rhoades proved the following interesting result.

Theorem 1.6 (see [28]). *Let (X, d) be a complete metric space. Let $f : X \to X$, and suppose that there exist decreasing functions $\alpha_i : (0, +\infty) \to [0, 1)$, $i = 1, \ldots, 5$, such that $\sum_{i=1}^{5} \alpha_i(t) < 1$ for each $t \in (0, +\infty)$ and satisfying*

$$\begin{aligned} d(fx, fy) \leq{}& \alpha_1(d(x, y))d(x, y) + \alpha_2(d(x, y))d(x, fx) + \alpha_3(d(x, y))d(y, fy) \\ &+ \alpha_4(d(x, y))d(fy, x) + \alpha_5(d(x, y))d(fx, y), \end{aligned} \tag{1.4}$$

for all $x, y \in X$, $x \neq y$. Then f has a unique fixed point z and for each $x_0 \in X$ the sequence $\{f^n x_0\}$ converges to z.

We generalize in this paper Theorems 1.4 and 1.5 by removing normality condition in their formulations. An example will show that these generalizations are proper. Further,

some results of Abbas and Rhoades about periodic points of selfmaps from [29] are extended to abstract metric spaces. Theorem 1.6 is also presented in this new setting, with a slightly shorter proof.

Note that it was shown in [15, 30, 31] that some of the fixed point results in abstract metric spaces can be directly reduced to the respective metric results. However, the results of the present paper do not fall into this category, since some of them are new even in the context of metric spaces.

2. Fixed Point Theorems

In this section we will prove Theorems 1.4 and 1.5 by omitting the assumption of normality in the results. We use only the definition of convergence in terms of the relation "$\ll$". The only assumption is that K is a solid cone, so we use neither continuity of the vector metric d, nor Sandwich Theorem. We begin with the following.

Theorem 2.1. *Let (X, d) be an abstract metric space over a solid cone K. Suppose that f, g, S, and T are four self-maps on X such that $fX \subset TX$ and $gX \subset SX$ and suppose that at least one of these four subsets of X is complete. Let*

$$d(fx, gy) \preceq \alpha d(Sx, Ty) + \beta[d(Sx, fx) + d(Ty, gy)] + \gamma[d(Sx, gy) + d(Ty, fx)], \tag{2.1}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then the pairs (f, S) and (g, T) have a unique common point of coincidence. If, moreover, pairs (f, S) and (g, T) are weakly compatible, then f, g, S, and T have a unique common fixed point.

For definitions of terms like "point of coincidence" and "weakly compatible pair" see, for example, [11].

Remark 2.2. In the papers [9] and [27], the cone K is supposed to be normal and solid. In that case the proof is essentially the same as in the setting of usual metric spaces.

We now give the proof of Theorem 2.1.

Proof. Suppose $x_0 \in X$ is an arbitrary point, and define the sequence $\{y_n\}$ by $y_{2n} = fx_{2n} = Tx_{2n+1}$, $y_{2n+1} = gx_{2n+1} = Sx_{2n+2}$, $n = 0, 1, 2, \ldots$. Now, as in [27], by (2.1), we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(fx_{2n}, gx_{2n+1}) \\ &\preceq \alpha d(Sx_{2n}, Tx_{2n+1}) + \beta[d(Sx_{2n}, fx_{2n}) + d(Tx_{2n+1}, gx_{2n+1})] \\ &\quad + \gamma[d(Sx_{2n}, gx_{2n+1}) + d(Tx_{2n+1}, fx_{2n})] \\ &\preceq (\alpha + \beta + \gamma) d(y_{2n-1}, y_{2n}) + (\beta + \gamma) d(y_{2n}, y_{2n+1}), \end{aligned} \tag{2.2}$$

which implies that $d(y_{2n}, y_{2n+1}) \preceq \delta d(y_{2n-1}, y_{2n})$, where $\delta = (\alpha + \beta + \gamma)/(1 - (\beta + \gamma)) < 1$. Similarly it can be shown that

$$d(y_{2n+1}, y_{2n+2}) \preceq \delta d(y_{2n}, y_{2n+1}). \tag{2.3}$$

Therefore, for all n,

$$d(y_n, y_{n+1}) \preceq \delta d(y_{n-1}, y_n) \preceq \cdots \preceq \delta^n d(y_0, y_1). \tag{2.4}$$

Now, for any $m > n$,

$$\begin{aligned} d(y_n, y_m) &\preceq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m) \\ &\preceq \left[\delta^n + \delta^{n+1} + \cdots + \delta^{m-1}\right] d(y_0, y_1) \\ &\preceq \frac{\delta^n}{1-\delta} d(y_0, y_1) \longrightarrow \theta \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{2.5}$$

Thus, by properties (p_4) and (p_3) and Definition 1.3, $\{y_n\}$ is a Cauchy sequence.

Suppose, for example, that SX is a complete subset of X. Then $y_n \to u = Sv$, $n \to \infty$, for some $v \in X$. Of course, subsequences $\{y_{2n-1}\}$ and $\{y_{2n}\}$ also converge to u. Let us prove that $fv = u$. Using (2.1) we get that

$$\begin{aligned} d(fv, u) = d(fv, gx_{2n-1}) + d(gx_{2n-1}, u) &\preceq \alpha d(Sv, Tx_{2n-1}) + \beta\left[d(Sv, fv) + d(Tx_{2n-1}, gx_{2n-1})\right] \\ &+ \gamma\left[d(Sv, gx_{2n-1}) + d(Tx_{2n-1}, fv)\right] + d(gx_{2n-1}, u), \end{aligned} \tag{2.6}$$

which further implies that

$$d(fv, u) \preceq \frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)} d(u, Tx_{2n-1}) + \frac{1 + \beta + \gamma}{1 - (\beta + \gamma)} d(u, gx_{2n-1}). \tag{2.7}$$

Let $\theta \ll c$ be given. Since $y_n \to u$ as $n \to \infty$, choose a natural number n_0 such that for all $n > n_0$ (Definition 1.3) we have that

$$d(u, Tx_{2n-1}) \ll \frac{1 - (\beta + \gamma)}{2(\alpha + \beta + \gamma)} c, \qquad d(u, gx_{2n-1}) \ll \frac{1 - (\beta + \gamma)}{2(1 + \beta + \gamma)} c. \tag{2.8}$$

Thus, according to (2.7) we obtain $d(fv, u) \ll (c/2) + (c/2) = c$. Therefore, $d(fv, u) \ll c$ for all $c \in \operatorname{int} K$. Using property ($p_2$), it follows that $d(fv, u) = \theta$ and so $fv = u = Sv$. Since $u \in fX \subset TX$, we get that there exists $w \in X$ such that $Tw = u$. Let us prove that also $gw = u$. By triangle inequality and (2.1), we have

$$\begin{aligned} d(gw, u) &\preceq d(gw, fx_{2n}) + d(fx_{2n}, u) \\ &\preceq \alpha d(Sx_{2n}, Tw) + \beta\left[d(Sx_{2n}, fx_{2n}) + d(Tw, gw)\right] \\ &\quad + \gamma\left[d(Sx_{2n}, gw) + d(Tw, fx_{2n})\right] + d(gx_{2n-1}, u), \end{aligned} \tag{2.9}$$

which further implies that

$$d(gw,u) \preceq \frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)} d(u,Sx_{2n}) + \frac{1+\beta+\gamma}{1-(\beta+\gamma)} d(u,fx_{2n}). \tag{2.10}$$

Now, for given $\theta \ll c$, since $y_n \to u$ as $n \to \infty$, choose a natural n_1 such that for all $n > n_1$ we have that

$$d(u,Sx_{2n}) \ll \frac{1-(\beta+\gamma)}{2(\alpha+\beta+\gamma)} c, \qquad d(u,fx_{2n}) \ll \frac{1-(\beta+\gamma)}{2(1+\beta+\gamma)} c. \tag{2.11}$$

According to (2.10) we obtain $d(gw,u) \ll (c/2) + (c/2) = c$. Therefore, $d(gw,u) \ll c$ for all $c \in \operatorname{int} K$. Using property (p_2), it follows that $d(gw,u) = \theta$ and so $gw = u = Tw$. We have proved that u is a common point of coincidence for pairs (f,S) and (g,T).

If now these pairs are weakly compatible, then $fu = fSv = Sfv = Su = z_1$ (say) and $gu = gTw = Tgw = Tu = z_2$ (say). Moreover,

$$\begin{aligned} d(z_1,z_2) = d(fu,gu) &\preceq \alpha d(Su,Tu) + \beta\left[d(Su,fu) + d(Tu,gu)\right] \\ &\quad + \gamma\left[d(Su,gu) + d(Tu,fu)\right] \\ &= \alpha d(z_1,z_2) + \beta[d(z_1,z_1) + d(z_2,z_2)] + \gamma[d(z_1,z_2) + d(z_2,z_1)] \\ &= (\alpha+2\gamma) d(z_1,z_2) \preceq (\alpha+\beta+\gamma) d(z_1,z_2) \prec d(z_1,z_2) \end{aligned} \tag{2.12}$$

implies that $z_1 = z_2$. So we have that $fu = gu = Su = Tu$. It remains to prove that, for example, $u = gu$. Indeed,

$$\begin{aligned} d(u,gu) = d(fv,gu) &\preceq \alpha d(Sv,Tu) + \beta\left[d(Sv,fv) + d(Tu,gu)\right] \\ &\quad + \gamma\left[d(Sv,gu) + d(Tu,fv)\right] \\ &= \alpha d(u,gu) + \beta[\theta+\theta] + \gamma\left[d(u,gu) + d(gu,u)\right] \\ &= (\alpha+2\gamma) d(u,gu) \preceq (\alpha+\beta+\gamma) d(u,gu) \prec d(u,gu), \end{aligned} \tag{2.13}$$

implying that $gu = u$. The uniqueness follows from (2.1). The proofs for cases in which fX, gX, or TX is complete are similar and are therefore omitted. The theorem is proved. □

We present now two examples showing that Theorem 2.1 is a proper extension of the known results. In both examples, the conditions of Theorem 2.1 are fulfilled, but in the first one (because of nonnormality of the cone) the main theorems from [9, 27] cannot be applied.

Example 2.3 (the case of a nonnormal cone). Let $X = [0,1]$, and let E be the set of all real-valued functions on X which also have continuous derivatives on X. Note that E is a vector space over $\mathbb{R}$ under usual function operations. Let τ be the strongest vector (locally convex) topology on E. Then (E,τ) is a topological vector space which is not normable and is not even metrizable. Let $K = \{\varphi \in E : \varphi(t) \geq 0, t \in \mathbb{R}\}$ and $d : X \times X \to E$ be defined by

$d(x,y)(t) = |x-y| \cdot e^t$. Then (X,d) is an abstract metric space over a nonnormal solid cone (Example 1.1). Consider the four mappings $f,g,T,S : X \to X$ defined by

$$fx = \frac{x}{8}, \qquad gx = \frac{x}{12}, \qquad Tx = \frac{x}{2}, \qquad Sx = \frac{x}{3}. \tag{2.14}$$

Clearly $f(X) \subset T(X)$ and $g(X) \subset S(X)$.

For $x,y \in X$,

$$\begin{aligned} d(fx,gy)(t) &= \left|\frac{x}{8} - \frac{y}{12}\right| e^t = \frac{1}{8}\left|x - \frac{2y}{3}\right| e^t, \\ d(Sx,Ty)(t) &= \left|\frac{x}{3} - \frac{y}{2}\right| e^t, \\ d(fx,Sx)(t) + d(gy,Ty)(t) &= \left|\frac{x}{8} - \frac{x}{3}\right| e^t + \left|\frac{y}{12} - \frac{y}{2}\right| e^t = \left(\frac{5x}{24} + \frac{5y}{12}\right) e^t, \\ d(fx,Ty) + d(gy,Sx)(t) &= \left(\left|\frac{x}{8} - \frac{y}{2}\right| + \left|\frac{y}{12} - \frac{x}{3}\right|\right) e^t. \end{aligned} \tag{2.15}$$

Now

$$\begin{aligned} d(fx,gy)(t) &= \frac{1}{8}\left|x - \frac{2y}{3}\right| e^t \\ &\le \frac{1}{6}\left|\frac{x}{3} - \frac{y}{2}\right| e^t + \frac{1}{6}\left(\frac{5x}{24} + \frac{5y}{12}\right) e^t + \frac{1}{6}\left(\left|\frac{x}{8} - \frac{y}{2}\right| + \left|\frac{y}{12} - \frac{x}{3}\right|\right) e^t \\ &= \alpha d(Sx,Ty)(t) + \beta\left[d(fx,Sx)(t) + rd(gy,Ty)(t)\right] \\ &\quad + \gamma\left[d(fx,Ty)(t) + d(gy,Sx)(t)\right]. \end{aligned} \tag{2.16}$$

Thus all the conditions of Theorem 2.1 are satisfied with $\alpha + 2\beta + 2\gamma = 5/6 < 1$. Note that 0 is the unique common fixed point of f, g, S, and T.

Example 2.4 (the case of a normal cone). Let $E = C_{\mathbb{R}}[0,1]$ with $K = \{\varphi \in E : \varphi(t) \ge 0, t \in [0,1]\}$ (this cone is normal; see [5]). Let $X = [0,\infty)$, and let $d : X \times X \to E$ be defined as $d(x,y)(t) := |x-y|e^t$. Take the functions $fx = x/3$, $gx = 0$, $SX = TX = x$ which map the set X into X. All the conditions of Theorem 2.1 are fulfilled with $\gamma = 1/3$, $\alpha + 2\beta < 1/3$. Obviously, f, g, S, and T have the unique common fixed point $x = 0$.

Remark 2.5. Taking $S = T = i_X$ and appropriate choices of f, g, α, β, and γ in Theorem 2.1, one easily gets [9, Corollaries 2.2–2.8]. In each of the following cases (1)–(7), (X,d) is a complete abstract metric space, K a solid cone, and f is a selfmap on X.

(1) Let

$$d(f^p x, f^q y) \preceq \alpha d(x,y) + \beta\left[d(x,f^p x) + d(y,f^q y)\right] + \gamma\left[d(x,f^q y) + d(y,f^p x)\right] \tag{2.17}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$, and p and q are fixed positive integers. Then f has a unique fixed point in X.

(2) If

$$d(fx, fy) \preceq \alpha d(x, y) + \beta[d(x, fx) + d(y, fy)] + \gamma[d(x, fy) + d(y, fx)] \tag{2.18}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$, then f has a unique fixed point in X.

(3) If

$$d(fx, fy) \preceq a_1 d(x, y) + a_2 d(x, fx) + a_3 d(y, fy) + a_4 d(x, fy) + a_5 d(y, fx) \tag{2.19}$$

for all $x, y \in X$, where $a_i \geq 0$ for each $i \in \{1, 2, \ldots, 5\}$ and $a_1 + a_2 + \cdots + a_5 < 1$, then f has a unique fixed point in X.

(4) If

$$d(fx, fy) \preceq \alpha d(x, y) \tag{2.20}$$

for all $x, y \in X$, where $\alpha \in [0, 1)$, then f has a unique fixed point in X.

(5) If

$$d(fx, fy) \preceq \beta[d(x, fx) + d(y, fy)] \tag{2.21}$$

for all $x, y \in X$, where $\beta \in [0, 1/2)$, then f has a unique fixed point in X.

(6) If

$$d(fx, fy) \preceq \gamma[d(x, fy) + d(y, fx)] \tag{2.22}$$

for all $x, y \in X$, where $\gamma \in [0, 1/2)$, then f has a unique fixed point in X.

(7) If

$$d(fx, fy) \preceq \alpha d(x, y) + \beta[d(x, fx) + d(y, fy)] \tag{2.23}$$

for all $x, y \in X$, where $\alpha, \beta \geq 0$ and $\alpha + 2\beta < 1$, then f has a unique fixed point in X.

We add an example of a Banach-type contraction on a nonnormal abstract metric space.

Example 2.6. Let $X = [0, 1]$, $E = C^1_{\mathcal{R}}[0, 1]$, $K = \{\varphi \in E : \varphi(t) \geq 0\}$. An abstract metric d on X is defined by $d(x, y)(t) := |x - y| \cdot \phi(t)$ where $\phi \in K$ is an arbitrary function (e.g., $\phi(t) = 2^t$). It is easy to see that (X, d) is a complete abstract metric space. Suppose that mapping $f : X \to X$ satisfies

$$d(fx, fy) \preceq \lambda d(x, y), \tag{2.24}$$

for all $x, y \in X$, where $\lambda \in [0,1)$. All the conditions from Remark 2.5(4) hold, and f has a unique fixed point in X.

This example verifies that Theorem 2.1 is a proper extension of the results from [7]. Indeed, we know (see Example 1.1) that the cone K is nonnormal. So, in this example Theorem 1 from [7] cannot be applied.

Corollary 2.7. *Let (X,d) be an abstract metric space and K a solid cone. Suppose that the mappings f, g, S and T are four selfmaps of X such that $f(X) \subset T(X)$ and $g(X) \subset S(X)$, and suppose that at least one of these four subsets of X is complete. Let*

$$d(f^n x, g^n y) \preceq \alpha d(Sx, Ty) \tag{2.25}$$

for all $x, y \in X$, where $\alpha \in [0,1)$ and (f,S) and (g,T) commute. Then f, g, S, and T have a unique common fixed point.

Proof. By Theorem 2.1, we obtain $u \in X$ such that

$$f^n u = g^n u = Su = Tu = u. \tag{2.26}$$

The result then follows from the fact that

$$d(fu, gu) = d(f^n fu, g^n gu) \preceq \alpha d(Sfu, Tgu) = \alpha d(fu, gu) \tag{2.27}$$

since $\alpha < 1$ so that $fu = gu$ by property (p_1). Again

$$d(fu, u) = d(f^n fu, g^n u) \preceq \alpha d(Sfu, Tu) = \alpha d(fu, u), \tag{2.28}$$

implies that $fu = u$. And hence u is the unique common fixed point of f, g, S, and T. □

3. Periodic Point Theorems

It is clear that if f is a map which has a fixed point p, then p is also a fixed point of f^n for every $n \in \mathbb{N}$. However the converse is not true. For example, consider $X = \mathbb{R}$ and f defined by $fx = a - x$, $a \neq 0$. Then f has a unique fixed point at $a/2$, but every even iterate of f is the identity map, which has each point of $\mathbb{R}$ as its fixed point. On the other hand, if $X = [0, +\infty)$, $fx = x^2$, then every iterate of f has the same fixed point as f. If a map f satisfies $F(f) = F(f^n)$ for each $n \in \mathbb{N}$, where $F(f)$ stands for the set of all fixed points of f, then it is said to have property P [29]. We will say that $f, g : X \to X$ have property Q if $F(f) \cap F(g) = F(f^n) \cap F(g^n)$ for each $n \in \mathbb{N}$.

The next result is a generalization of the corresponding result in metric spaces (see [29, Theorem 1.1]). It will be deduced also without using normality of the cone.

Theorem 3.1. *Let (X,d) be an abstract metric space over a solid cone K, and let $f : X \to X$ be such that $F(f) \neq \emptyset$ and that*

$$d\left(fx, f^2 x\right) \preceq \lambda d(x, fx) \tag{3.1}$$

holds for some $\lambda \in (0,1)$ *and either (i) for all* $x \in X$ *or (ii) for all* $x \in X$, $x \neq fx$. *Then* f *has property* P.

Proof. We will always assume that $n > 1$, since the statement for $n = 1$ is trivial. Let $u \in F(f^n)$. Suppose that f satisfies (i). Then

$$\begin{aligned} d(u, fu) &= d\left(ff^{n-1}u, f^2f^{n-1}u\right) \preceq \lambda d\left(f^{n-1}u, f^n u\right) = \lambda d\left(ff^{n-2}u, f^2f^{n-2}u\right) \\ &\preceq \lambda^2 d\left(f^{n-2}u, f^{n-1}u\right) \preceq \cdots \preceq \lambda^n d(u, fu) \preceq \lambda d(u, fu). \end{aligned} \tag{3.2}$$

According to property (p_1) it follows that $d(u, fu) = \theta$, that is, $fu = u$. Suppose that f satisfies (ii). If $fu = u$, then there is nothing to prove. Suppose, if possible, that $fu \neq u$. Then, similarly as in case (i) we get that $d(u, fu) = d(ff^{n-1}u, f^2f^{n-1}u)$. In order to use (3.1) we need that $f^{n-1}u \neq ff^{n-1}u = f^n u$. But, if this is not the case, then $f^{n-1}u = u$ and so $u = f^n u = fu$, a contradiction. Hence, applying (3.1) we obtain that

$$d(u, fu) = d\left(ff^{n-1}u, f^2f^{n-1}u\right) \preceq \lambda d\left(f^{n-1}u, f^n u\right) = \lambda d\left(ff^{n-2}u, f^2f^{n-2}u\right). \tag{3.3}$$

Repeating the same argument several times we finally obtain, similarly as in case (i), that $d(u, fu) \preceq \lambda^n d(u, fu)$, which again implies $u = fu$ since $\lambda \in (0,1)$, a contradiction. □

Corollary 3.2. *Let* (X, d) *be an abstract metric space over a solid cone* K. *Suppose that a mapping* $f : X \to X$ *satisfies*

$$d(fx, fy) \preceq \alpha d(x, y) + \beta[d(x, fx) + d(y, fy)] + \gamma[d(x, fy) + d(y, fx)] \tag{3.4}$$

for all $x, y \in X$, *where* $\alpha, \beta, \gamma \geq 0$ *and* $\alpha + 2\beta + 2\gamma < 1$. *Then* f *has property* P.

Proof. From Remark 2.5(2), $F(f) \neq \emptyset$. We will prove that f satisfies the condition (i) of Theorem 3.1. Indeed,

$$\begin{aligned} d\left(fx, f^2x\right) &= d(fx, ffx) \preceq \alpha d(x, fx) + \beta\left[d(x, fx) + d\left(fx, f^2x\right)\right] \\ &\quad + \gamma\left[d\left(x, f^2x\right) + d(fx, fx)\right] \\ &\preceq \alpha d(x, fx) + \beta\left[d(x, fx) + d\left(fx, f^2x\right)\right] + \gamma\left[d(x, fx) + d\left(fx, f^2x\right)\right], \end{aligned} \tag{3.5}$$

which implies that $d(fx, f^2x) \preceq \lambda d(x, fx)$, where $\lambda = (\alpha + \beta + \gamma)/(1 - (\beta + \gamma)) < 1$. Hence, f has property P. □

The method of proof of the following result differs to the one from [29] (see also [9, Theorem 3.2]).

Theorem 3.3. *Let* (X, d) *be a complete abstract metric space over a solid cone* K. *Suppose that mappings* $f, g : X \to X$ *satisfy* (2.1) *(with* $S = T = i_X$*). Then* f *and* g *have property* Q.

Proof. By Theorem 2.1, we have that $F(f) \cap F(g) = \{u\}$, where u is the unique common fixed point of f and g. So $F(f^n) \cap F(g^n) \neq \emptyset$ for each $n \in \mathbb{N}$. Let $v \in F(f^n) \cap F(g^n)$, where $n > 1$ is arbitrary. Then, we obtain

$$\begin{aligned}
d(u,v) &= d(f^n u, g^n v) = d\left(f f^{n-1} u, g g^{n-1} v\right) \\
&\preceq \alpha d\left(f^{n-1}u, g^{n-1}v\right) + \beta\left[d\left(f^{n-1}u, f^n u\right) + d\left(g^{n-1}v, g^n v\right)\right] \\
&\quad + \gamma\left[d\left(f^{n-1}u, g^n v\right) + d\left(g^{n-1}v, f^n u\right)\right] \\
&= \alpha d\left(u, g^{n-1}v\right) + \beta d\left(g^{n-1}v, g^n v\right) + \gamma\left[d(u, g^n v) + d\left(u, g^{n-1}v\right)\right] \\
&\preceq \alpha d\left(u, g^{n-1}v\right) + \beta d\left(g^{n-1}v, u\right) + \beta d(u, g^n v) + \gamma\left[d(u, g^n v) + d\left(u, g^{n-1}v\right)\right],
\end{aligned} \tag{3.6}$$

wherefrom it follows that $d(u, g^n v) \preceq \delta d(u, g^{n-1}v)$, where $\delta = (\alpha + \beta + \gamma)/(1 - (\beta + \gamma))$.

Further, we have that

$$d\left(u, g^{n-1}v\right) \preceq \delta d\left(u, g^{n-2}v\right). \tag{3.7}$$

Indeed,

$$\begin{aligned}
d\left(u, g^{n-1}v\right) &= d\left(f^{n-1}u, g^{n-1}v\right) = d\left(f f^{n-2}u, g g^{n-2}v\right) \\
&\preceq \alpha d\left(f^{n-2}u, g^{n-2}v\right) + \beta\left[d\left(f^{n-2}u, f^{n-1}u\right) + d\left(g^{n-2}v, g^{n-1}v\right)\right] \\
&\quad + \gamma\left[d\left(f^{n-2}u, g^{n-1}v\right) + d\left(g^{n-2}v, f^{n-1}u\right)\right] \\
&= \alpha d\left(u, g^{n-2}v\right) + \beta d\left(g^{n-2}v, g^{n-1}v\right) + \gamma\left[d\left(u, g^{n-1}v\right) + d\left(u, g^{n-2}v\right)\right] \\
&\preceq \alpha d\left(u, g^{n-2}v\right) + \beta d\left(g^{n-2}v, u\right) + \beta d\left(u, g^{n-1}v\right) \\
&\quad + \gamma\left[d\left(u, g^{n-1}v\right) + d\left(u, g^{n-2}v\right)\right],
\end{aligned} \tag{3.8}$$

which implies (3.7). Hence,

$$d(u,v) = d(u, g^n v) \preceq \delta d\left(u, g^{n-1}v\right) \preceq \delta^2 d\left(u, g^{n-2}v\right) \preceq \cdots \preceq \delta^n d(u,v). \tag{3.9}$$

Since $\delta^n \in [0,1)$, according to property (p_1) it follows $d(u,v) = \theta$. Hence $v = u$, which implies that f and g have property Q. □

Corollary 3.4. *Let (X,d) be a complete abstract metric space over a solid cone K. Suppose that the mapping $f : X \to X$ satisfies one of the conditions (3)–(6) of Remark 2.5. Then f has property P.*

Remark 3.5. In the paper [9], the space (X, d) is supposed to be a complete cone metric space over a normal and solid cone K. Hence, our Theorems 3.1, 3.3, and Corollary 3.2 are proper extensions of Theorems 3.1, 3.2. and 3.3 from [9].

In the following result the cone K is regular, hence also normal (for the definition see, e.g., [5]).

Theorem 3.6. *Let (X, d) be an abstract metric space over a regular cone K. Let $f, g : X \to X$ be two mappings such that $fX \subset gX$ and one of these subset of X is complete. Suppose that there exist decreasing functions $\alpha_i : K \to [0, 1)$, $i = 1, \ldots, 5$, such that $\sum_{i=1}^{5} \alpha_i(t) < 1$ for each $t \in K$ and satisfying*

$$\begin{aligned} d(fx, fy) \preceq{}& \alpha_1(d(gx, gy))d(gx, gy) + \alpha_2(d(gx, gy))d(gx, fx) + \alpha_3(d(gx, gy))d(gy, fy) \\ &+ \alpha_4(d(gx, gy))d(fy, gx) + \alpha_5(d(gx, gy))d(fx, gy) \end{aligned} \tag{3.10}$$

for all $x, y \in X$, $x \neq y$. Then f and g have a unique point of coincidence. If, moreover, the pair (f, g) is weakly compatible, then f and g have a unique common fixed point.

Proof. Suppose, for example, that gX is complete. Take an arbitrary $x_0 \in X$ and, using that $fX \subset gX$, construct a Jungck sequence $\{y_n\}$ defined by $y_n = fx_n = gx_{n+1}$, $n = 0, 1, 2, \ldots$. Let us prove that this is a Cauchy sequence. If $y_n = y_{n-1}$ for some n, then it is easy to prove that the sequence $\{y_n\}$ becomes eventually constant and so convergent.

Suppose that $y_n \neq y_{n-1}$ for each $n \in \mathbb{N}$. Using (3.10), we obtain that

$$\begin{aligned} d(y_n, y_{n+1}) &= d(fx_n, fx_{n+1}) \\ &\preceq \alpha_1(d(y_{n-1}, y_n))d(y_{n-1}, y_n) + \alpha_2(d(y_{n-1}, y_n))d(y_{n-1}, y_n) \\ &\quad + \alpha_3(d(y_{n-1}, y_n))d(y_n, y_{n+1}) + \alpha_4(d(y_{n-1}, y_n))d(y_{n+1}, y_{n-1}) \\ &\quad + \alpha_5(d(y_{n-1}, y_n))d(y_n, y_n), \end{aligned} \tag{3.11}$$

for each $n \in \mathbb{N}$. Also,

$$\begin{aligned} d(y_{n+1}, y_n) &= d(fx_n, fx_{n+1}) \\ &\preceq \alpha_1(d(y_{n-1}, y_n))d(y_n, y_{n-1}) + \alpha_2(d(y_{n-1}, y_n))d(y_{n+1}, y_n) \\ &\quad + \alpha_3(d(y_{n-1}, y_n))d(y_{n-1}, y_n) + \alpha_4(d(y_{n-1}, y_n))d(y_n, y_n) \\ &\quad + \alpha_5(d(y_{n-1}, y_n))d(y_{n+1}, y_{n-1}). \end{aligned} \tag{3.12}$$

Adding the last two relations (and putting temporarily $\alpha_i = \alpha_i(d(y_{n-1}, y_n))$, $i = 1, \ldots, 5$) we obtain

$$d(y_n, y_{n+1}) \preceq \beta(d(y_{n-1}, y_n))d(y_{n-1}, y_n), \tag{3.13}$$

where

$$\beta(t) = \frac{2\alpha_1(t) + \alpha_2(t) + \alpha_3(t) + \alpha_4(t) + \alpha_5(t)}{2 - (\alpha_2(t) + \alpha_3(t) + \alpha_4(t) + \alpha_5(t))}. \tag{3.14}$$

It is easy to see that monotonicity of all α_i's implies that β is also a decreasing function and that $0 < \beta(t) < 1$ for each $t \in K$. In particular, $d(y_n, y_{n+1}) \prec d(y_{n-1}, y_n)$ and so the sequence $\{d(y_n, y_{n+1})\}$ is strictly decreasing (and bounded from below).

Since the cone K is regular, there exists $\lim_{n\to\infty} d(y_n, y_{n+1}) = p$ and $\theta \preceq p \preceq d(y_n, y_{n+1})$ for each n. Then $1 > \beta(p) > \beta(d(y_n, y_{n+1}))$ for each n, and hence

$$d(y_n, y_{n+1}) \preceq \beta(p) d(y_{n-1}, y_n) \preceq (\beta(p))^2 d(y_{n-2}, y_{n-1}) \preceq \cdots \preceq (\beta(p))^n d(y_0, y_1), \tag{3.15}$$

where $\beta(p)$ is a fixed scalar belonging to $[0, 1)$.

Now we prove that $\{y_n\}$ is a Cauchy sequence in the usual way: for $m > n$ it is

$$\begin{aligned} d(y_n, y_m) &\preceq d(y_n, y_{n+1}) + \cdots + d(y_{m-1}, y_m) \\ &\preceq \left((\beta(p))^n + \cdots (\beta(p))^{m-1}\right) d(y_0, y_1) \\ &\preceq \frac{(\beta(p))^n}{1 - \beta(p)} d(y_0, y_1) \longrightarrow \theta \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.16}$$

Thus, by properties (p_4) and (p_3) and Definition 1.3, $\{y_n\}$ is a Cauchy sequence in gX and so there is $z \in X$ such that $fx_n = gx_{n+1} \to gz$ when $n \to \infty$. We will prove that $fz = gz$.

Put $x = x_n$, $y = z$ in the contractive condition. We obtain (writing temporarily $\alpha_i = \alpha_i(d(x_n, z))$) that

$$\begin{aligned} d(fx_n, fz) &\preceq \alpha_1 d(gx_n, gz) + \alpha_2 d(gx_n, fx_n) + \alpha_3 d(gz, fz) \\ &\quad + \alpha_4 d(fz, gx_n) + \alpha_5 d(fx_n, gz) \\ &\preceq \alpha_1 d(gx_n, gz) + (\alpha_2 + \alpha_4) d(gx_n, fx_n) + \alpha_3 d(gz, fz) \\ &\quad + \alpha_4 d(fz, fx_n) + \alpha_5 d(fx_n, gz). \end{aligned} \tag{3.17}$$

Taking into account that all α_i's are bounded in $[0, 1)$ and that the abstract metric d is continuous (because the cone K is normal), passing to the limit in the last vector inequality, we obtain that

$$d(gz, fz) \preceq \alpha_1 \cdot \theta + (\alpha_2 + \alpha_4) \cdot \theta + \alpha_3 d(gz, fz) + \alpha_4 d(fz, gz) + \alpha_5 \cdot \theta, \tag{3.18}$$

that is, $d(gz, fz) \preceq (\alpha_3 + \alpha_4) d(gz, fz)$. Since $\alpha_3 + \alpha_4 < 1$, it follows that $gz = fz = w$ and f and g have a point of coincidence w.

Suppose that $w_1 = fz_1 = gz_1$ is another point of coincidence for f and g. Then (3.10) implies that

$$\begin{aligned} d(w_1, w_2) &= d(fz, fz_1) \\ &\preceq \alpha_1 d(gz, gz_1) + \alpha_2 d(gz, fz) + \alpha_3 d(gz_1, fz_1) + \alpha_4 d(fz_1, gz) + \alpha_5 d(fz, gz_1) \\ &= \alpha_1 d(w, w_1) + \alpha_2 \cdot \theta + \alpha_3 \cdot \theta + \alpha_4 d(w_1, w) + \alpha_5 d(w, w_1) \\ &= (\alpha_1 + \alpha_4 + \alpha_5) d(w, w_1). \end{aligned} \tag{3.19}$$

Since $\alpha_1 + \alpha_4 + \alpha_5 < 1$, the last relation is possible only if $w = w_1$. So, the point of coincidence is unique.

If (f, g) is weakly compatible, then [8, Proposition 1.4] implies that f and g have a unique common fixed point. The proof for the case in which fX is complete is similar and is therefore omitted. □

Remark 3.7. Taking $E = \mathbb{R}$, $K = [0, +\infty)$, $g(x) = x$, we obtain a shorter proof of Theorem 1.6 (i.e., [28, Theorem 4]).

Remark 3.8. Taking appropriate choices of f, g and α_i, $i = 1, \ldots, 5$ in Theorem 3.6, one can easily get the results of Reich (see relations (7), (8) in [28]), Hardy-Rogers (see relation (18) in [28]) and Ćirić (see relation (21) in [28]) in the setting of abstract metric spaces.

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