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Research Article

Regularity of Global Attractor for the Reaction-Diffusion Equation

Hong Luo

College of Mathematics and Software Science, Sichuan Normal University, Sichuan, Chengdu 610066, China

Correspondence should be addressed to Hong Luo, lhscnu@hotmail.com

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By using an iteration procedure, regularity estimates for the linear semigroups, and a classical existence theorem of global attractor, we prove that the reaction-diffusion equation possesses a global attractor in Sobolev space H^k for all $k > 0$, which attracts any bounded subset of $H^k(\Omega)$ in the H^k-norm.

1. Introduction

This paper is concerned with the following initial-boundary problem of reaction-diffusion systems involving an unknown function $u = u(x,t)$:

$$\begin{aligned} &\frac{\partial u}{\partial t} = a\Delta u - g(u) \quad \text{in } \Omega \times (0,\infty), \\ &u = 0, \quad \text{in } \partial\Omega \times (0,\infty), \\ &u(x,0) = \varphi, \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $a > 0$ is a given constant. Δ is the Laplace operator. Ω denotes an open bounded set of R^n $(n = 1,2,3)$ with smooth boundary $\partial\Omega$. $g(s)$ is a polynomial on $s \in R^1$, which is given by

$$g(s) = \sum_{k=1}^{p} a_k s^k, \tag{1.2}$$

where p should be an odd number, that is, $p = 2m + 1$ $(m \geq 1)$, and

$$a_p > 0. \tag{1.3}$$

The reaction-diffusion systems (1.1) have been extensively studied during the last decades, one of the motivations being that such systems could account for phenomena occurring in living organisms. In 1952, Turing [1] proposed that a combination of chemical reaction and diffusion produces spatial patterns of chemical concentration, under certain conditions. Such patterns are of interest because they give a possible explanation for the development of pattern and form in developmental biology [2–4] and experimental chemical systems [5]. Schneider et al. [6–9] have studied existence of periodic travelling wave solutions and positive periodic solution of reaction-diffusion systems. In [10–17], asymptotic behaviour of the nonlinear reaction-diffusion equation, such as global attractors, inertial manifolds, and approximate inertial manifolds, has been studied.

The global asymptotical behaviors of solutions and existence of global attractors are important for the study of the dynamical properties of general nonlinear dissipative dynamical systems. So, many authors are interested in the existence of global attractors such as [12–19]. As for the reaction-diffusion equation (1.1), the existence of global solutions and global attractors in $L^2(\Omega)$ has been proved by Temam [16], Marion [18, 19], and Zhong et al. [17]. For convenience, we introduce the main results as follows.

Lemma 1.1. *Under the conditions* (1.2) *and* (1.3) *for* $\varphi \in H$, *the following three claims hold.*

(1) *Equation* (1.1) *has a unique global weak solution* $u \in C((0,\infty); H) \bigcap L^2((0,T); H_{1/2})$, *for* $T > 0$;

(2) *Equation* (1.1) *has a unique strong solution* $u \in C([0,T); H_{1/2}) \bigcap L^2((0,T); H_1)$ *for any* $T > 0$;

(3) *Equation* (1.1) *has a global attractor* $\mathcal{A} \subset H$, *which attracts any bounded set of* H *in the* H*-norm.*

Here the spaces H, $H_{1/2}$ *and* H_1 *are defined as follows:*

$$H = L^2(\Omega), \qquad H_{1/2} = H_0^1(\Omega), \qquad H_1 = H_0^1(\Omega) \bigcap H^2(\Omega). \tag{1.4}$$

In this paper, we shall use the regularity estimates for the linear semigroups, combining with the classical existence theorem of global attractors, to prove that the reaction-diffusion equation possesses, in any kth differentiable function spaces $H^k(\Omega)$, a global attractor, which attracts any bounded set of $H^k(\Omega)$ in H^k-norm. The basic idea is an iteration procedure, which is from recent books and papers [20–24].

2. Preliminaries

Let X and X_1 be two Banach spaces and $X_1 \subset X$ a compact and dense inclusion. Consider the abstract nonlinear evolution equation defined on X, given by

$$\begin{aligned} \frac{du}{dt} &= Lu + G(u), \\ u(x,0) &= u_0, \end{aligned} \tag{2.1}$$

where $u(t)$ is an unknown function, $L : X_1 \to X$ a linear operator, and $G : X_1 \to X$ a nonlinear operator.

A family of operators $S(t) : X \to X$ $(t \geq 0)$ is called a semigroup generated by (2.1) provided that $S(t)$ satisfies the properties:

(1) $S(t) : X \to X$ is a continuous mapping for any $t \geq 0$;

(2) $S(0) = \mathrm{id} : X \to X$ the identity;

(3) $S(t+s) = S(t) \cdot S(s)$ for all $t, s \geq 0$, and the solution of (2.1) can be expressed as

$$u(t, u_0) = S(t)u_0. \tag{2.2}$$

Next, we introduce the concepts and definitions of invariant sets, global attractors, ω-limit sets for the semigroup $S(t)$.

Definition 2.1. Let $S(t)$ be a semigroup defined on X. A set $\Sigma \subset X$ is called an invariant set of $S(t)$ if $S(t)\Sigma = \Sigma$ for all $t \geq 0$. An invariant set Σ is an attractor of $S(t)$ if Σ is compact, and there exists a neighborhood $U \subset X$ of Σ such that for any $u_0 \in U$:

$$\inf_{v \in \Sigma} \|S(t)u_0 - v\|_X \longrightarrow 0, \quad \text{as } t \longrightarrow \infty. \tag{2.3}$$

In this case, we say that Σ attracts U. Especially, if Σ attracts any bounded set of X, Σ is called a global attractor of $S(t)$.

For a set $D \subset X$, we define the ω-limit set of D as follows:

$$\omega(D) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)D}, \tag{2.4}$$

where the closure is taken in the X-norm. The following Lemma 2.2 is the classical existence theorem of global attractor by Temam [16].

Lemma 2.2. *Let $S(t) : X \to X$ be the semigroup generated by* (2.1). *Assume that the following conditions hold:*

(1) *$S(t)$ has a bounded absorbing set $B \subset X$, that is, for any bounded set $A \subset X$ there exists a time $t_A \geq 0$ such that $S(t)u_0 \in B$ for all $u_0 \subset A$ and $t > t_A$;*

(2) *$S(t)$ is uniformly compact, that is, for any bounded set $U \subset X$ and some $T > 0$ sufficiently large, the set $\overline{\bigcup_{t \geq T} S(t)U}$ is compact in X.*

Then, that the ω-limit set $\mathcal{A} = \omega(B)$ of B is a global attractor of (2.1), *and $\mathcal{A}$ is connected providing B is connected.*

Note that we used to assume that the linear operator L in (2.1) *is a sectorial operator which generates an analytic semigroup e^{tL}. It is known that there exists a constant $\lambda \geq 0$ such that $L - \lambda I$ generates the fractional power operators $\mathcal{L}^\alpha$ and fractional order spaces X_α for $\alpha \in R^1$, where $\mathcal{L} = -(L - \lambda I)$. Without loss of generality, we assume that L generates the fractional power operators $\mathcal{L}^\alpha$ and fractional order spaces X_α as follows:*

$$\mathcal{L}^\alpha = (-L)^\alpha : X_\alpha \longrightarrow X, \quad \alpha \in R^1, \tag{2.5}$$

where $X_\alpha = D(\mathcal{L}^\alpha)$ is the domain of $\mathcal{L}^\alpha$. By the semigroup theory of linear operators (Pazy [25]), we know that $X_\beta \subset X_\alpha$ is a compact inclusion for any $\beta > \alpha$.

Thus, Lemma 2.2 can be equivalently expressed in the following Lemma 2.3 [24].

Lemma 2.3. *Let $u(t, u_0) = S(t)u_0$ ($u_0 \in X$, $t \geq 0$) be a solution of* (2.1) *and $S(t)$ the semigroup generated by* (2.1). *Let X_α be the fractional order space generated by L. Assume*

(1) *for some $\alpha \geq 0$, there is a bounded set $B \subset X_\alpha$; for any $u_0 \in X_\alpha$, there exists $t_{u_0} > 0$ such that*

$$u(t, u_0) \in B, \quad \forall t > t_{u_0}; \tag{2.6}$$

(2) *there is a $\beta > \alpha$, for any bounded set $U \subset X_\beta$ there are $T > 0$ and $C > 0$ such that*

$$\|u(t, u_0)\|_{X_\beta} \leq C, \quad \forall t > T, \ u_0 \in U. \tag{2.7}$$

Then (2.1) *has a global attractor $\mathcal{A} \subset X_\alpha$, which attracts any bounded set of X_α in the X_α-norm.*

For sectorial operators, we also have the following properties, which can be found in [25].

Lemma 2.4. *Let $L : X_1 \to X$ be a sectorial operator which generates an analytic semigroup $T(t) = e^{tL}$. If all eigenvalues λ of L satisfy $\operatorname{Re}\lambda < -\lambda_0$ for some real number $\lambda_0 > 0$, then for $\mathcal{L}^\alpha$ ($\mathcal{L} = -L$) we have*

(1) *$T(t) : X \to X_\alpha$ is bounded for all $\alpha \in R^1$ and $t > 0$;*

(2) *$T(t)\mathcal{L}^\alpha x = \mathcal{L}^\alpha T(t)x$ for all $x \in X_\alpha$;*

(3) *for each $t > 0$, $\mathcal{L}^\alpha T(t) : X \to X$ is bounded, and*

$$\|\mathcal{L}^\alpha T(t)\| \leq C_\alpha t^{-\alpha} e^{-\delta t}, \tag{2.8}$$

where some $\delta > 0$, $C_\alpha > 0$ is a constant only depending on α;

(4) *the X_α-norm can be defined by*

$$\|x\|_{X_\alpha} = \|\mathcal{L}^\alpha x\|_X; \tag{2.9}$$

(5) *if $\mathcal{L}$ is symmetric, for any $\alpha, \beta \in R^1$, we have*

$$\langle \mathcal{L}^{\alpha} u, v \rangle_H = \left\langle \mathcal{L}^{\alpha-\beta} u, \mathcal{L}^{\beta} v \right\rangle_H. \tag{2.10}$$

3. Main Results

Let H and H_1 be the spaces defined as in (1.4). We define the operators $L : H_1 \to H$ and $G : H_1 \to H$ by

$$\begin{aligned} Lu &= a\Delta u, \\ Gu &= -g(u), \end{aligned} \tag{3.1}$$

where $g(u)$ is the same as one of (1.2). Thus, the reaction-diffusion equation (1.1) can be written in the abstract form (2.1). It is well known that the linear operator $L : H_1 \to H$ given by (3.1) is a sectorial operator and $\mathcal{L} = -L$. The space H_1 is the same as (1.4), $H_{1/2}$ is given by $H_{1/2}$ = closure of H_1 in $H^1(\Omega)$, and $H_k = H^{2k}(\Omega) \cap H_1$ for $k \geq 1$.

The main result in this paper is given by the following theorem, which provides the existence of global attractors of the reaction-diffusion equation (1.1) in any kth order space H_k.

Theorem 3.1. *Let the function g be a polynomial of order p*

$$g(u) = \sum_{k=1}^{p} a_k u^k, \qquad p = 2m + 1 \ (m \geq 1,\ m \in N), \tag{3.2}$$

with leading coefficient

$$a_p > 0. \tag{3.3}$$

Assume $p = 3$ for $n = 3$. Then, for any $\alpha \geq 0$ (1.1) has a global attractor $\mathcal{A}$ in H_α, and $\mathcal{A}$ attracts any bounded set of H_α in the H_α-norm.

Proof. From Lemma 1.1, we know that the solution of system (1.1) is a weak solution for any $\varphi \in H$. Hence, the solution $u(t, \varphi)$ of system (1.1) can be written as

$$u(t, \varphi) = e^{tL}\varphi + \int_0^t e^{(t-\tau)L} G(u) d\tau. \tag{3.4}$$

By (3.1), we rewrite (3.4) as

$$u(t, \varphi) = e^{tL}\varphi - \int_0^t e^{(t-\tau)L} g(u) d\tau. \tag{3.5}$$

Next, according to Lemma 2.3, we prove Theorem 3.1 in the following six steps.

Step 1. We prove that for any bounded set $U \subset H_{1/2}$ there is a constant $C > 0$ such that the solution $u(t,\varphi)$ of system (1.1) is uniformly bounded by the constant C for any $\varphi \in U$ and $t \geq 0$. To do that, we firstly check that system (1.1) has a global Lyapunov function as follows:

$$F(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + f(u)\right) dx, \tag{3.6}$$

where

$$f(z) = \int_0^z g(z)dz = \sum_{k=1}^{p} \frac{1}{k+1} a_k z^{k+1}. \tag{3.7}$$

In fact, if $u(t,\cdot)$ is a weak solution of system (1.1), we have

$$\frac{d}{dt} F(u(t,\varphi)) = \left\langle DF(u), \frac{du}{dt} \right\rangle_H. \tag{3.8}$$

By (3.1) and (3.6), we get

$$\frac{du}{dt} = Lu + G(u) = -DF(u). \tag{3.9}$$

Hence, it follows from (3.8) and (3.9) that

$$\frac{dF(u)}{dt} = \langle DF(u), -DF(u)\rangle_H = -\|DF(u)\|_H^2, \tag{3.10}$$

which implies that (3.6) is a Lyapunov function.

Integrating (3.10) from 0 to t gives

$$F(u(t,\varphi)) = -\int_0^t \|DF(u)\|_H^2 dt + F(\varphi). \tag{3.11}$$

Using (1.2) and (3.6), we have

$$\begin{aligned} F(u) &= \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + f(u)\right) dx = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{p+1} a_p u^{p+1} + \sum_{k=1}^{p-1} \frac{1}{k+1} a_k u^{k+1}\right) dx \\ &\geq \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{p+1} a_p u^{p+1} - \sum_{k=1}^{p-1} \frac{1}{k+1} |a_k||u|^{k+1}\right) dx \end{aligned}$$

$$\geq \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{p+1} a_p u^{p+1} - \sum_{k=1}^{p-1} \frac{1}{k+1} |a_k| \left(\varepsilon |u|^{p+1} + \varepsilon^{-(k+1)/(p-k)} \right) \right) dx$$
$$= \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{p+1} a_p u^{p+1} - \varepsilon \left(\sum_{k=1}^{p-1} \frac{1}{k+1} |a_k| \right) |u|^{p+1} - \sum_{k=1}^{p-1} \frac{1}{k+1} |a_k| \varepsilon^{-(k+1)/(p-k)} \right) dx. \tag{3.12}$$

Choosing ε such that $\varepsilon(\sum_{k=1}^{p-1}(1/(k+1))|a_k|) = (1/2(p+1))a_p$, and noting that p is an odd number, that is, $p = 2m+1$ $(m \geq 1)$, we get

$$F(u) \geq \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{p+1} a_p u^{p+1} - \frac{1}{2(p+1)} a_p |u|^{p+1} - \sum_{k=1}^{p-1} \frac{1}{k+1} |a_k| \varepsilon^{-(k+1)/(p-k)} \right) dx$$
$$= \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2(p+1)} a_p |u|^{p+1} - \sum_{k=1}^{p-1} \frac{1}{k+1} |a_k| \varepsilon^{-(k+1)/(p-k)} \right) dx$$
$$\geq C_1 \int_\Omega \left(|\nabla u|^2 + |u|^{p+1} \right) dx - C_2. \tag{3.13}$$

Combining with (3.11) yields

$$C_1 \int_\Omega \left(|\nabla u|^2 + |u|^{p+1} \right) dx - C_2 \leq - \int_0^t \|DF(u)\|_H^2 dt + F(\varphi),$$
$$C_1 \int_\Omega \left(|\nabla u|^2 + |u|^{p+1} \right) dx + \int_0^t \|DF(u)\|_H^2 dt \leq F(\varphi) + C_2, \tag{3.14}$$
$$\int_\Omega \left(|\nabla u|^2 + |u|^{p+1} \right) dx \leq C, \quad \forall t \geq 0,\ \varphi \in U,$$

which implies

$$\|u(t,\varphi)\|_{H_{1/2}} \leq C, \quad \forall t \geq 0,\ \varphi \in U \subset H_{1/2}. \tag{3.15}$$

where C_1, C_2, and C are positive constants. C only depends on φ.

Step 2. We prove that for any bounded set $U \subset H_\alpha$ $(1/2 \leq \alpha < 1)$ there exists $C > 0$ such that

$$\|u(t,\varphi)\|_{H_\alpha} \leq C, \quad \forall t \geq 0,\ \varphi \in U,\ \alpha < 1. \tag{3.16}$$

By $H_{1/2} \hookrightarrow L^{2p}(\Omega)$, we have

$$\begin{aligned}\|g(u)\|_H^2 = \int_\Omega |g(u)|^2 dx = \int_\Omega \left|\sum_{k=1}^{p} a_k u^k\right|^2 dx &\le \int_\Omega \left[a_p|u|^p + \sum_{k=1}^{p-1}|a_k|\left(\varepsilon|u|^p + \varepsilon^{-k/(p-k)}\right)\right]^2 dx \\ &\le C\left(\int_\Omega |u|^{2p} dx + 1\right) \le C\left(\|u\|_{H_{1/2}}^{2p} + 1\right),\end{aligned} \tag{3.17}$$

which implies that $g : H_{1/2} \to H$ is bounded.

Hence, it follows from (2.9) and (3.5) that

$$\begin{aligned}\|u(t,\varphi)\|_{H_\alpha} &= \left\|e^{tL}\varphi + \int_0^t e^{(t-\tau)L} g(u) d\tau\right\|_{H_\alpha} \le \|\varphi\|_{H_\alpha} + \int_0^t \left\|\mathcal{L}^\alpha e^{(t-\tau)L} g(u)\right\|_H d\tau \\ &\le \|\varphi\|_{H_\alpha} + \int_0^t \left\|\mathcal{L}^\alpha e^{(t-\tau)L}\right\| \|g(u)\|_H d\tau \\ &\le \|\varphi\|_{H_\alpha} + C\int_0^t \left\|\mathcal{L}^\alpha e^{(t-\tau)L}\right\| \left(\|u\|_{H_{1/2}}^{2p} + 1\right) d\tau \\ &\le \|\varphi\|_{H_\alpha} + C\int_0^t \tau^\beta e^{-\delta t} d\tau \le C, \quad \forall t \ge 0,\ \varphi \in U \subset H_\alpha,\end{aligned} \tag{3.18}$$

where $\beta = \alpha$ $(0 < \beta < 1)$. Hence, (3.16) holds.

Step 3. We prove that for any bounded set $U \subset H_\alpha$ $(1 \le \alpha < 3/2)$ there exists $C > 0$ such that

$$\|u(t,\varphi)\|_{H_\alpha} \le C, \quad \forall t \ge 0,\ \varphi \in U \subset H_\alpha,\ \alpha < \frac{3}{2}. \tag{3.19}$$

In fact, by the embedding theorems of fractional order spaces [25]:

$$H_\alpha \hookrightarrow C^0(\Omega) \cap H^1(\Omega), \quad \alpha \ge \frac{3}{4}, \tag{3.20}$$

we have

$$\begin{aligned}\|g(u)\|_{H_{1/2}}^2 = \int_\Omega |\nabla g(u)|^2 dx = \int_\Omega \left|\nabla\left(\sum_{k=1}^{p} a_k u^k\right)\right|^2 dx &\le \int_\Omega \left|\sum_{k=1}^{p} k a_k u^{(k-1)} \nabla u\right|^2 dx \\ &\le \int_\Omega \left[p a_p |u|^{p-1} + \sum_{k=1}^{p-1} k|a_k|\left(\varepsilon|u|^{p-1} + \varepsilon^{-(k-1)/(p-k)}\right)\right]^2 |\nabla u|^2 dx\end{aligned}$$

$$
\begin{aligned}
&\le C\int_{\Omega}\left(|u|^{2p-2}+1\right)|\nabla u|^2dx\\
&\le C\int_{\Omega}\left(\sup_{x\in\Omega}|u|^{2p-2}+1\right)|\nabla u|^2dx\\
&\le C\left(\|u\|_{H_\alpha}^{2p-2}+1\right)\|u\|_{H_{1/2}}^2\le C\left(\|u\|_{H_\alpha}^{2p-2}+1\right)\|u\|_{H_\alpha}^2,
\end{aligned}
\tag{3.21}
$$

which implies

$$
g:H_\alpha\longrightarrow H_{1/2}\text{ is bounded for }\alpha\ge\frac{3}{4}. \tag{3.22}
$$

Therefore, it follows from (3.16) and (3.22) that

$$
\|g(u)\|_{H_{1/2}}<C,\quad \forall t\ge 0,\ \varphi\in U\subset H_\alpha,\ \frac{3}{4}\le\alpha<1. \tag{3.23}
$$

Then, by using same method as that in Step 2, we get from (3.23) that

$$
\begin{aligned}
\|u(t,\varphi)\|_{H_\alpha}&=\left\|e^{tL}\varphi+\int_0^t e^{(t-\tau)L}g(u)d\tau\right\|_{H_\alpha}\le\|\varphi\|_{H_\alpha}+\int_0^t\left\|\mathcal{L}^\alpha e^{(t-\tau)L}g(u)\right\|_H d\tau\\
&\le\|\varphi\|_{H_\alpha}+\int_0^t\left\|\mathcal{L}^\alpha e^{(t-\tau)L}\right\|\|g(u)\|_H d\tau\\
&\le\|\varphi\|_{H_\alpha}+C\int_0^t\left\|\mathcal{L}^{\alpha-(1/2)}e^{(t-\tau)L}\right\|\|g(u)\|_{H_{1/2}}d\tau\\
&\le\|\varphi\|_{H_\alpha}+C\int_0^t\tau^{-\beta}e^{-\delta t}d\tau\le C,\quad\forall t\ge 0,\ \varphi\in U\subset H_\alpha,
\end{aligned}
\tag{3.24}
$$

where $\beta=\alpha-1/2$ $(0<\beta<1)$. Hence, (3.19) holds.

Step 4. We prove that for any bounded set $U\subset H_\alpha$ $(3/2\le\alpha<2)$ there exists $C>0$ such that

$$
\|u(t,\varphi)\|_{H_\alpha}\le C,\quad\forall t\ge 0,\ \varphi\in U\subset H_\alpha,\ \alpha<2. \tag{3.25}
$$

In fact, by the embedding theorems of fractional order spaces [25]:

$$
H^2\hookrightarrow W^{1,6}\hookrightarrow W^{1,4},\qquad H_\alpha\hookrightarrow C^0(\Omega)\cap H^2(\Omega),\quad\alpha\ge 1, \tag{3.26}
$$

we have

$$
\begin{aligned}
\|g(u)\|_{H_1}^2 &= \int_\Omega |\Delta g(u)|^2 dx = \int_\Omega \left|\Delta\left(\sum_{k=1}^{p} a_k u^k\right)\right|^2 dx \\
&= \int_\Omega \left|\sum_{k=2}^{p}\left[k(k-1)a_k u^{(k-2)}(\nabla u)^2 + k a_k u^{k-1}\Delta u\right] + a_1 \Delta u\right|^2 dx \\
&\le \int_\Omega \Big\{ p(p-1)a_p |u|^{p-2}(\nabla u)^2 + p a_p |u|^{p-1}\Delta u \\
&\qquad + \sum_{k=2}^{p-1}\Big[k(k-1)a_k\left(\varepsilon|u|^{p-2} + \varepsilon^{-(k-2)/(p-k)}\right)(\nabla u)^2 \\
&\qquad\qquad + k a_k\left(\varepsilon|u|^{p-1} + \varepsilon^{-(k-1)/(p-k)}\right)\Delta u\Big] + a_1|\Delta u|\Big\}^2 dx \\
&\le C\int_\Omega \left(|u|^{p-2}|\nabla u|^2 + |\nabla u|^2 + |u|^{p-1}|\Delta u| + |\Delta u|\right)^2 dx \\
&\le C\int_\Omega \left(|u|^{2p-4}|\nabla u|^4 + |\nabla u|^4 + |u|^{2p-2}|\Delta u|^2 + |\Delta u|^2\right) dx \\
&\le C\int_\Omega \left(\sup_{x\in\Omega}|u|^{2p-4}|\nabla u|^4 + |\nabla u|^4 + \sup_{x\in\Omega}|u|^{2p-2}|\Delta u|^2 + |\Delta u|^2\right) dx \\
&\le C\left(|u|_{H_\alpha}^{2p-4}\|u\|_{W^{1,4}}^4 + \|u\|_{W^{1,4}}^4 + |u|_{H_\alpha}^{2p-2}\|u\|_{H^2}^2 + \|u\|_{H^2}^2\right) \\
&\le C\left(|u|_{H^\alpha}^{2p-4}\|u\|_{H^2}^4 + \|u\|_{H^2}^4 + |u|_{H_\alpha}^{2p-2}\|u\|_{H^\alpha}^2 + \|u\|_{H^\alpha}^2\right) \\
&\le C\left(|u|_{H^\alpha}^{2p} + \|u\|_{H^\alpha}^4 + \|u\|_{H^\alpha}^2\right),
\end{aligned}
\tag{3.27}
$$

which implies that

$$
g : H_\alpha \longrightarrow H_1 \text{ is bounded for } \alpha \ge 1. \tag{3.28}
$$

Therefore, it follows from (3.19) and (3.28) that

$$
\|g(u)\|_{H_1} < C, \quad \forall t \ge 0,\ \varphi \in U \subset H_\alpha,\ 1 \le \alpha < \frac{3}{2}. \tag{3.29}
$$

Then, we get from (3.29) that

$$\begin{aligned}
\|u(t,\varphi)\|_{H_\alpha} &= \left\| e^{tL}\varphi + \int_0^t e^{(t-\tau)L} g(u)\,d\tau \right\|_{H_\alpha} \le \|\varphi\|_{H_\alpha} + \int_0^t \left\| \mathcal{L}^\alpha e^{(t-\tau)L} g(u) \right\|_H d\tau \\
&\le \|\varphi\|_{H_\alpha} + \int_0^t \left\| \mathcal{L}^\alpha e^{(t-\tau)L} \right\| \|g(u)\|_H d\tau \\
&\le \|\varphi\|_{H_\alpha} + \int_0^t \left\| \mathcal{L}^{\alpha-1} e^{(t-\tau)L} \right\| \|g(u)\|_{H_1} d\tau \\
&\le \|\varphi\|_{H_\alpha} + C\int_0^t \tau^{-\beta} e^{-\delta t} d\tau \le C, \quad \forall t \ge 0,\ \varphi \in U \subset H_\alpha,
\end{aligned} \tag{3.30}$$

where $\beta = \alpha - 1$ $(0 < \beta < 1)$. Hence, (3.25) holds.

Step 5. We prove that for any bounded set $U \subset H_\alpha$ $(\alpha \le 0)$ there exists $C > 0$ such that

$$\|u(t,\varphi)\|_{H_\alpha} \le C, \quad \forall t \ge 0,\ \varphi \in U \subset H_\alpha,\ \alpha \ge 0. \tag{3.31}$$

In fact, by the embedding theorems of fractional order spaces [25]:

$$H^3 \hookrightarrow W^{2,6} \hookrightarrow W^{1,6}, \qquad H^3 \hookrightarrow W^{2,6} \hookrightarrow W^{2,4}, \qquad H_\alpha \hookrightarrow C^0(\Omega) \cap H^3(\Omega), \quad \alpha \ge \frac{3}{2}, \tag{3.32}$$

we have

$$\begin{aligned}
\|g(u)\|_{H_{3/2}}^2 &= \int_\Omega |\nabla\Delta g(u)|^2 dx = \int_\Omega \left| \nabla\Delta\left(\sum_{k=1}^p a_k u^k \right) \right|^2 dx \\
&= \int_\Omega \Bigg(\sum_{k=2}^p \Big[k(k-1)(k-2) a_k |u|^{k-3} |\nabla u|^3 + 3k(k-1) a_k |u|^{k-2} |\nabla u||\Delta u| \\
&\qquad + k a_k |u|^{k-1} |\nabla u| \Big] + a_1 |\nabla u||\Delta u| \Bigg)^2 dx \\
&\le \int_\Omega \Big\{ p(p-1)(p-2) a_p |u|^{p-3} |\nabla u|^3 + 3p(p-1) a_p |u|^{p-2} |\nabla u||\Delta u| + p a_p |u|^{p-1} |\nabla u| \\
&\qquad + \sum_{k=3}^{p-1} \Big[k(k-1)(k-2) a_k \left(\varepsilon |u|^{p-3} + \varepsilon^{-(k-3)/(p-k)} \right) |\nabla u|^3 \\
&\qquad\qquad + 3k(k-1) a_k \left(\varepsilon |u|^{p-2} + \varepsilon^{-(k-2)/(p-k)} \right) |\nabla u||\Delta u| \\
&\qquad\qquad + k a_k \left(\varepsilon |u|^{p-1} + \varepsilon^{-(k-1)/(p-k)} \right) |\nabla\Delta u| \Big] \\
&\qquad + 6a_2 |\nabla u||\Delta u| + 2a_2 |u||\nabla u| + a_1 |\nabla\Delta u| \Big\}^2 dx
\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_{\Omega}\Big(|u|^{p-3}|\nabla u|^3+|\nabla u|^3+|u|^{p-2}|\nabla u||\Delta u|+|\nabla u||\Delta u| \\
&\qquad +|u|^{p-1}|\nabla\Delta u|+|u||\nabla u|+|\nabla\Delta u|\Big)^2 dx \\
&\leq C\int_{\Omega}\Big(|u|^{2p-6}|\nabla u|^6+|\nabla u|^6+|u|^{2p-4}|\nabla u|^2|\Delta u|^2+|\nabla u|^2|\Delta u|^2 \\
&\qquad +|u|^{2p-2}|\nabla\Delta u|^2+|u|^2|\nabla u|^2+|\nabla\Delta u|^2\Big)dx \\
&\leq C\int_{\Omega}\Bigg(\sup_{x\in\Omega}|u|^{2p-6}|\nabla u|^6+|\nabla u|^6+\sup_{x\in\Omega}|u|^{2p-4}|\nabla u|^2|\Delta u|^2+|\nabla u|^2|\Delta u|^2 \\
&\qquad +\sup_{x\in\Omega}|u|^{2p-2}|\nabla\Delta u|^2+|u|^2|\nabla u|^2+|\nabla\Delta u|^2\Bigg)dx \\
&\leq C\Big(\|u\|_{H_\alpha}^{2p-6}\|u\|_{W^{1,6}}^6+\|u\|_{W^{1,6}}^6+\|u\|_{H_\alpha}^{2p-4}\|u\|_{W^{1,2}}^2\|u\|_{W^{2,2}}^2+\|u\|_{W^{2,4}}^4 \\
&\qquad +\|u\|_{H_\alpha}^{2p-2}\|u\|_{W^{3,2}}^2+\|u\|^2\|u\|_{W^{1,2}}^2+\|u\|_{W^{3,2}}^2\Big) \\
&\leq C\Big(\|u\|_{H_\alpha}^{2p-6}\|u\|_{H_\alpha}^6+\|u\|_{H_\alpha}^{2p-4}\|u\|_{W^{2,4}}^4+\|u\|_{H_\alpha}^{2p-2}|u|_{H^3}^2+\|u\|_{H^3}^6+\|u\|_{W^{2,4}}^4+\|u\|_{H^3}^2\Big) \\
&\leq C\Big(\|u\|_{H_\alpha}^{2p}+\|u\|_{H_\alpha}^{2p-4}\|u\|_{H_\alpha}^4+\|u\|_{H_\alpha}^{2p-2}\|u\|_{H_\alpha}^2+\|u\|_{H_\alpha}^6+\|u\|_{H_\alpha}^4+\|u\|_{H_\alpha}^2\Big) \\
&\leq C\Big(\|u\|_{H_\alpha}^{2p}+\|u\|_{H_\alpha}^6+\|u\|_{H_\alpha}^4+\|u\|_{H_\alpha}^2\Big),
\end{aligned}
\tag{3.33}
$$

which implies that

$$
g : H_\alpha \longrightarrow H_{3/2} \text{ is bounded for } \alpha \geq \frac{3}{2}. \tag{3.34}
$$

Therefore, it follows from (3.25) and (3.34) that

$$
\|g(u)\|_{H_{3/2}} < C, \quad \forall t \geq 0,\ \varphi \in U \subset H_\alpha,\ \frac{3}{2} \leq \alpha < 2. \tag{3.35}
$$

Then, we get from (3.35) that

$$\begin{aligned}\|u(t,\varphi)\|_{H_\alpha} &= \left\| e^{tL}\varphi + \int_0^t e^{(t-\tau)L} g(u)d\tau \right\|_{H_\alpha} \le \|\varphi\|_{H_\alpha} + \int_0^t \left\| \mathcal{L}^\alpha e^{(t-\tau)L} g(u) \right\|_H d\tau \\ &\le \|\varphi\|_{H_\alpha} + C\int_0^t \left\| \mathcal{L}^{\alpha-(3/2)} e^{(t-\tau)L} \right\| \|g(u)\|_{H_{3/2}} d\tau \\ &\le \|\varphi\|_{H_\alpha} + C\int_0^t \tau^{-\beta} e^{-\delta t} d\tau \le C, \quad \forall t \ge 0,\ \varphi \in U \subset H_\alpha,\end{aligned} \tag{3.36}$$

where $\beta = \alpha - 3/2$ $(0 < \beta < 1)$. Hence, (3.31) holds for $2 \le \alpha < 5/2$.

By doing the same procedures as Steps 1–4, we can prove that (3.31) holds for all $\alpha \ge 0$.

Step 6. We show that for any $\alpha \ge 0$, system (1.1) has a bounded absorbing set in H_α. We first consider the case of $\alpha = 1/2$.

It is well known that the reaction-diffusion equation possesses a global attractor in H space, and the global attractor of this equation consists of equilibria with their stable and unstable manifolds. Thus, each trajectory has to converge to a critical point. From (3.31) and (3.10), we deduce that for any $\varphi \in H_{1/2}$ the solution $u(t,\varphi)$ of system (1.1) converges to a critical point of F. Hence, we only need to prove the following two properties:

(1) $F(u) \to \infty \Leftrightarrow \|u\|_{H_{1/2}} \to \infty$;

(2) the set $S = \{u \in H_{1/2} \mid DF(u) = 0\}$ is bounded.

Property (1) is obviously true, we now prove property (2) in the following. It is easy to check if $DF(u) = 0$, u is a solution of the following equation:

$$\begin{aligned} a\Delta u - g(u) &= 0, \\ u|_{\partial\Omega} &= 0, \end{aligned} \tag{3.37}$$

where $g(u)$ is given by (1.2). Taking the scalar product of (3.37) with u, then we derive that

$$\int_\Omega \left(|\nabla u|^2 + \sum_{k=1}^{2m+1} a_k u^{k+1} \right) dx = 0. \tag{3.38}$$

By (1.3) and (3.38), we have

$$\int_\Omega \left(|\nabla u|^2 + a_{2m+1} u^{2(m+1)} - \sum_{k=1}^{2m} |a_k| |u|^{k+1} \right) dx \le 0. \tag{3.39}$$

Using Hölder inequality and the above inequality, we have

$$\int_\Omega \left(|\nabla u|^2 + u^{2(m+1)} \right) dx \le C, \tag{3.40}$$

where $C > 0$ is a constant. Thus, property (2) is proved.

Now, we show that system (1.1) has a bounded absorbing set in H_α for any $\alpha \geq 1/2$, that is, for any bounded set $U \subset H_\alpha$ there are $T > 0$ and a constant $C > 0$ independent of φ such that

$$\|u(t,\varphi)\|_{H_\alpha} \leq C, \quad \forall t \geq T,\ \varphi \in U. \tag{3.41}$$

From the above discussion, we know that (3.41) holds as $\alpha = 1/2$. By (3.5), we have

$$u(t,\varphi) = e^{(t-T)L}u(T,\varphi) - \int_0^t e^{(t-\tau)L}g(u)d\tau. \tag{3.42}$$

Let $B \subset H_{1/2}$ be the bounded absorbing set of system (1.1), and $T_0 > 0$ such that

$$u(t,\varphi) \in B, \quad \forall t \geq T_0,\ \varphi \in U \subset H_\alpha \left(\alpha \geq \frac{1}{2}\right). \tag{3.43}$$

It is well known that

$$\left\|e^{tL}\right\| \leq Ce^{-t\lambda_1}, \tag{3.44}$$

where $\lambda_1 > 0$ is the first eigenvalue of the equation:

$$\begin{gathered} a\Delta u - g(u) = 0, \\ u|_{\partial\Omega} = 0. \end{gathered} \tag{3.45}$$

Hence, for any given $T > 0$ and $\varphi \in U \subset H_\alpha$ $(\alpha \geq 1/2)$, we have

$$\left\|e^{(t-\tau)L}u(t,\varphi)\right\|_{H_\alpha} = \left\|\mathcal{L}^\alpha e^{(t-\tau)L}u(t,\varphi)\right\|_H \longrightarrow 0, \quad \text{as } t \longrightarrow \infty. \tag{3.46}$$

From (3.42)-(3.43) and Lemma 2.4 for any $1/2 \leq \alpha < 1$, we get that

$$\begin{aligned} \|u(t,\varphi)\|_{H_\alpha} &\leq \left\|e^{(t-T_0)L}u(T_0,\varphi)\right\|_{H_\alpha} + \int_{T_0}^t \left\|\mathcal{L}^\alpha e^{(t-\tau)L}g(u)\right\|d\tau \\ &\leq \left\|e^{(t-T_0)L}u(T_0,\varphi)\right\|_{H_\alpha} + C\int_0^{t-T_0} \tau^{-\alpha}e^{-\lambda_1\tau}d\tau, \end{aligned} \tag{3.47}$$

where $C > 0$ is a constant independent of φ.

Then, we infer from (3.46) and (3.47) that (3.41) holds for all $1/2 \leq \alpha < 1$. By the iteration method, we have that (3.41) holds for all $\alpha \geq 1/2$.

Finally, this theorem follows from (3.31)–(3.41) and Lemma 2.3. The proof is completed. □

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