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Research Article

The Systems of Nonlinear Gradient Flows on Metric Spaces and Its Gamma-Convergence

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We first establish the explicit structure of nonlinear gradient flow systems on metric spaces and then develop Gamma-convergence of the systems of nonlinear gradient flows, which is a scheme meant to ensure that if a family of energy functionals of several variables depending on a parameter Gamma-converges, then the solutions to the associated systems of gradient flows converge as well. This scheme is a nonlinear system edition of the notion initiated by Sylvia Serfaty in 2011.

1. Introduction

The theory of Gamma-convergence was introduced by De Giorgi in the 1970s. It has become both a standard criterion for the study of variational problems and one of the extremely important topics in the calculus of variations. Gradient flows which are defined on metric spaces were also noticed by De Giorgi, that is, the notion replacing "gradient flows in a differentiable structure" then that of "curve of maximal slope in metric space." This notion was introduced in [1] then further developed in [2–5]. It turns out to be useful in many applications, in particular for defining gradient flows over the probability measure spaces equipped with the Wasserstein metric.

In 2004, Gamma-convergence of gradient flows on Hilbert spaces was introduced by Sandier and Serfaty in [6]. This abstract method states that if a family of energy functionals $\{J_\varepsilon\}_{\varepsilon>0}$ Γ-converges to a limiting functional J, then, under suitable conditions, solutions of the gradient flow of J_ε converge to solutions of the gradient flow of J. This scheme was used successfully for the dynamics of Ginzburg-Landau vortices (cf. [6]), the Cahn-Hilliard equation (cf. [7, 8]), and the Allen-Cahn equation (cf. [9]).

The notion of Gamma-convergence of gradient flows on metric spaces was initiated by Serfaty [9] in 2011. She presented and proved the following.

Proposition 1.1 (cf. [9] Gamma-convergence of gradient flows in the metric spaces setting). *Let $(X_\varepsilon, d_\varepsilon)$ and (X, d) be complete metric spaces. Let Φ_ε and Φ be functionals defined on metric spaces $(X_\varepsilon, d_\varepsilon)$ and (X, d), respectively. Assume that there is a sense of convergence $\mathcal{S}$ of $u_\varepsilon \in X_\varepsilon$ to $u \in X$ which can be general and with respect to the Γ-liminf convergence of Φ_ε to Φ:*

$$u_\varepsilon \xrightarrow{\mathcal{S}} u \Longrightarrow \liminf_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon) \geq \Phi(u). \tag{1.1}$$

Let g_ε and g be strong upper gradients of Φ_ε and Φ, respectively. Assume in addition the following relations.

(1) *Lower bound on the metric derivatives: if $u_\varepsilon(t) \xrightarrow{\mathcal{S}} u(t)$, for $t \in [0, T)$ then for all $s \in [0, T)$*

$$\liminf_{\varepsilon\to 0} \int_0^s \left|u'_\varepsilon\right|^p_{d_\varepsilon}(t)dt \geq \int_0^s \left|u'\right|^p_d(t)dt. \tag{1.2}$$

(2) *Lower bound on the slopes: if $u_\varepsilon \xrightarrow{\mathcal{S}} u$, then*

$$\liminf_{\varepsilon\to 0} g_\varepsilon(u_\varepsilon) \geq g(u). \tag{1.3}$$

Let $u_\varepsilon(t)$ be a p-curve of maximal slope on $(0, T)$ for Φ_ε with respect to g_ε, such that $u_\varepsilon(t) \xrightarrow{\mathcal{S}} u(t)$, which is well prepared in the sense that

$$\lim_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon(0)) = \Phi(u(0)). \tag{1.4}$$

Then u is a p-curve of maximal slope with respect to g and

$$\begin{gathered}
\lim_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon(t)) = \Phi(u(t)) \quad \forall t \in [0, T), \\
g_\varepsilon(u_\varepsilon) \longrightarrow g(u) \quad \text{in } L^p_{\text{loc}}(0, T), \\
\left|u'_\varepsilon\right|_{d_\varepsilon} \longrightarrow \left|u'\right|_d \quad \text{in } L^p_{\text{loc}}(0, T).
\end{gathered} \tag{1.5}$$

Obviously, this scheme (Proposition 1.1.) can be applied to single gradient flow problems only. To the best of our knowledge, nonlinear equations are more difficult than linear equations, the problem of system of differential equations is more complicated and more important than the problem of scalar equations, and the system of nonlinear gradient flows on metric spaces has not appeared elsewhere. This gives us a motivation for studying the systems of nonlinear gradient flows on metric spaces and then establishing its Gamma-convergence structure which can be applied to the problems of nonlinear gradient flow systems on metric spaces. This scheme can be regarded as a "nonlinear system" edition of the notion initiated by Sylvia Serfaty.

This paper is organized as follows. In Section 2, we introduce some necessary knowledge on gradient flows, basic definitions of absolutely continuous curve, metric derivative, strong upper gradient, and curve of maximal slope for functional. In Section 3, we establish the explicit structure of nonlinear gradient flow systems and Gamma-convergence of the systems of nonlinear gradient flows on metric spaces. Finally, we give two examples to illustrate a special case of our main results in Section 4.

2. Basic Definitions and Preliminaries

Let $(X, \langle\cdot,\cdot\rangle)$ be a real Hilbert space with the corresponding norm $\|\cdot\|$ and let $E : X \to \mathbb{R}$ be a functional defined on X. We say that E is Fréchet differentiable at $x \in X$ if there exists $x^* \in X^* \equiv \mathcal{L}(X;\mathbb{R})$ (the space of all bounded linear functionals on X) such that

$$E(x+h) - E(x) = x^*(h) + o(\|h\|), \tag{2.1}$$

where $\lim_{\|h\|\to 0}(o(\|h\|)/\|h\|) = 0$.

Note that if such an x^* exists, then it is unique and we denote $DE(x) \equiv x^*$. In view of the Riesz representation theorem there exists a unique element $y \in X$ such that

$$\langle y, h\rangle = x^*(h) \quad \forall h \in X. \tag{2.2}$$

Moreover $\|x^*\|_{X^*} = \|y\|$. $DE(x)$ is called the differentiable of E at x (notice that it is a bounded linear functional on X). We denote $\nabla_X E(x) \equiv y$ and we call $\nabla_X E(x)$ the gradient of E at x. Hence we have

$$DE(x)(h) = \langle \nabla_X E(x), h\rangle \quad \forall h \in X. \tag{2.3}$$

We say that E is of class C^1 on X (i.e., $E \in C^1(X;\mathbb{R})$) if the map $x \to DE(x)$ is continuous on X. If $E \in C^1(X;\mathbb{R})$, then the directional derivative of E at $u(\in X)$ in direction φ exists and is given by

$$\left.\frac{d}{dt}\right|_{t=0} E(u+t\varphi) = \langle \nabla_X E(u), \varphi\rangle. \tag{2.4}$$

Let $\gamma : \mathbb{R} \to X$ be a differentiable curve in X with $\gamma(t_0) = x \in X$. Then

$$\left.\frac{d}{dt}\right|_{t=t_0} E(\gamma(t)) = \langle \nabla_X E(\gamma(t_0)), \gamma'(t_0)\rangle. \tag{2.5}$$

The evolution equation

$$\frac{dx}{dt} = -\nabla_X E(x) \tag{2.6}$$

is called the gradient flow of E on Hilbert space $(X, \|\cdot\|_X)$. Using the definitions of Hilbert spaces and Gradient flows, it is easy to show the following basic and useful lemma.

Lemma 2.1. *Suppose that for each $1 \le i \le n$, $(Y_i, \langle\cdot,\cdot\rangle_{Y_i})$ is an inner product space with induced norm $\|\cdot\|_{Y_i}$. Let one defines*

$$\langle x, y\rangle_Y \equiv \sum_{i=1}^{n} \langle x_i, y_i\rangle_{Y_i} \tag{2.7}$$

for each $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ in $Y \equiv Y_1 \times \cdots \times Y_n$. Then one has the following.

(i) *We can see that $(Y, \langle\cdot,\cdot\rangle_Y)$ is an inner product space with induced norm*

$$\|x\|_Y^2 = \sum_{i=1}^{n} \|x_i\|_{Y_i}^2 \quad \forall x = (x_1, \ldots, x_n) \in Y. \tag{2.8}$$

Thus, $(Y, \|\cdot\|_Y)$ is a Hilbert space.

(ii) *Let F be a C^1 functional defined on $Y \equiv Y_1 \times \cdots \times Y_n$ and let $x = (x_1, \ldots, x_n) \in Y$. An element $y = (y_1, \ldots, y_n) \in Y$ (and denote $y = \nabla_Y F(x)$) is called the gradient of F at x on the Hilbert space Y if for every differentiable curve $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$ in Y satisfying $\gamma(t_0) = x$,*

$$\frac{d}{dt}\left(F(\gamma(t))\right)\bigg|_{t=t_0} = \left\langle \nabla_Y F(\gamma(t_0)), \gamma'(t_0)\right\rangle_Y = \sum_{i=1}^{n} \left\langle y_i, \gamma_i'(t_0)\right\rangle_{Y_i}. \tag{2.9}$$

Let one denotes

$$y_i = \nabla_{Y_i} F(\gamma(t_0)) = \nabla_{Y_i} F(x) \quad \forall 1 \le i \le n. \tag{2.10}$$

$\nabla_{Y_i} F(x)$ is called the gradient of F at x with respect to Y_i. Hence one has

$$\frac{d}{dt}\left(F(\gamma(t))\right)\bigg|_{t=t_0} = \sum_{i=1}^{n} \left\langle \nabla_{Y_i} F(\gamma(t_0)), \gamma_i'(t_0)\right\rangle_{Y_i}. \tag{2.11}$$

The evolution equation (the gradient flow of F on $(Y, \|\cdot\|_Y)$)

$$\frac{dx}{dt} = -\nabla_Y F(x), \tag{2.12}$$

can be expressed as the following system of gradient flows on Hilbert spaces:

$$
\begin{aligned}
\frac{dx_1}{dt} &= -\nabla_{Y_1} F(x_1(t),\dots,x_n(t)) \in Y_1,\\
\frac{dx_2}{dt} &= -\nabla_{Y_2} F(x_1(t),\dots,x_n(t)) \in Y_2,\\
&\vdots\\
\frac{dx_n}{dt} &= -\nabla_{Y_n} F(x_1(t),\dots,x_n(t)) \in Y_n.
\end{aligned}
\tag{2.13}
$$

Definition 2.2 (p-absolutely continuous curve). Let (X,d) be a complete metric space equipped with the distance d. A mapping $v:(a,b)\to X$ is called a p-absolutely continuous curve or belongs to $AC^p(a,b;X)$, $p\geq 1$, if there exists an $L^p(a,b)$ function m such that

$$d(v(s),v(t)) \leq \int_s^t m(r)dr \quad \forall a<s\leq t<b. \tag{2.14}$$

Proposition 2.3 (cf. [5]). *Let (X,d) be a metric space and let $u:[a,b]\to X$. Then*

(i) *$u\in AC([a,b];X)$ if and only if there exists $m\in L^1(a,b)$, $m\geq 0$, such that*

$$d(u(s),u(t)) \leq \int_s^t m(\tau)d\tau \quad \forall a\leq s<t\leq b. \tag{2.15}$$

(ii) *If $u\in AC([a,b];X)$, the metric derivative*

$$|u'|_d(t) \equiv \lim_{h\to 0}\frac{d(u(t+h),u(t))}{|h|} \tag{2.16}$$

exists for a.e. $t\in(a,b)$, $|u'|_d\in L^1(a,b)$,

$$d(u(s),u(t)) \leq \int_s^t |u'|_d d\tau \quad \forall a\leq s<t\leq b, \tag{2.17}$$

and if m satisfies (2.15), *then $|u'|_d\leq m$ a.e. on (a,b).*

In the following, let us give a motivation for defining "gradient flow" on metric spaces. Here we completely follow the nice contents of Section 1.3 in [5]. Note that every solution u of the gradient flow

$$\frac{du}{dt} = -\nabla_X E(u(t)) \tag{2.18}$$

can be characterized by the following the scalar equations:

$$\begin{aligned}\frac{d}{dt}(E(u(t))) = (E \circ u)'(t) &= \langle \nabla_X E(u(t)), u'(t) \rangle \\ &= -\langle \nabla_X E(u(t)), \nabla_X E(u(t)) \rangle \\ &= -\langle u'(t), u'(t) \rangle \\ &= -\|\nabla_X E(u(t))\| \|u'(t)\|,\end{aligned} \tag{2.19}$$

$$\|u'(t)\| = \|\nabla_X E(u(t))\|. \tag{2.20}$$

Using Young's inequality, (2.19) and (2.20) are equivalent to

$$(E \circ u)'(t) = -\frac{1}{2}\|u'(t)\|^2 - \frac{1}{2}\|\nabla_X E(u(t))\|^2. \tag{2.21}$$

We can impose (2.18), (2.19), (2.20), and (2.21) as a system of differential inequalities in the couple (u, g) by using the following strategies.

(i) The function g is an upper bound for the modulus of the gradient

$$|(E \circ v)'| \leq g(v) \cdot \|v'\| \tag{2.22}$$

for every regular curve $v : (0, +\infty) \mapsto X$.

(ii) Impose that the functional E decreasing along u as much as possible compatibly with (2.22), that is,

$$(E \circ u)'(t) \leq -g(u(t)) \cdot \|u'(t)\| \quad \text{in } (0, +\infty). \tag{2.23}$$

(iii) Prescribe the dependence of $\|u'\|$ on $g(u)$,

$$\|u'\| = g(u) \quad \text{in } (0, +\infty), \tag{2.24}$$

or even in a single formula

$$(E \circ u)'(t) \leq -\frac{1}{2}\|u'(t)\|^2 - \frac{1}{2}(g(u(t)))^2 \quad \text{in } (0, +\infty). \tag{2.25}$$

Whereas (2.18), (2.19), and (2.20) make sense only in a Hilbert space framework, the formulas (2.21)~(2.25) are of purely metric nature and can be extended to more general metric space, provided we understand $\|u'\|$ as the metric derivative of u, $|u'|_d$. Of course, the concept of upper gradient provides only an upper estimate for the modulus of $\nabla_X E$ in the regular case, but it is enough to define steepest descent curves, that is, curves which realize the minimal selection of $d/dt(E\circ u)(t)$ compatible with

$$\left|(E\circ v)'(t)\right| \le g(v(t))\left|v'\right|_d(t) \tag{2.26}$$

for a.e. $t\in(0,+\infty)$.

Suppose that $u\in AC(a,b;X)$, and $g\circ u$ is Borel. Using (2.26), we have for any $a<s<t<b$,

$$\begin{aligned}|(E\circ u)(t)-(E\circ u)(s)| &= \left|\int_s^t (E\circ u)'(\tau)d\tau\right| \\ &\le \int_s^t \left|(E\circ u)'(\tau)\right|d\tau \\ &\le \int_s^t g(u(\tau))\left|u'\right|_d(\tau)d\tau.\end{aligned} \tag{2.27}$$

Therefore we say that g is a strong upper gradient for E if for each $u\in AC(a,b;X)$, $g\circ u$ is Borel and (2.27) holds for all $a<s<t<b$. Using the ideas (ii) and (iii) and Young's inequality, we say that a locally absolutely continuous function $u:(a,b)\mapsto X$ is a curve of maximal slope for E with respect to its strong upper gradient g if $E\circ u$ is a.e. equal to a nonincreasing map φ and

$$(E\circ u)'(t)=\varphi'(t)\le -\frac{|u'|_d^2(t)}{2}-\frac{\left(g(u(t))\right)^2}{2} \tag{2.28}$$

for a.e. $t\in(a,b)$. Let us present the main definitions and three lemmas as follows.

Definition 2.4 (strong upper gradient). Suppose that (X_i,d_i) is a complete metric space equipped with the distance d_i for $i=1,\dots,n$. Let $g_i: X_1\times\cdots\times X_n \to [0,+\infty]$ for each i with $1\le i\le n$. We say that $g=(g_1,\dots,g_n)$ is a strong upper gradient for $\Phi: X_1\times\cdots\times X_n\to\mathbb{R}$ if for each $(u_1,\dots,u_n)\in AC(a,b;X_1\times\cdots\times X_n)$, $g_i\circ(u_1,\dots,u_n)$ is Borel for $i=1,\dots,n$, and

$$|\Phi(u_1,\dots,u_n)(t)-\Phi(u_1,\dots,u_n)(s)|\le\int_s^t\left(\sum_{i=1}^n\left|u_i'\right|_{d_i}(\tau)g_i(u_1,\dots,u_n)(\tau)\right)d\tau \quad \forall a<s\le t<b. \tag{2.29}$$

Definition 2.5 (a pair of Young's functions). Suppose that $F^*$and $G^*:[0,+\infty)\to[0,+\infty)$ are two differentiable functions. We say that (F^*,G^*) is a pair of Young's functions if they satisfy

(1) Young's inequality: $st\le F^*(t)+G^*(s)$ for all $s,t\ge 0$,

(2) Young's equality: $st=F^*(t)+G^*(s)\Leftrightarrow s=f^*(t)$ or $t=g^*(s)$,

where $f^*=(F^*)'$ and $g^*=(G^*)'$.

Definition 2.6 (curve of maximal slope). Let (F_i^*, G_i^*) be a pair of Young's functions for each i with $1 \leq i \leq n$. We say that $(u_1, \dots, u_n) : (a,b) \to X_1 \times \cdots \times X_n$ is a $(F_1^*, \dots, F_n^*)$-curve of maximal slope for the functional Φ with respect to the strong upper gradient $g = (g_1, \dots, g_n)$ if $\Phi(u_1, \dots, u_n)$ is L^1-a.e. equal to a nonincreasing map φ and

$$\varphi'(t) \leq -\sum_{i=1}^{n} \left[F_i^* \left(|u_i'|_{d_i}(t) \right) + G_i^* \left(g_i(u_1, \dots, u_n)(t) \right) \right] \tag{2.30}$$

for L^1-a.e. $t \in (a,b)$.

Definition 2.7 (Γ-liminf convergence). Let $(X_{i,\varepsilon}, d_{i,\varepsilon})$ and (X_i, d_i) be complete metric spaces for all $1 \leq i \leq n$ and $\varepsilon > 0$. Suppose that $\Phi_\varepsilon : X_{1,\varepsilon} \times \cdots \times X_{n,\varepsilon} \mapsto (-\infty, +\infty]$ and $\Phi : X_1 \times \cdots \times X_n \mapsto (-\infty, +\infty]$ are functionals defined on $X_{1,\varepsilon} \times \cdots \times X_{n,\varepsilon}$ and $X_1 \times \cdots \times X_n$, respectively. We say that $\Phi_\varepsilon \Gamma$-liminf converges to Φ if

$$(u_{1,\varepsilon}, \dots, u_{n,\varepsilon}) \xrightarrow{\mathcal{S}} (u_1, \dots, u_n), \tag{2.31}$$

then

$$\liminf_{\varepsilon \to 0} \Phi_\varepsilon(u_{1,\varepsilon}, \dots, u_{n,\varepsilon}) \geq \Phi(u_1, \dots, u_n), \tag{2.32}$$

where $(u_{1,\varepsilon}, \dots, u_{n,\varepsilon}) \in X_{1,\varepsilon} \times \cdots \times X_{n,\varepsilon}$, $(u_1, \dots, u_n) \in X_1 \times \cdots \times X_n$, and the sense of convergence $\mathcal{S}$ can be general.

Lemma 2.8. *Suppose that $\alpha \geq 0$ and $a_n \geq 0$ for each $n \in \mathbb{N}$. If $\liminf_{n \to \infty} a_n \geq \alpha$ and if f is a continuous nondecreasing function on $[0, +\infty)$, then*

$$\liminf_{n \to \infty} f(a_n) \geq f(\alpha). \tag{2.33}$$

Proof. (i) For each $k \in \mathbb{N}$, $a_k^- \equiv \inf_{n \geq k} a_n \leq a_n$, $\forall n \geq k$.

(ii) Since f is non-decreasing on $[0, \infty)$, and using (i), we have

$$f\left(a_k^-\right) \leq f(a_n), \quad \forall n \geq k. \tag{2.34}$$

Therefore,

$$f\left(a_k^-\right) \text{ is a lower bound for } \left\{ f(a_n) \mid n \geq k \right\}, \tag{2.35}$$

and so

$$f\left(a_k^-\right) \leq \inf_{n \geq k} f(a_n), \quad \text{for each } k \in \mathbb{N}. \tag{2.36}$$

(iii) Since f is continuous on $[0,\infty)$, and using (ii), we have

$$\begin{aligned} f(\alpha) &\leq f\left(\liminf_{n\to\infty} a_n\right) = f\left(\lim_{k\to\infty}\left(\inf_{n\geq k} a_n\right)\right) \\ &= f\left(\lim_{k\to\infty} a_k^-\right) = \lim_{k\to\infty} f\left(a_k^-\right) \\ &\leq \lim_{k\to\infty}\left(\inf_{n\geq k} f(a_n)\right) = \liminf_{k\to\infty} f(a_k). \end{aligned} \tag{2.37}$$

□

Lemma 2.9 (see [10, Theorem 5.11]). *Let f be nonnegative and measurable on E. Then $\int_E f\,dx = 0$ if and only if $f = 0$ a.e. in E.*

Lemma 2.10. *Let (X_i, d_i) be a metric space for each $1 \leq i \leq n$. Let $X \equiv X_1 \times \cdots \times X_n = \{(x_1, \cdots, x_n) \mid x_i \in X_i$ for $1 \leq i \leq n\}$. Define the function $d : X \times X \mapsto \mathbb{R}$ by*

$$d(x, y) \equiv \sum_{i=1}^{n} d_i(x_i, y_i) \tag{2.38}$$

for each $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in X$. Then

(i) *(X, d) is a metric space,*

(ii) *suppose that $v = (v_1, \ldots, v_n) \in AC(a, b; X)$. The metric derivative $|v'|_d$ can be expressed as*

$$|v'|_d(t) = \sum_{i=1}^{n} |v_i'|_{d_i}(t) \tag{2.39}$$

for a.e. $t \in (a, b)$.

3. Main Results

In the following theorem we introduce the systems of explicit nonlinear gradient flows of energy functional with respect to the strong upper gradient on metric spaces and investigate an upper control for some form of velocity of solutions by its dissipation rate of the energy functional. Using this idea, we can see that if motion is driven by energy dissipation and if there are solutions that move without losing much energy, then they must move very slowly for each component solution.

Theorem 3.1. *Let (X_i, d_i) be a complete metric spaces equipped with distance d_i for $i = 1, 2, \ldots, n$. Let Φ be a functional defined on $X_1 \times X_2 \times \cdots \times X_n$ and let $g = (g_1, \ldots, g_n)$ be a strong upper gradient for Φ. Assume that $f_i : [0, +\infty) \mapsto [0, +\infty)$ is a continuous, strictly increasing and surjective function for each $1 \leq i \leq n$. Let F_i and G_i be defined by*

$$F_i(t) = \int_0^t f_i(\tau)\,d\tau, \qquad G_i(t) = \int_0^t f_i^{-1}(\tau)\,d\tau \tag{3.1}$$

for each $t \geq 0$ *and* $1 \leq i \leq n$. *Suppose that* $(u_1,,\ldots,u_n) \in AC(a,b;X_1 \times \cdots \times X_n)$ *and that is a* $(F_1,\ldots,F_n)$*-curve of maximal slope for the functional* Φ *with respect to the strong upper gradient* g *on* $(a,b) \subset \mathbb{R}$. *Then one has the following.*

(i)

$$\begin{aligned}\Phi(u_1,\ldots,u_n)(s) - \Phi(u_1,\ldots,u_n)(t) \\ &= \int_s^t \sum_{i=1}^n \Big[F_i\Big(|u_i'|_{d_i}(\tau)\Big) + G_i(g_i(u_1,\ldots,u_n))(\tau)\Big]d\tau \\ &= \int_s^t \sum_{i=1}^n |u_i'|_{d_i}(\tau) g_i(u_1,\ldots,u_n)(\tau) d\tau\end{aligned} \tag{3.2}$$

for a.e. $a < s \leq t < b$.

(ii) $(u_1,\ldots,u_n)$ *satisfies the system of explicit nonlinear gradient flows of* Φ *with respect to the structure* $((f_1,\ldots,f_n),(g_1,\ldots,g_n),X_1 \times \cdots \times X_n)$

$$\begin{aligned} f_1\Big(|u_1'|_{d_1}(t)\Big) &= g_1(u_1(t),\ldots,u_n(t)), \\ f_2\Big(|u_2'|_{d_2}(t)\Big) &= g_2(u_1(t),\ldots,u_n(t)), \\ &\vdots \\ f_n\Big(|u_n'|_{d_n}(t)\Big) &= g_n(u_1(t),\ldots,u_n(t)), \end{aligned} \tag{3.3}$$

for a.e. $t \in (a,b)$.

(iii) *Assume that function* $\varphi_i(t) \equiv f_i(t) \cdot t$ *is convex and strictly increasing on* $[0,+\infty)$ *for each* $1 \leq i \leq n$, *then one has*

(a)

$$\sum_{i=1}^n \varphi_i\left(\frac{d_i(u_i(t),u_i(s))}{t-s}\right) \leq \frac{\Phi(u_1,\ldots,u_n)(s) - \Phi(u_1,\ldots,u_n)(t)}{t-s}, \tag{3.4}$$

for all $a < s < t < b$,

(b)

$$\sum_{i=1}^n \varphi_i\Big(|u_i'|_{d_i}(t)\Big) \leq (\Phi \circ (u_1,\ldots,u_n))'(t), \tag{3.5}$$

for a.e. $t \in (a,b)$.

(iv) *If* $f_i(t) = t$ *for* $1 \le i \le n$*, then* (ii) *can be expressed as*

$$\begin{aligned} |u_1'|_{d_1} &= g_1(u_1, \dots, u_n), \\ |u_2'|_{d_2} &= g_2(u_1, \dots, u_n), \\ &\vdots \\ |u_n'|_{d_n} &= g_n(u_1, \dots, u_n), \end{aligned} \tag{3.6}$$

which is the system of explicit linear gradient flows of Φ *with respect to the structure* $((f_1, \dots, f_n), (g_1, \dots, g_n), X_1 \times \cdots \times X_n)$.

Proof. According to the definitions of F_i and G_i, (F_i, G_i) is a pair of Young's functions. Recall we assume that $(u_1, \dots, u_n)$ is a $(F_1, \dots, F_n)$-curve of maximal slope for Φ with respect to g on (a, b). For a.e. $a < s < t < b$, we have

$$\begin{aligned} -\int_s^t (\Phi \circ (u_1, \dots, u_n))'(\tau) d\tau &\ge \int_s^t \sum_{i=1}^n \left[F_i\left(|u_i'|_{d_i}(\tau) \right) + G_i(g_i(u_1, \dots, u_n))(\tau) \right] d\tau \\ &\ge \int_s^t \sum_{i=1}^n |u_i'|_{d_i}(\tau) g_i(u_1, \dots, u_n)(\tau) d\tau \\ &\ge |\Phi \circ (u_1, \dots, u_n)(t) - \Phi \circ (u_1, \dots, u_n)(s)|. \end{aligned} \tag{3.7}$$

The last inequality holds due to the assumption of strong upper gradient for g. Thus we easily obtain the following formula:

$$\begin{aligned} &\int_s^t \sum_{i=1}^n \left[F_i\left(|u_i'|_{d_i}(\tau) \right) + G_i(g_i(u_1, \dots, u_n))(\tau) \right] d\tau \\ &\quad = \int_s^t \sum_{i=1}^n |u_i'|_{d_i}(\tau) g_i(u_1, \dots, u_n)(\tau) d\tau \\ &\quad = \Phi \circ (u_1, \dots, u_n)(s) - \Phi \circ (u_1, \dots, u_n)(t), \end{aligned} \tag{3.8}$$

for L^1 a.e. $a < s < t < b$. Using the Young's inequality and the Vanishing theorem (Lemma 2.9), we obtain

$$F_i\left(|u_i'|_{d_i}(\tau) \right) + G_i(g_i(u_1, \dots, u_n))(\tau) = |u_i'|_{d_i}(\tau) g_i(u_1, \dots, u_n)(\tau) \tag{3.9}$$

for a.e. $a < \tau < b$ and for $1 \le i \le n$. By Young's equality, we discover that the system of explicit gradient flows of Φ with respect to the structure $((f_1, \dots, f_n), g, X_1 \times \cdots \times X_n)$ holds.

We now prove assertion (iii). By using assertions (i), (ii), and the assumption for ψ_i, we see that

$$\sum_{i=1}^{n}\int_{s}^{t}\psi_i\left(\left|u_i'\right|_{d_i}(\tau)\right)d\tau = \Phi\circ(u_1,\dots,u_n)(s)-\Phi\circ(u_1,\dots,u_n)(t), \tag{3.10}$$

for a.e. $a<s<t<b$. Owing to the metric derivative $|u_i'|_{d_i}$ which is the smallest admissible function m_i satisfying

$$d_i(u_i(s),u_i(t)) \le \int_{s}^{t} m_i(\tau)d\tau \quad \forall a<s\le t<b \tag{3.11}$$

and ψ_i is convex and strictly increasing on $[0,+\infty)$, by using Jensen's inequality, we find

$$\begin{aligned}\sum_{i=1}^{n}\psi_i\left(\frac{d_i(u_i(t),u_i(s))}{t-s}\right) &\le \sum_{i=1}^{n}\psi_i\left(\frac{1}{t-s}\int_{s}^{t}\left|u_i'\right|_{d_i}(\tau)d\tau\right)\\ &\le \frac{1}{t-s}\sum_{i=1}^{n}\int_{s}^{t}\psi_i\left(\left|u_i'\right|_{d_i}(\tau)\right)d\tau\\ &= \frac{\Phi\circ(u_1,\dots,u_n)(s)-\Phi\circ(u_1,\dots,u_n)(t)}{t-s}.\end{aligned} \tag{3.12}$$

This completes the proof of (a). By passing to the limit $s\to t$ in (a), we deduce that (b) holds. □

In our second main result, we study and establish the abstract structure of "Gamma-convergence of gradient flow systems on metric spaces" which is a nonlinear system edition of the notion (Proposition 1.1) and which can be applied to the problems involving a system of nonlinear gradient flows on metric spaces.

Theorem 3.2 (Gamma-convergence of systems of gradient flows on metric spaces). *Let $(X_{i,\varepsilon}, d_{i,\varepsilon})$ and (X_i,d_i) be complete metric spaces for all $1\le i\le n$ and $\varepsilon>0$. Let Φ_ε and Φ be functionals defined on spaces $X_{1,\varepsilon}\times\cdots\times X_{n,\varepsilon}$ and $X_1\times\cdots\times X_n$, respectively. Suppose that the* Γ-lim inf *convergence of Φ_ε to Φ holds. Let $g_\varepsilon=(g_{1,\varepsilon},\dots,g_{n,\varepsilon})$ and $g=(g_1,\dots,g_n)$ be strong upper gradients of Φ_ε and Φ, respectively. Let (F_i^*,G_i^*) be a pair of Young's functions having continuous, strictly increasing and surjective derivative $(F_i^*)'(=f_i^*)$ for $i=1,\dots,n$. Assume in addition the following relations.*

(1) *Lower bound on the metric derivatives: if $(u_{1,\varepsilon},\dots,u_{n,\varepsilon})(t)\overset{\mathcal{S}}{\rightharpoonup}(u_1,\dots,u_n)(t)$, for $t\in[0,T)$ then for all $s\in[0,T)$*

$$\liminf_{\varepsilon\to0}\left|u_{i,\varepsilon}'\right|_{d_{i,\varepsilon}}(s)\ge\left|u_i'\right|_{d_i}(s),\quad \textit{for } i=1,\dots,n. \tag{3.13}$$

(2) *Lower bound on the strong upper gradients: If $(u_{1,\varepsilon},\dots,u_{n,\varepsilon})\overset{\mathcal{S}}{\rightharpoonup}(u_1,\dots,u_n)$, then*

$$\liminf_{\varepsilon\to0} g_{i,\varepsilon}(u_{1,\varepsilon},\dots,u_{n,\varepsilon})\ge g_i(u_1,\dots,u_n)\quad \textit{for } i=1,\dots,n. \tag{3.14}$$

Let $(u_{1,\varepsilon},\ldots,u_{n,\varepsilon})$ *be a* $(F_1^*,\ldots,F_n^*)$*-curve of maximal slope on* $(0,T)$ *for* Φ_ε *with respect to* $g_\varepsilon = (g_{1,\varepsilon},\ldots,g_{n,\varepsilon})$ *such that* $(u_{1,\varepsilon},\ldots,u_{n,\varepsilon})(t) \xrightarrow{\mathcal{S}} (u_1,\ldots,u_n)(t)$*, which is well-prepared in the sense that*

$$\lim_{\varepsilon\to 0}\Phi_\varepsilon(u_{1,\varepsilon},\ldots,u_{n,\varepsilon})(0) = \Phi(u_1,\ldots,u_n)(0). \tag{3.15}$$

Then

(i) $(u_1,\ldots,u_n)$ *is a* $(F_1^*,\ldots,F_n^*)$*-curve of maximal slope on* $(0,T)$ *for* Φ *with respect to* $g = (g_1,\ldots,g_n)$*,*

(ii)

$$\lim_{\varepsilon\to 0}\Phi_\varepsilon(u_{1,\varepsilon},\ldots,u_{n,\varepsilon})(t) = \Phi(u_1,\ldots,u_n)(t) \quad \forall t\in[0,T), \tag{3.16}$$

(iii)

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^s\sum_{i=1}^n\left[F_i^*\left(\left|u'_{i,\varepsilon}\right|_{d_{i,\varepsilon}}(t)\right)+G_i^*\left(f_i^*\left(\left|u'_{i,\varepsilon}\right|_{d_{i,\varepsilon}}(t)\right)\right)\right]dt\\
&\quad=\int_0^s\sum_{i=1}^n\left[F_i^*\left(|u'_i|_{d_i}(t)\right)+G_i^*\left(f_i^*\left(|u'_i|_{d_i}(t)\right)\right)\right]dt,
\end{aligned} \tag{3.17}$$

(iv)

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^s\sum_{i=1}^n\left[F_i^*\left(g_i^*\left(g_{i,\varepsilon}(u_{1,\varepsilon},\ldots,u_{n,\varepsilon})(t)\right)\right)+G_i^*\left(g_{i,\varepsilon}(u_{1,\varepsilon},\ldots,u_{n,\varepsilon})(t)\right)\right]dt\\
&\quad=\int_0^s\sum_{i=1}^n\left[F_i^*\left(g_i^*\left(g_i(u_1,\ldots,u_n)(t)\right)\right)+G_i^*\left(g_i(u_1,\ldots,u_n)(t)\right)\right]dt,
\end{aligned} \tag{3.18}$$

where $g_i^* = (f_i^*)^{-1}$*.*

Proof. Owing to the fact that $(u_{1,\varepsilon},\ldots,u_{n,\varepsilon})$ is a $(F_1^*,\ldots,F_n^*)$-curve of maximal slope for Φ_ε with respect to g_ε on $[0,T)$, we recall that Theorem 3.1-(i) and (ii) yields

$$\begin{aligned}
&\Phi_\varepsilon(u_{1,\varepsilon},\ldots,u_{n,\varepsilon})(0)-\Phi_\varepsilon(u_{1,\varepsilon},\ldots,u_{n,\varepsilon})(s)\\
&\quad=\int_0^s\sum_{i=1}^n\left[F_i^*\left(\left|u'_{i,\varepsilon}\right|_{d_{i,\varepsilon}}(t)\right)+G_i^*\left(g_{i,\varepsilon}(u_{1,\varepsilon},\ldots,u_{n,\varepsilon})(t)\right)\right]dt\\
&\quad=\int_0^s\sum_{i=1}^n\left|u'_{i,\varepsilon}\right|_{d_{i,\varepsilon}}(t)\,g_{i,\varepsilon}(u_{1,\varepsilon},\ldots,u_{n,\varepsilon})(t)\,dt,
\end{aligned} \tag{3.19}$$

for a.e. $0 < s < T$;

$$f_i^*\left(\left|u'_{i,\varepsilon}\right|_{d_{i,\varepsilon}}(t)\right) = g_{i,\varepsilon}(u_{1,\varepsilon},\dots,u_{n,\varepsilon})(t), \tag{3.20}$$

for a.e. $t \in [0,T)$ and for $1 \le i \le n$.

Passing to the $\liminf_{\varepsilon\to 0}$ to (3.19) and applying Fatou's lemma, we deduce that

$$\begin{aligned}
&\liminf_{\varepsilon\to 0}\,[\Phi_\varepsilon(u_{1,\varepsilon},\dots,u_{n,\varepsilon})(0) - \Phi_\varepsilon(u_{1,\varepsilon},\dots,u_{n,\varepsilon})(s)]\\
&\quad\ge \sum_{i=1}^n\left[\liminf_{\varepsilon\to 0}\int_0^s F_i^*\left(\left|u'_{i,\varepsilon}\right|_{d_{i,\varepsilon}}(t)\right)dt + \liminf_{\varepsilon\to 0}\int_0^s G_i^*(g_{i,\varepsilon}(u_{1,\varepsilon},\dots,u_{n,\varepsilon})(t))dt\right]\\
&\quad\ge \sum_{i=1}^n\left[\int_0^s \liminf_{\varepsilon\to 0} F_i^*\left(\left|u'_{i,\varepsilon}\right|_{d_{i,\varepsilon}}(t)\right)dt + \int_0^s \liminf_{\varepsilon\to 0} G_i^*(g_{i,\varepsilon}(u_{1,\varepsilon},\dots,u_{n,\varepsilon})(t))dt\right]\\
&\quad\ge \sum_{i=1}^n\left[\int_0^s F_i^*\left(\left|u'_i\right|_{d_i}(t)\right)dt + \int_0^s G_i^*(g_i(u_1,\dots,u_n)(t))dt\right].
\end{aligned} \tag{3.21}$$

The last inequality is achieved by the assumptions of (1) and (2) as well as Lemma 2.8.

Using the fact that each (F_i^*, G_i^*) is a pair of Young's functions (hence Young's inequality holds), (3.21), and the strong upper gradient assumption of g for Φ, we can check that

$$\begin{aligned}
&\liminf_{\varepsilon\to 0}\,[\Phi_\varepsilon(u_{1,\varepsilon},\dots,u_{n,\varepsilon})(0) - \Phi_\varepsilon(u_{1,\varepsilon},\dots,u_{n,\varepsilon})(s)]\\
&\quad\ge \sum_{i=1}^n\left[\int_0^s F_i^*\left(\left|u'_i\right|_{d_i}(t)\right)dt + \int_0^s G_i^*(g_i(u_1,\dots,u_n)(t))dt\right]\\
&\quad\ge \int_0^s \sum_{i=1}^n \left|u'_i\right|_{d_i}(t) g_i(u_1,\dots,u_n)(t)dt\\
&\quad\ge \Phi(u_1,\dots,u_n)(0) - \Phi(u_1,\dots,u_n)(s).
\end{aligned} \tag{3.22}$$

We recall that $(u_{1,\varepsilon},\dots,u_{n,\varepsilon})$ is well prepared, and using (3.22), we can deduce that

$$\limsup_{\varepsilon\to 0}\Phi_\varepsilon(u_{1,\varepsilon},\dots,u_{n,\varepsilon})(s) \le \Phi(u_1,\dots,u_n)(s). \tag{3.23}$$

Using the Γ-liminf convergence of Φ_ε to Φ and (3.23), we obtain

$$\lim_{\varepsilon\to 0}\Phi_\varepsilon(u_{1,\varepsilon},\dots,u_{n,\varepsilon})(s) = \Phi(u_1,\dots,u_n)(s), \quad \forall s \in [0,T). \tag{3.24}$$

Combining (3.24) with (3.22), we can now conclude that, for each $s \in [0,T)$,

$$\begin{aligned}
&\int_0^s \sum_{i=1}^n \left[F_i^*\left(|u_i'|_{d_i}(t)\right) + G_i^*(g_i(u_1,\ldots,u_n)(t))\right]dt \\
&= \int_0^s \sum_{i=1}^n |u_i'|_{d_i}(t) g_i(u_1,\ldots,u_n)(t)dt \\
&= \Phi(u_1,\ldots,u_n)(0) - \Phi(u_1,\ldots,u_n)(s).
\end{aligned} \tag{3.25}$$

Using the Young's inequality and the Vanishing theorem (Lemma 2.9) again, we conclude that, for a.e. $t \in [0,T)$,

$$F_i^*\left(|u_i'|_{d_i}(t)\right) + G_i^*(g_i(u_1,\ldots,u_n)(t)) = |u_i'|_{d_i}(t) g_i(u_1,\ldots,u_n)(t), \tag{3.26}$$

for all $1 \leq i \leq n$. Moreover, by Young's equality, we have, for a.e. $t \in [0,T)$,

$$f_i^*\left(|u_i'|_{d_i}(t)\right) = g_i(u_1,\ldots,u_n)(t), \tag{3.27}$$

for each $1 \leq i \leq n$.

Next, differentiating formulas (3.25) with respect to variable s, we see that $(u_1,\ldots,u_n)$ is a $(F_1^*,\ldots,F_n^*)$-curve of maximal slope for Φ with respect to g on $[0,T)$. Using formula (3.19), (3.24), and (3.25) we can check that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_0^s \sum_{i=1}^n \left[\left(F_i^* \left|u_{i,\varepsilon}'\right|_{d_{i,\varepsilon}}(t)\right) + G_i^*(g_{i,\varepsilon}(u_{1,\varepsilon},\ldots,u_{n,\varepsilon})(t))\right]dt \\
&\quad= \int_0^s \sum_{i=1}^n \left[F_i^*\left(|u_i'|_{d_i}(t)\right) + G_i^*(g_i(u_1,\ldots,u_n)(t))\right]dt,
\end{aligned} \tag{3.28}$$

for all $s \in [0,T)$. Finally, we recall formulas (3.20) and (3.27), and using (3.28), we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_0^s \sum_{i=1}^n \left[F_i^*\left(\left|u_{i,\varepsilon}'\right|_{d_{i,\varepsilon}}(t)\right) + G_i^*\left(f_i^*\left(\left|u_{i,\varepsilon}'\right|_{d_{i,\varepsilon}}(t)\right)\right)\right]dt \\
&\quad= \int_0^s \sum_{i=1}^n \left[F_i^*\left(|u_i'|_{d_i}(t)\right) + G_i^*\left(f_i^*\left(|u_i'|_{d_i}(t)\right)\right)\right]dt, \\
&\lim_{\varepsilon\to 0} \int_0^s \sum_{i=1}^n [F_i^*(g_i^*(g_{i,\varepsilon}(u_{1,\varepsilon},\ldots,u_{n,\varepsilon})(t))) + G_i^*(g_{i,\varepsilon}(u_{1,\varepsilon},\ldots,u_{n,\varepsilon})(t))]dt \\
&\quad= \int_0^s \sum_{i=1}^n [F_i^*(g_i^*(g_i(\mathrm{u}_1,\ldots,u_n)(t))) + G_i^*(g_i(u_1,\ldots,u_n)(t))]dt
\end{aligned} \tag{3.29}$$

and complete the proof of Theorem 3.2. □

4. Examples

In this section, we present two examples to illustrate a special case of our main results.

Example 4.1. Let $p > 1$. If $f_i^*(t) = t^{p-1}$ for each $1 \le i \le n$, then $(f_i^*)^{-1}(t) = t^{q-1}$ for $i = 1, 2, \ldots, n$, where $(1/p) + (1/q) = 1$. Hence

$$F_i^*(t) = \frac{t^p}{p}, \qquad G_i^*(t) = \frac{t^q}{q} \quad \text{for each } 1 \le i \le n. \tag{4.1}$$

In this case, Theorem 3.1 can be expressed as following.

(i) One has

$$\begin{aligned}\Phi(u_1, \ldots, u_n)(s) - \Phi(u_1, \ldots, u_n)(t) &= \int_s^t \sum_{i=1}^n \left[\frac{|u_i'|_{d_i}^p(\tau)}{p} + \frac{g_i^q(u_1, \ldots, u_n)(\tau)}{q} \right] d\tau \\ &= \int_s^t \sum_{i=1}^n |u_i'|_{d_i}(\tau) g_i(u_1, \ldots, u_n)(\tau) d\tau.\end{aligned} \tag{4.2}$$

(ii) The system of explicit nonlinear gradient flows of Φ with respect to the structure $((f_1^*, \ldots, f_n^*), (g_1, \ldots, g_n), X_1 \times \cdots \times X_n)$ is

$$|u_i'|_{d_i}^{p-1}(t) = g_i(u_1(t), \ldots, u_n(t)), \quad \text{for } i = 1, 2, \ldots, n. \tag{4.3}$$

(iii) $\psi_i(t) = t^p$ for $i = 1, \ldots, n$ (ψ_i is convex and strictly increasing on $[0, +\infty)$ for each $1 \le i \le n$). Then

(a)

$$\sum_{i=1}^n \left(\frac{d_i(u_i(t), u_i(s))}{t - s} \right)^p \le \frac{\Phi(u_1, \ldots, u_n)(s) - \Phi(u_1, \ldots, u_n)(t)}{t - s} \tag{4.4}$$

for all $a < s < t < b$,

(b)

$$\sum_{i=1}^n \left(|u_i'|_{d_i}(t) \right)^p \le (\Phi \circ (u_1, \ldots, u_n))'(t) \quad \text{for a.e. } t \in (a, b). \tag{4.5}$$

Under the hypotheses of Theorem 3.2, we have

(i)

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^s\sum_{i=1}^n\left[\frac{\left(\left|u'_{i,\varepsilon}\right|_{d_{i,\varepsilon}}(t)\right)^p}{p}+\frac{\left(\left|u'_{i,\varepsilon}\right|_{d_{i,\varepsilon}}^{p-1}(t)\right)^q}{q}\right]dt\\=\int_0^s\sum_{i=1}^n\left[\frac{\left(\left|u'_i\right|_{d_i}(t)\right)^p}{p}+\frac{\left(\left|u'_i\right|_{d_i}^{p-1}(t)\right)^q}{q}\right]dt,\end{aligned}\tag{4.6}$$

that is, $\lim_{\varepsilon\to 0}\int_0^s\sum_{i=1}^n|u'_{i,\varepsilon}|_{d_{i,\varepsilon}}^p(t)dt=\int_0^s\sum_{i=1}^n|u'_i|_{d_i}^p(t)dt$.

(ii)

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^s\sum_{i=1}^n\left[\frac{\left((g_{i,\varepsilon}(u_{1,\varepsilon},\dots,u_{n,\varepsilon})(t))^{q-1}\right)^p}{p}+\frac{(g_{i,\varepsilon}(u_{1,\varepsilon},\dots,u_{n,\varepsilon})(t))^q}{q}\right]dt\\=\int_0^s\sum_{i=1}^n\left[\frac{\left((g_i(u_1,\dots,u_n)(t))^{q-1}\right)^p}{p}+\frac{(g_i(u_1,\dots,u_n)(t))^q}{q}\right]dt,\end{aligned}\tag{4.7}$$

that is, $\lim_{\varepsilon\to 0}\int_0^s\sum_{i=1}^n(g_{i,\varepsilon}(u_{1,\varepsilon},\dots,u_{n,\varepsilon})(t))^q dt=\int_0^s\sum_{i=1}^n(g_i(u_1,\dots,u_n)(t))^q dt$.

Example 4.2. Considering the case $p=2$ in Example 4.1, we have

(i)

$$\begin{aligned}\Phi(u_1,\dots,u_n)(s)-\Phi(u_1,\dots,u_n)(t)&=\int_s^t\sum_{i=1}^n\left[\frac{|u'_i|_{d_i}^2(\tau)}{2}+\frac{(g_i(u_1,\dots,u_n)(\tau))^2}{2}\right]d\tau\\&=\int_s^t\sum_{i=1}^n|u'_i|_{d_i}(\tau)g_i(u_1,\dots,u_n)(\tau)d\tau.\end{aligned}\tag{4.8}$$

(ii) The system of explicit linear gradient flows of Φ with respect to the structure is

$$\begin{aligned}|u'_1|_{d_1}(t)&=g_1(u_1(t),\dots,u_n(t))\\|u'_2|_{d_2}(t)&=g_2(u_1(t),\dots,u_n(t))\\&\vdots\\|u'_n|_{d_n}(t)&=g_n(u_1(t),\dots,u_n(t)).\end{aligned}\tag{4.9}$$

(iii)

$$\sum_{i=1}^{n}\left(\frac{d_i(u_i(t),u_i(s))}{t-s}\right)^2 \le \frac{\Phi(u_1,\ldots,u_n)(s)-\Phi(u_1,\ldots,u_n)(t)}{t-s} \tag{4.10}$$

for all $a < s < t < b$. Moreover, we have

$$\sum_{i=1}^{n}\left|u_i'\right|_{d_i}^2(t) \le (\Phi\circ(u_1,\ldots,u_n))'(t) \quad \text{for a.e. } t\in(a,b). \tag{4.11}$$

(iv)

$$\lim_{\varepsilon\to 0}\int_0^s\sum_{i=1}^{n}\left|u_{i,\varepsilon}'\right|_{d_{i,\varepsilon}}^2(t)dt = \int_0^s\sum_{i=1}^{n}\left|u_i'\right|_{d_i}^2(t)dt. \tag{4.12}$$

(v)

$$\lim_{\varepsilon\to 0}\int_0^s\sum_{i=1}^{n}\left(g_{i,\varepsilon}(u_{1,\varepsilon},\ldots,u_{n,\varepsilon})(t)\right)^2dt = \int_0^s\sum_{i=1}^{n}\left(g_i(u_1,\ldots,u_n)(t)\right)^2dt. \tag{4.13}$$

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References

[1] E. De Giorgi, A. Marino, and M. Tosques, "Problems of evolution in metric spaces and maximal decreasing curve," *Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali. Serie VIII*, vol. 68, no. 3, pp. 180–187, 1980.

[2] M. Degiovanni, A. Marino, and M. Tosques, "Evolution equations with lack of convexity," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 9, no. 12, pp. 1401–1443, 1985.

[3] A. Marino, C. Saccon, and M. Tosques, "Curves of maximal slope and parabolic variational inequalities on nonconvex constraints," *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie IV*, vol. 16, no. 2, pp. 281–330, 1989.

[4] L. Ambrosio, "Minimizing movements," *Accademia Nazionale delle Scienze detta dei XL. Rendiconti. V.*, vol. 19, pp. 191–246, 1995.

[5] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, Lectures in Mathematics ETH Zürich, Birkhäuser, Basel, Switzerland, 2nd edition, 2008.

[6] E. Sandier and S. Serfaty, "Gamma-convergence of gradient flows with applications to Ginzburg-Landau," *Communications on Pure and Applied Mathematics*, vol. 57, no. 12, pp. 1627–1672, 2004.

[7] N. Q. Le, "A Gamma-convergence approach to the Cahn-Hilliard equation," *Calculus of Variations and Partial Differential Equations*, vol. 32, no. 4, pp. 499–522, 2008.

[8] N. Q. Le, "On the convergence of the Ohta-Kawasaki equation to motion by nonlocal Mullins-Sekerka law," *SIAM Journal on Mathematical Analysis*, vol. 42, no. 4, pp. 1602–1638, 2010.

[9] S. Serfaty, "Gamma-convergence of gradient flows on Hilbert and metric spaces and applications," *Discrete and Continuous Dynamical Systems A*, vol. 31, no. 4, pp. 1427–1451, 2011.

[10] R. L. Wheeden and A. Zygmund, *Measure and Integral*, Marcel Dekker, New York, NY, USA, 1977.