Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 918529, 24 pages
doi:10.1155/2012/918529

Research Article

Fractional Difference Equations with Real Variable

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Received 20 June 2012; Revised 14 October 2012; Accepted 25 October 2012

Academic Editor: Dumitru Băleanu

We independently propose a new kind of the definition of fractional difference, fractional sum, and fractional difference equation, give some basic properties of fractional difference and fractional sum, and give some examples to demonstrate several methods of how to solve certain fractional difference equations.

1. Introduction

Fractional calculus is an emerging field recently drawing attention from both theoretical and applied disciplines. During the last two decades, it has been successfully applied to several fields [1–6], and it is well known that there is a large quantity of research on what is usually called integer-order difference equations [7, 8]. However, discrete fractional calculus and fractional difference equations represent a very new area for scientists. A pioneering work has been done by Atici et al. [9–12], Anastassiou [13, 14], Bastos et al. [15], Abdeljawad et al. [16–20], and Cheng [21–23], and so forth. In this paper, limited to the length of the paper, we will introduce some of our basic works about discrete fractional calculus and fractional difference equations. Some proofs and results of the theorems and examples in Sections 3–5 are well proved by a more concise method. We refer to the monographer [23] for more further results. In [23] we also aim at presenting some basic properties about discrete fractional calculus and, in a systematic manner, results including the existence and uniqueness of solutions for the Cauchy Type and Cauchy problems, involving nonlinear fractional difference equations, explicit solutions of linear difference equations and linear difference system by their deduction to Volterra sum equation and by using operational methods, applications of Z-transform, R-transform, N-transform, Adomian decomposition method, method of undetermined coefficients, Jordan matrix theory method, and by discrete Mittag-Leffler function and discrete Green' function, and a theory of so-called sequential linear fractional difference equations, as well as some introduction for discrete fractional difference variational problem, and so forth.

2. Integer-Order Difference and Sum with Real Variable

Let us start from sum and difference of the integer order. Define

$$ {}_h\sum_{s=a}^{t} x(s) \triangleq [x(a) + x(a+h) + x(a+2h) + \cdots + x(t)], \tag{2.1} $$

where $t = a + jh,\ j \in N_0 = \{0, 1, 2, \dots\}$.

Definition 2.1. Let a, t be real numbers, and let h be a positive number, we call

$$ {}_a\nabla_h^{-1} x(t) = {}_h\sum_{s=a}^{t} x(s)h \tag{2.2} $$

one-order backward sum of $x(t)$, where $t = a + jh,\ j \in N_0 = \{0, 1, 2, \dots\}$. We call

$$ {}_a\nabla_h^{-k} x(t) = {}_a\nabla_h^{-1}\left({}_a\nabla_h^{-(k-1)} x(t)\right) \tag{2.3} $$

k-order backward sum of $x(t)$, where k is a positive integer number.

Definition 2.2. Let a, t be real numbers, and let h be a positive number, we call

$$ {}_a\Delta_h^{-1} x(t) = {}_h\sum_{s=a}^{t-h} x(s)h \tag{2.4} $$

one-order forward sum of $x(t)$, where $t = a + jh,\ j \in N_1 = \{1, 2, \dots\}$. We call

$$ {}_a\Delta_h^{-k} x(t) = {}_a\Delta_h^{-1}\left({}_a\Delta_h^{-(k-1)} x(t)\right) \tag{2.5} $$

k-order forward difference of $x(t)$, where k is a positive integer number.

Definition 2.3. Let t be a real number, and let h be a positive number, we call

$$ \nabla_h x(t) = \frac{x(t) - x(t-h)}{h} \tag{2.6} $$

one-order backward difference of $x(t)$, where h is step. We call

$$ \nabla_h^k x(t) = \nabla_h\left(\nabla_h^{k-1} x(t)\right) \tag{2.7} $$

k-order backward difference of $x(t)$, where k is a positive integer number.

Similarly, we can define forward difference as follows.

Definition 2.4. Let t be a real number, and let h be a positive number, we call

$$\Delta_h x(t) = \frac{x(t+h) - x(t)}{h} \tag{2.8}$$

one-order forward difference of $x(t)$, where h is step. We call

$$\Delta_h^k x(t) = \Delta_h\left(\Delta_h^{k-1} x(t)\right) \tag{2.9}$$

k-order forward difference of $x(t)$, where k is a positive integer number.

Theorem 2.5. *The following two equalities hold:*

(1) $\nabla_h({}_a\nabla_h^{-1} x(t)) = x(t)$,

(2) $\Delta_h({}_a\Delta_h^{-1} x(t)) = x(t)$.

Definition 2.6. If k, t are real numbers, and let h be a positive number, define

$$t_h^{\overline{k}} = h^k \frac{\Gamma(t/h + k)}{\Gamma(t/h)}, \quad (k \in R) \tag{2.10}$$

rising factorial function, and set $t_h^{\overline{0}} = 1$. If k is a positive integer number, then we have

$$t_h^{\overline{k}} = t(t+h)(t+2h)\cdots(t+(k-1)h). \tag{2.11}$$

Definition 2.7. Let k, t be real numbers, and let h be a positive number, define

$$t_h^{(k)} = h^k \frac{\Gamma(t/h + 1)}{\Gamma(t/h + 1 - k)}, \quad (k \in R) \tag{2.12}$$

down factorial function, and set $t_h^{(0)} = 1$. If k is an positive integer number, then

$$t_h^{(k)} = t(t-h)(t-2h)\cdots(t-(k-1)h). \tag{2.13}$$

In Definitions 2.6 and 2.7, if $h = 1$, we can simply denote $t_h^{\overline{k}}$, $t_h^{(k)}$ as $t^{\overline{k}}$, $t^{(k)}$.

Definition 2.8. For any $k, \gamma \in R$, $h > 0$, we define

$$\begin{bmatrix}\gamma \\ k\end{bmatrix} \triangleq \frac{\Gamma(k+\gamma)}{\Gamma(\gamma)\Gamma(k+1)}, \qquad \begin{bmatrix}\gamma \\ k\end{bmatrix}_h \triangleq h^\gamma \begin{bmatrix}\gamma \\ \dfrac{k}{h}\end{bmatrix}. \tag{2.14}$$

If $k \in N$, $h = 1$, then it is easy to see that

$$\begin{bmatrix} \gamma \\ k \end{bmatrix} = \frac{\gamma(\gamma+1)\cdots(\gamma+k-1)}{k!}. \tag{2.15}$$

If we let $t/h = \tilde{t}$, or $t = \tilde{t}h$, then we clearly have the following.

Theorem 2.9. *Assume that $k \in R$, $h > 0$, $t/h = \tilde{t}$; then*

$$t_h^{\overline{k}} = h^k \tilde{t}^{\overline{k}}; \qquad t_h^{(k)} = h^k \tilde{t}^{(k)}. \tag{2.16}$$

Theorem 2.10. *Let $k \in R$, $h > 0$, then, following equality holds:*

$$\lim_{h\to 0} t_h^{\overline{k}} = \lim_{h\to 0} t_h^{(k)} = t^k. \tag{2.17}$$

3. Fractional Sum and Difference with Real Variable

Before giving the definitions of fractional sum ${}_a\nabla_h^{-\gamma} x(t)$, $\gamma > 0$, let us revisit the calculation of the sum of the integer order. By Definition 2.1, we have

$${}_a\nabla_h^{-1} x(t) = {}_h\sum_{s=a}^{t} x(s)h, \quad t = a + jh,\ j \in N_0, \tag{3.1}$$

then

$$\begin{aligned} {}_a\nabla_h^{-2} x(t) &= {}_a\nabla_h^{-1}\left[{}_a\nabla_h^{-1} x(t)\right] = {}_h\sum_{s=a}^{t} {}_a\nabla_h^{-1} x(s)h = h\left[{}_h\sum_{s=a}^{t} {}_h\sum_{r=a}^{s} x(r)h\right] \\ &= h^2\left[{}_h\sum_{r=a}^{t} {}_h\sum_{s=r}^{t} x(r)\right] = h^2\left[{}_h\sum_{r=a}^{t} \frac{t-r+h}{h} x(r)\right] = {}_h\sum_{r=a}^{t} (t-r+h)_h^{\overline{1}} x(r)h, \\ {}_a\nabla_h^{-3} x(t) &= {}_a\nabla_h^{-1}\left[{}_a\nabla_h^{-2} x(t)\right] = {}_h\sum_{s=a}^{t} {}_a\nabla_h^{-2} x(s)h = h^2\left[{}_h\sum_{s=a}^{t} {}_h\sum_{r=a}^{s} \frac{t-r+h}{h} x(r)h\right] \\ &= \frac{h^3}{2}\left[{}_h\sum_{r=a}^{t}\left(\frac{t-r+h}{h}\right)\left(\frac{t-r+2h}{h}\right)x(r)\right] = \frac{1}{2!}\left[{}_h\sum_{r=a}^{t} (t-r+h)_h^{\overline{2}} x(r)h\right] \ldots. \end{aligned} \tag{3.2}$$

By recursive, it is not hard to obtain

$$\begin{aligned} {}_a\nabla_h^{-m} x(t) &= \frac{h^m}{(m-1)!}\left[{}_h\sum_{s=a}^{t}\left(\frac{t-s+h}{h}\right)\left(\frac{t-s+2h}{h}\right)\cdots\left(\frac{t-s+(m-1)h}{h}\right)x(s)\right] \\ &= \frac{1}{\Gamma(m)}\left[{}_h\sum_{s=a}^{t} (t-s+h)_h^{\overline{m-1}} x(s)h\right] = \frac{1}{\Gamma(m)}\left[{}_h\sum_{s=a}^{t} (t-\rho_h(s))_h^{\overline{m-1}} x(s)h\right], \end{aligned} \tag{3.3}$$

where $\rho_h(s) = s - h$.

Obviously, the right side of formula (3.3) is also meaningful for all real $m > 0$, so we define fractional sum as follows.

Definition 3.1. Let $\gamma > 0$, $a \in R$, $h > 0$, $t = a + kh$, $k \in N_0$, we call

$$ {}_a\nabla_h^{-\gamma} x(t) = \frac{1}{\Gamma(\gamma)} \left[{}_h\sum_{s=a}^{t} (t - \rho_h(s))_h^{\overline{\gamma-1}} x(s) h \right] \tag{3.4}$$

γ order fractional sum of $x(t)$.

For any positive number order fractional difference, we take the following.

Definition 3.2. Let $\mu > 0$, and assume that $m - 1 < \mu < m$, where m denotes a positive integer. Define

$$ {}_a\nabla_h^{\mu} x(t) = \nabla_h^m \left({}_a\nabla_h^{-(m-\mu)} x(t) \right) \tag{3.5}$$

as μ order *R-L* type backward fractional difference. Meantime, define

$$ {}_a^C\nabla_h^{\mu} x(t) = \left({}_a\nabla_h^{-(m-\mu)} \right) \nabla_h^m x(t) \tag{3.6}$$

as μ order Caputo type backward fractional difference.

If we start from Definition 2.2,

$$ {}_a\Delta_h^{-1} x(t) = {}_h\sum_{s=a}^{t-h} x(s) h, \quad t = a + jh,\ j \in N_1, \tag{3.7}$$

completely in a similar way, we get positive integer m-order forward sum

$$ {}_a\Delta_h^{-m} x(t) = \frac{1}{\Gamma(m)} \left[{}_h\sum_{s=a}^{t-mh} (t - \sigma_h(s))_h^{(m-1)} x(s) h \right], \tag{3.8}$$

where $\sigma_h(s) = s + h$.

The right side of (3.8) is meaningful for all real $m > 0$, so we can define forward fractional sum as follows.

Definition 3.3. Let $\gamma > 0$, $a \in R$, $h > 0$, $t = a + \gamma h + kh$, $k \in N_0$, define

$$ {}_a\Delta_h^{-\gamma} x(t) = \frac{1}{\Gamma(\nu)} \left[{}_h\sum_{s=a}^{t-\gamma h} (t - \sigma_h(s))_h^{(\gamma-1)} x(s) h \right] \tag{3.9}$$

as γ order fractional sum of $x(t)$, where $\sigma_h(s) = s + h$.

Definition 3.4. Let $\mu > 0$, and assume that $m-1 < \mu < m$, where m denotes a positive integer. Define

$$ {}_a\Delta_h^{\mu} x(t) = \Delta_h^m \left({}_a\Delta_h^{-(m-\mu)} x(t) \right) \tag{3.10}$$

as μ order *R-L* type forward fractional difference. Meantime, define

$$ {}_a^C\Delta_h^{\mu} x(t) = \left({}_a\Delta_h^{-(m-\mu)} \right) \Delta_h^m x(t) \tag{3.11}$$

as μ order Caputo type forward fractional difference.

In Definitions 3.1–3.4, if step $h = 1$, it is a kind of important situation. At this time, we simply denote ${}_a\nabla_h^{-\gamma}$, ${}_a\Delta_h^{-\gamma}$; ∇_h^{μ}, Δ_h^{μ} as ${}_a\nabla^{-\gamma}$, ${}_a\Delta^{-\gamma}$; ∇^{μ}, Δ^{μ}. When $h = 1$, backward fractional sum is defined as follows.

Definition 3.5. Let $\gamma > 0$, and define

$$ {}_a\nabla^{-\gamma} x(t) = \frac{1}{\Gamma(\gamma)} \sum_{s=a}^{t} (t-\rho(s))^{\overline{\gamma-1}} x(s) \tag{3.12}$$

as γ order fractional sum of $x(t)$, where $t = a \bmod (1)$, $\rho(s) = s-1$.

For any positive number order fractional difference, we can take the following way.

Definition 3.6. Let $\mu > 0$ and assume that $m-1 < \mu < m$, where m denotes a positive integer. Define

$$ {}_a\nabla^{\mu} x(t) = \nabla^m \left({}_a\nabla^{-(m-\mu)} x(t) \right) \tag{3.13}$$

as μ order *R-L* type backward fractional difference. Meantime, define

$$ {}_a^C\nabla^{\mu} x(t) = \left({}_a\nabla^{-(m-\mu)} \right) \nabla^m x(t) \tag{3.14}$$

as μ order Caputo type backward fractional difference.

We can define forward fractional sum as follows.

Definition 3.7. Let $\gamma > 0$, and define

$$ {}_a\Delta^{-\gamma} x(t) = \frac{1}{\Gamma(\gamma)} \sum_{s=a}^{t-\gamma} (t-\sigma(s))^{(\gamma-1)} x(s) \tag{3.15}$$

as γ order forward fractional sum of $x(t)$, where $t-\gamma = a \bmod (1)$, $\sigma(s) = s+1$.

Definition 3.8. Let $\mu > 0$, and assume that $m - 1 < \mu < m$, where m denotes a positive integer. Define

$$ {}_a\Delta^{\mu} x(t) = \Delta^m \left({}_a\Delta^{-(m-\mu)} x(t) \right) \tag{3.16} $$

as μ order R-L type forward fractional difference. Meantime, define

$$ {}_a^C\Delta^{\mu} x(t) = \left({}_a\Delta^{-(m-\mu)} \right) \Delta^m x(t) \tag{3.17} $$

as μ order Caputo type forward fractional difference.

By Definition 2.8, it is easy to calculate

$$ \begin{gathered} \begin{bmatrix} \gamma \\ t-s \end{bmatrix} = \frac{1}{\Gamma(\gamma)} (t - \rho(s))^{\overline{\gamma-1}}, \\ \begin{bmatrix} \gamma \\ t-\gamma-s \end{bmatrix} = \frac{1}{\Gamma(\gamma)} (t - \sigma(s))^{\gamma-1}. \end{gathered} \tag{3.18} $$

By Theorem 2.9 we have

$$ \begin{gathered} \frac{(t-s+h)_h^{\overline{\gamma-1}}}{\Gamma(\gamma)} = h^{\gamma-1} \frac{((t-s)/h+1)^{\overline{\gamma-1}}}{\Gamma(\gamma)} = h^{\gamma-1} \begin{bmatrix} \gamma \\ \dfrac{t-s}{h} \end{bmatrix}, \\ \frac{(t-s-h)_h^{(\gamma-1)}}{\Gamma(\gamma)} = h^{\gamma-1} \frac{((t-s)/h-1)^{(\gamma-1)}}{\Gamma(\gamma)} = h^{\gamma-1} \begin{bmatrix} \gamma \\ \dfrac{t-s}{h} - \gamma \end{bmatrix}. \end{gathered} \tag{3.19} $$

Therefore, if we adopt Definition 2.8, then Definitions 3.1, 3.3, 3.5, and 3.7 can be rewritten as follows.

Definition 3.9. Assume that $\gamma > 0$, let $a \in R$, $h > 0$, $t = a + kh$, $k \in N_0$, and define

$$ {}_a\nabla_h^{-\gamma} x(t) = h \sum_{s=a}^{t} \begin{bmatrix} \gamma \\ t-s \end{bmatrix}_h x(s) \tag{3.20} $$

as γ order backward fractional sum of $x(t)$.

Definition 3.10. Assume that $\gamma > 0$, let $a \in R$, $h > 0$, $t = a + \gamma h + kh$, $k \in N_0$, and define

$$ {}_a\Delta_h^{-\gamma} x(t) = h \sum_{s=a}^{t-\gamma h} \begin{bmatrix} \gamma \\ t-s-\gamma h \end{bmatrix}_h x(s) \tag{3.21} $$

as γ order forward fractional sum of $x(t)$.

Definition 3.11. Assume that $\gamma > 0$, $t, a \in R$, and $t = a \bmod (1)$, and define

$$ {}_a\nabla^{-\gamma}x(t) = \sum_{s=a}^{t} \begin{bmatrix} \gamma \\ t-s \end{bmatrix} x(s) \tag{3.22} $$

as γ order backward fractional sum of $x(t)$.

Definition 3.12. Assume that $\gamma > 0$, $t, a \in R$, and $t - \gamma = a \bmod (1)$, and define

$$ {}_a\Delta^{-\gamma}x(t) = \sum_{s=a}^{t-\gamma} \begin{bmatrix} \gamma \\ t-\gamma-s \end{bmatrix} x(s) \tag{3.23} $$

as γ order forward fractional sum of $x(t)$.

Set $a/h = \tilde{a}$, $t/h = \tilde{t}$, or $a = \tilde{a}h, t = \tilde{t}h$, and set $x(t) = x(\tilde{t}h) = y(\tilde{t})$; then by Theorem 2.9 and Definitions 3.1–3.4, one obtains the following.

Theorem 3.13. *For any $\gamma, \mu > 0$, the following equalities hold:*

(1) ${}_a\nabla_h^{-\gamma}x(t) = h^{\gamma}[{}_{\tilde{a}}\nabla^{-\gamma}y(\tilde{t})]$; ${}_a\Delta_h^{-\gamma}x(t) = h^{\gamma}[{}_{\tilde{a}}\Delta^{-\gamma}y(\tilde{t})]$,

(2) ${}_a\nabla_h^{\mu}x(t) = h^{-\mu}[{}_{\tilde{a}}\nabla^{\mu}y(\tilde{t})]$; ${}_a\Delta_h^{\mu}x(t) = h^{-\mu}[{}_{\tilde{a}}\Delta^{\mu}y(\tilde{t})]$,

(3) ${}_a^C\nabla_h^{\mu}x(t) = h^{-\mu}[{}_{\tilde{a}}^C\nabla^{\mu}y(\tilde{t})]$; ${}_a^C\Delta_h^{\mu}x(t) = h^{-\mu}[{}_{\tilde{a}}^C\Delta^{\mu}y(\tilde{t})]$.

From Theorem 3.13 we can see, by stretching $t = \tilde{t}h$, the functions ${}_a\nabla_h^{-\gamma}x(t)$ and ${}_a\nabla_h^{\mu}x(t)$, with common step h, can be convert into the functions ${}_a\nabla^{-\gamma}y(\tilde{t})$ and ${}_a\nabla^{\mu}y(\tilde{t})$ with step $h = 1$, respectively. In essence, nothing arises much different, but the latter is more convenient in research.

In view of Definitions 3.1–3.4 and Theorem 2.10, if we let $h \to 0$, then we can obtain the following.

Corollary 3.14. *Assume that $x(t)$ is integrable, then:*

(1) $\lim_{h\to 0}({}_a\nabla_h^{-\gamma}x(t)) = \lim_{h\to 0}({}_a\nabla_h^{-\gamma}x(t)) = (1/\Gamma(\gamma))\int_a^t (t-s)^{\gamma-1}x(t)ds \triangleq D_t^{-\gamma}x(t)$,

(2) $\lim_{h\to 0}({}_a\nabla_h^{\mu}x(t)) = \lim_{h\to 0}({}_a\nabla_h^{\mu}x(t)) = D^m({}_aD_t^{-(m-\mu)}x(t)) \triangleq {}_a D_t^{\mu}x(t)$,

(3) $\lim_{h\to 0}({}_a^C\nabla_h^{\mu}x(t)) = \lim_{h\to 0}({}_a^C\nabla_h^{\mu}x(t)) = D^m({}_aD_t^{-(m-\mu)}x(t)) \triangleq {}_a^C D_t^{\mu}x(t)$.

4. Some Basic Properties

We sometimes only list some basic results here, for more detailed results and their proofs can been seen in monographer [23].

Theorem 4.1. *Assume that the following function is well defined; then*

(1) $\nabla_h t_h^{\overline{\gamma}} = \gamma t_h^{\overline{\gamma-1}}$, $\Delta_h t_h^{(\gamma)} = \gamma t_h^{(\gamma-1)}$,

(2) $(t+\gamma h)t_h^{\overline{\gamma}} = t_h^{\overline{\gamma+1}}$, $(t-\gamma h)t_h^{(\gamma)} = t_h^{(\gamma+1)}$, $\gamma \in R$,

(3) *If* $0 < \gamma < 1$, *then* $t_h^{\overline{\alpha\gamma}} \le (t_h^{\overline{\alpha}})^\gamma$, $t_h^{(\alpha\gamma)} \ge (t_h^{(\alpha)})^\gamma$,

(4) $t_h^{\overline{\alpha+\beta}} = (t+\beta)_h^{\overline{\alpha}} t_h^{\overline{\beta}}$, $t_h^{(\alpha+\beta)} = (t-\beta)_h^{(\alpha)} t_h^{(\beta)}$,

(5) *Let* $0 < t \le r$, *if* $\gamma > 0$, *then* $t_h^{\overline{\gamma}} \le r_h^{\overline{\gamma}}$, $t_h^{(\gamma)} \le r_h^{\gamma}$; *If* $\gamma < 0$, *then* $t_h^{\overline{\gamma}} \ge r_h^{\overline{\gamma}}$, $t_h^{(\gamma)} \ge r_h^{\gamma}$.

Theorem 4.2. *Let* $0 \le m-1 < \gamma \le m, m \in N$, *where* $x(t)$ *is defined in* $N_{h,a} = \{a, a+h, a+2h, \ldots\}$, *then*

(1) ${}_a\nabla_h^{-\gamma} x(t) = {}_a\Delta_h^{-\gamma} x(t+\gamma h)$, $t \in N_{h,a}$,

(2) ${}_a\nabla^{\gamma} x(t) = {}_a\Delta^{\gamma} x(t-\gamma h)$, $t \in N_{h,m+a}$.

Theorem 4.3. *Let* $0 \le m-1 < \gamma \le m$, $m \in N$, $x(t)$ *is defined in* $N_{h,a} = \{a, a+h, a+2h, \ldots\}$, *then*

(1) ${}_a\Delta_h^{-\gamma} x(t) = {}_a\nabla_h^{-\gamma} x(t-\gamma h), t \in N_{h,a+\gamma}$,

(2) ${}_a\Delta_h^{\gamma} x(t) = {}_a\nabla_h^{\gamma} x(t+\gamma h), t \in N_{h,a-\gamma+m}$.

Theorem 4.4. *For any real* γ, *the following equality holds:*

(1) ${}_a\nabla_h^{-\gamma} \nabla_h x(t) = \nabla_h({}_a\nabla_h^{-\gamma}) x(t) - ((t-a-1)_h^{\overline{\gamma-1}}/\Gamma(\gamma)) x(a-h)$,

(2) ${}_a\Delta_h^{-\gamma} \Delta_h x(t) = \Delta_h({}_a\Delta_h^{-\gamma}) x(t) - ((t-a)_h^{(\gamma-1)}/\Gamma(\gamma)) x(a)$.

Theorem 4.5. *For any real* γ *and* $p > 0$, *the following equality holds:*

(1) ${}_a\nabla_h^{-\gamma} \nabla_h^p x(t) = \nabla_h^p({}_a\nabla_h^{-\gamma} x(t)) - \sum_{k=0}^{p-1}((t-a+1)_h^{\overline{\gamma-p+k}}/\Gamma(\gamma+k-p+1)) \nabla_h^k x(a-h)$,

(2) ${}_a\Delta_h^{-\gamma} \Delta_h^p x(t) = \Delta_h^p({}_a\Delta_h^{-\gamma} x(t)) - \sum_{k=0}^{p-1}((t-a)_h^{(\gamma-p+k)}/\Gamma(\gamma+k-p+1)) \Delta_h^k x(a)$.

Theorem 4.6. *Let* $p, \gamma > 0$, *then*

(1) $\nabla_h^p({}_a\nabla_h^{-\gamma} x(t)) = {}_a\nabla_h^{-(\gamma-p)} x(t)$,

(2) $\Delta_h^p({}_a\Delta_h^{-\gamma} x(t)) = {}_a\Delta_h^{-(\gamma-p)} x(t)$.

In the previous theorems, we only need to consider the simplest case $h = 1$, but actually the methods of proof and conclusions can also be extended for general step $h > 0$. In fact, we only need do a stretching transformation and then make use of Theorem 2.9.

Next, we discusses fractional sum transform such as: Z transform, N transform, R transform, and some properties of these transforms.

Definition 4.7. Let $f(t)$ be defined in $N_0 = \{0, 1, 2, \ldots\}$, we call

$$f(t) = \sum_{t=0}^{\infty} f(t) z^{-t} \tag{4.1}$$

is a Z transform of $f(t)$, denote it by $Z[f(t)]$.

Definition 4.8. Let $f(t)$ be defined in $N_{t_0} = \{t_0, t_0+1, t_0+2, \ldots\}$, $t_0 \in R$, and define N transform as follows:

$$N_{t_0}(f(t))(s) = \sum_{t=t_0}^{\infty} (1-s)^{t-1} f(t). \tag{4.2}$$

If the domain of the function $f(t)$ is N_1, then we use the notation $N(f(t))$.

If we set $t - t_0 = n \in N_0$, define

$$f_n^{\{t_0\}} = f(n + t_0) = f(t), \qquad f_{n-1}^{\{t_0\}} = f(n - 1 + t_0) = f(t - 1), \dots,$$
$$f_0^{\{t_0\}} = f(0 + t_0) = f(t_0). \tag{4.3}$$

Then, $f(t_0), f(t_0 + 1), \dots, f(t), \dots$ can be regarded as a sequence

$$f_0^{\{t_0\}}, f_1^{\{t_0\}}, \dots f_n^{\{t_0\}}, \dots. \tag{4.4}$$

Under this definition, N transform can be simply rewritten as

$$\begin{aligned} N_0(f(t))(s) &= \sum_{t=t_0}^{\infty} (1-s)^{t-1} f(s) \\ &= \sum_{n=0}^{\infty} (1-s)^{n+t_0-1} f(n+t_0) \\ &= (1-s)^{t_0-1} \sum_{n=0}^{\infty} (1-s)^n f_n^{\{t_0\}}. \end{aligned} \tag{4.5}$$

Set $z = 1/(1 - s)$, then we have

$$N_0(f(t))(s) = z^{1-t_0} \sum_{n=0}^{\infty} f_n^{\{t_0\}} z^{-n} = z^{1-t_0} F(z), \tag{4.6}$$

where $F(z)$ is Z transform of sequence $f_n^{\{t_0\}}$.

If $t_0 = 1$, then

$$N(f(t)) = F(z), \quad \left(z = \frac{1}{1-s}\right). \tag{4.7}$$

Theorem 4.9. *For any $\gamma \in R \setminus \{\dots, -2, -1, 0\}$, then*

(1) $N(t^{\overline{\gamma-1}})(s) = \Gamma(\gamma)/s^{\gamma}$, $|1 - s| < 1$,

(2) $N(t^{\overline{\gamma-1}} \alpha^{-t})(s) = \alpha^{\gamma-1}\Gamma(\gamma)/(s + \alpha - 1)^{\gamma}$, $|1 - s| < \alpha$.

Proof. (1) Making use of (4.7), we get

$$N\left(\frac{t^{\overline{\gamma-1}}}{\Gamma(\gamma)}\right) = F(z), \tag{4.8}$$

where $F(z)$ is Z transform of sequence $f_n^{\{1\}} = f(n+1)$,

$$f_n^{\{1\}} = f(n+1) = \frac{(n+1)^{\overline{\gamma-1}}}{\Gamma(\gamma)} = \begin{bmatrix} \gamma \\ n \end{bmatrix}. \tag{4.9}$$

Since (see [21–23])

$$F\left(\begin{bmatrix} \gamma \\ n \end{bmatrix}\right) = \left(\frac{z-1}{z}\right)^{-\gamma} = \frac{1}{s^{\gamma}}, \quad (|z| > 1, |1-s| < 1), \tag{4.10}$$

hence

$$N\left(\frac{t^{\overline{\gamma-1}}}{\Gamma(\gamma)}\right) = \frac{1}{s^{\gamma}}, \quad (|1-s| < 1). \tag{4.11}$$

(2) It is only to use

$$\sum_{t=1}^{\infty}(1-s)^{t-1} t^{\overline{\gamma-1}} \alpha^{-t} = \frac{1}{\alpha}\sum_{t=1}^{\infty}\left(1 - \frac{s+\alpha-1}{\alpha}\right)^{t-1} t^{\overline{\gamma-1}}, \tag{4.12}$$

then the proof of (2) follows from the proof of (1). □

Theorem 4.10. *Let $f(t)$ and $g(t)$ be defined in N_a, and define convolution of $f(t)$, $g(t)$ as follows:*

$$(h * g)_a(t) = \sum_{s=a}^{t} h(t-\rho(s))g(s). \tag{4.13}$$

For $h(t) = t^{\overline{\gamma-1}}/\Gamma(\gamma)$, then

$$(h * g)_a(t) = \frac{1}{\Gamma(\gamma)}\sum_{s=a}^{t}(t-\rho(s))^{\overline{\gamma-1}} g(s) = {}_a\nabla^{-\gamma} g(t). \tag{4.14}$$

Theorem 4.11. *Let f, g be defined in N_a, then*

$$N_a(f * g) = N_1(f) N_a(g). \tag{4.15}$$

Theorem 4.12. *For any real γ, one has*

$$N_a\left({}_a\nabla^{-\gamma} f(t)\right) = s^{-\gamma} N_a(f(t)). \tag{4.16}$$

Theorem 4.13. *For $0 < \gamma \le 1$, one has*

$$N_{a+1}\left({}_a\nabla^{-\gamma} f(t)\right) = s^{\gamma} N_a(f(t))(s) - (1-s)^{\alpha-1} f(a), \tag{4.17}$$

where f is defined in N_a.

Theorem 4.14. *Let $\mu \in R \setminus \{\dots,-2,-1,0\}$, $\gamma > 0$, then*

$$ {}_1\nabla^{-\gamma}\left(\frac{t^{\overline{\mu}}}{\Gamma(\mu+1)}\right) = \frac{t^{\overline{\mu+\gamma}}}{\Gamma(\mu+\gamma+1)}. \tag{4.18}$$

Theorem 4.15. *Let f be a real function, $\mu, \gamma > 0$, then*

$$ {}_a\nabla^{-\gamma}\left[{}_a\nabla^{-\mu} f(t)\right] = {}_a\nabla^{-(\mu+\gamma)} f(t) = {}_a\nabla^{-\mu}\left[{}_a\nabla^{-\gamma} f(t)\right]. \tag{4.19}$$

Definition 4.16. Let $f(t)$ be defined in N_{t_0}, and define R transform as follows:

$$ R_{t_0}(f(t)) = \sum_{t=t_0}^{\infty}\left(\frac{1}{s+1}\right)^{t+1} f(t). \tag{4.20}$$

In Definition 4.16, if we set $t - t_0 = n \in N_0$, and define:

$$\begin{gathered} f_n^{\{t_0\}} = f(n+t_0) = f(t), \qquad f_{n-1}^{\{t_0\}} = f(n-1+t_0) = f(t-1), \dots, \\ f_0^{\{t_0\}} = f(0+t_0) = f(t_0), \end{gathered} \tag{4.21}$$

then, $f(t_0), f(t_0+1), \dots, f(t), \dots$ can be regarded as a sequence

$$ f_0^{\{t_0\}}, f_1^{\{t_0\}}, \dots f_n^{\{t_0\}}, \dots. \tag{4.22}$$

Under this definition, R transform can be simply rewritten as

$$\begin{aligned} R_{t_0}(f(t))(s) &= \sum_{t=t_0}^{\infty}\left(\frac{1}{s+1}\right)^{t+1} f(t) \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{s+1}\right)^{n+t_0+1} f(n+t_0) \\ &= \left(\frac{1}{s+1}\right)^{t_0+1} \sum_{n=0}^{\infty}\left(\frac{1}{1+s}\right)^{n} f_n^{\{t_0\}}. \end{aligned} \tag{4.23}$$

Set $z = 1 + s$, then

$$ R_{t_0}(f(t))(s) = z^{-1-t_0}\sum_{n=0}^{\infty} f_n^{\{t_0\}} z^{-n} = z^{-1-t_0} F(z), \tag{4.24}$$

where $F(z)$ is a Z transform of sequence $f_n^{\{t_0\}}$.

Theorem 4.17. *For any $\gamma \in R \setminus \{\ldots, -2, -1, 0\}$, then*

(1) $R_{\gamma-1}(t^{(\gamma-1)})(s) = \Gamma(\gamma)/s^{\gamma}$,

(2) $R_{\gamma-1}(t^{(\gamma-1)}\alpha^{t})(s) = \alpha^{\gamma-1}\Gamma(\gamma)/(s+1-\alpha)^{\gamma}$.

Proof. (1) let $t_0 = \gamma - 1$, then

$$R_{\gamma-1}\left(\frac{t^{(\gamma-1)}}{\Gamma(\gamma)}\right) = z^{-\gamma}F(z), \tag{4.25}$$

where $F(z)$ is a Z transform of sequence $f_n^{\{\gamma-1\}}$. Since

$$f_n^{\{\gamma-1\}} = f(n+\gamma-1) = \frac{(n+\gamma-1)^{(\gamma-1)}}{\Gamma(\gamma)} = \begin{bmatrix}\gamma\\ n\end{bmatrix}, \tag{4.26}$$

and (see [22, 23])

$$F\left(\begin{bmatrix}\gamma\\ n\end{bmatrix}\right) = \left(\frac{z-1}{z}\right)^{-\gamma}, \tag{4.27}$$

hence

$$R_{\gamma-1}\left(\frac{t^{(\gamma-1)}}{\Gamma(\gamma)}\right) = (z-1)^{-\gamma} = s^{-\gamma}, \quad (|1+s| < 1), \tag{4.28}$$

or

$$R_{\gamma-1}\left(\frac{t^{(\gamma-1)}}{\Gamma(\gamma)}\right) = \frac{1}{s^{\gamma}}, \quad (|1+s| < 1). \tag{4.29}$$

(2) The proof of (2) follows from the proof of (1). □

Definition 4.18. Define convolution of $h(t)$ and $g(t)$ as follows:

$$(h * g)(t) = \sum_{s=a}^{t-\gamma} h(t-\sigma(s))g(s). \tag{4.30}$$

If $h(t) = t^{(\gamma-1)}/\Gamma(\gamma)$, then

$$(h * g)_a(t) = \frac{1}{\Gamma(\gamma)}\sum_{s=a}^{t-\gamma}(t-\rho(s))^{(\gamma-1)}g(s) = {}_a\Delta^{-\gamma}g(t). \tag{4.31}$$

Theorem 4.19. *For any* $\gamma \in R \setminus \{\dots, -2, -1, 0\}$*, then*

$$R_{\gamma+a}(h * g) = R_{\gamma-1}(h) R_a(g). \tag{4.32}$$

Theorem 4.20. *Let* $\mu > 0$*,* $m - 1 < \mu \leq m \in N_1$*, and let* $f(t)$ *be defined in* $N_{\mu-m} = \{\mu - m, \mu - m + 1, \dots\}$*, then*

$$R_0(\Delta^{\mu} f(t))(s) = s^{\mu} R_{\mu-m}(f(t))(s) - \sum_{k=0}^{m-1} s^{m-k-1} \Delta^{k-m+\mu} f(t) \Big|_{t=0}. \tag{4.33}$$

Theorem 4.21. *Let* $\mu \in R \setminus \{\dots, -2, -1, 0\}$*,* $\gamma > 0$*, then*

$$\Delta^{-\gamma}\left(\frac{t^{(\mu)}}{\Gamma(\mu+1)}\right) = \frac{t^{(\mu+\gamma)}}{\Gamma(\mu+\gamma+1)}. \tag{4.34}$$

Theorem 4.22. *Let* f *be a real function,* $\mu, \gamma > 0$*, then for all* $t = \mu + \gamma \bmod (1)$*, one has*

$$\Delta^{-\gamma}\left[\Delta^{-\mu} f(t)\right] = \Delta^{-(\mu+\gamma)} f(t) = \Delta^{-\mu}\left[\Delta^{-\gamma} f(t)\right]. \tag{4.35}$$

5. The Solution of the Fractional Difference Equations with Real Variable

In this section, we give examples to demonstrate the solving method of fractional difference equations and reveal the inner relationship between fractional differential equations and fractional differential equations.

Theorem 5.1. *Let* $\mu \in R$*,* $\gamma \in R$*, then*

(1) $\nabla^{\gamma} t^{\overline{\mu}} = \mu^{(\gamma)} t^{\overline{\mu-\gamma}}$, $\Delta^{\gamma} t^{(\mu)} = \mu^{(\gamma)} t^{(\mu-\gamma)}$,

(2) $\Delta^{\gamma} t^{\overline{\mu}} = \mu^{(\gamma)} (t+\gamma)^{\overline{\mu-\gamma}}$, $\nabla^{\gamma} t^{(\mu)} = \mu^{(\gamma)} (t-\gamma)^{(\mu-\gamma)}$.

Proof. (1) The proof of (1) directly follows from Theorem 4.1 and Theorem 4.2.

(2) By Theorem 4.2 and (1), we have

$$\begin{aligned} \Delta^{\gamma} t^{\overline{\mu}} &= \nabla^{\gamma} (t+\gamma)^{\overline{\mu}} = \mu^{(\gamma)} (t+\gamma)^{\overline{\mu-\gamma}}, \\ \nabla^{\gamma} t^{(\mu)} &= \Delta^{\gamma} (t-\gamma)^{(\mu)} = \mu^{(\gamma)} (t-\gamma)^{(\mu-\gamma)}. \end{aligned} \tag{5.1}$$

□

Example 5.2. Consider Euler type fractional difference equations

$$t^{\overline{2\alpha}} \Delta^{2\alpha} x(t) + a t^{\overline{\alpha}} \Delta^{\alpha} x(t) + b x(t) = 0, \quad (0 < \alpha < 1). \tag{5.2}$$

Set $x(t) = t^{\overline{\gamma}}$, and take it into previous equation, we get

$$t^{\overline{2\alpha}}\gamma^{(2\alpha)}(t+2\alpha)^{\overline{\gamma-2\alpha}} + at^{\overline{\alpha}}\gamma^{(\alpha)}(t+\alpha)^{\overline{\gamma-\alpha}} + bt^{\overline{\gamma}} = 0. \tag{5.3}$$

By Theorem 4.1 (4), we obtain

$$\gamma^{(2\alpha)}t^{\overline{\gamma}} + a\gamma^{(\alpha)}t^{\overline{\gamma}} + bt^{\overline{\gamma}} = 0, \tag{5.4}$$

and get indicator equation

$$\gamma^{(2\alpha)} + a\gamma^{(\alpha)} + b = 0. \tag{5.5}$$

Therefore, we can transform Euler type fractional difference equations into its indicator equation.

Example 5.3. Consider initial value problem of homogeneous linear γ order ($0 < \gamma \leq 1$) fractional difference equation with constant coefficient

$$\begin{gathered}\nabla^{\gamma}y(t) + a\nabla^{0}y(t) = 0, \quad t \in N_0, \\ \nabla^{\gamma-1}(t)\Big|_{t=-1} = a_0.\end{gathered} \tag{5.6}$$

Note that $\nabla^{\gamma-1}y(t)$ is defined in $N_{-1} = \{-1, 0, 1, 2, \dots\}$, since

$$\begin{aligned}{}_{-1}\nabla^{\gamma-1}f(t)\Big|_{t=-1} &= \frac{1}{\Gamma(1-\gamma)}\sum_{s=-1}^{t}(t-\rho(s))^{\overline{-\gamma}}y(s) \\ &= \frac{1^{\overline{-\gamma}}}{\Gamma(1-\gamma)}y(-1) = y(-1).\end{aligned} \tag{5.7}$$

Therefore, initial problem of (5.6) is equivalent to initial problem

$$\begin{gathered}\nabla^{\gamma}y(t) + a\nabla^{0}y(t) = 0, \quad t \in N, \\ y(-1) = a_0.\end{gathered} \tag{5.8}$$

The solution of initial problem of (5.6) is equivalent to the solution of sum equations

$$y(t) = \frac{(t+1)^{\overline{\gamma-1}}}{\Gamma(\gamma)}a_0 + a\sum_{s=0}^{t}(t-\rho(s))^{\overline{\gamma-1}}y(s). \tag{5.9}$$

We use approximation method to solve these sum equations. Set

$$\begin{aligned} y_0(t) &= \frac{(t+1)^{\overline{\gamma-1}}}{\Gamma(\gamma)} a_0, \\ y_m(t) &= y_0(t) + \frac{a}{\Gamma(\gamma)} \sum_{s=0}^{t} (t-\rho(s))^{\overline{\gamma-1}} y_{m-1}(s) \\ &= y_0(t) + a\nabla^{-\gamma} y_{m-1}(t), \quad m = 1, 2, \ldots. \end{aligned} \tag{5.10}$$

Applying power law (Theorem 4.22), we get

$$y_1(t) = y_0(t) + a\nabla^{-\gamma} y_0(t) = a_0 \left(\frac{(t+1)^{\overline{\gamma-1}}}{\Gamma(\gamma)} + a \frac{(t+1)^{\overline{2\gamma-1}}}{\Gamma(2\gamma)} \right). \tag{5.11}$$

Applying power law repeatedly, and by recursion, we obtain

$$y_m(t) = a_0 \sum_{i=0}^{m} \frac{a^i t^{\overline{i\gamma+\gamma-1}}}{\Gamma((i+1)\gamma)}, \quad m = 0, 1, 2, \ldots. \tag{5.12}$$

Let $m \to \infty$, then

$$y(t) = a_0 \sum_{i=0}^{\infty} \frac{a^i (t+1)^{\overline{i\gamma+\gamma-1}}}{\Gamma((i+1)\gamma)} = a_0 \sum_{i=0}^{\infty} a^i \begin{bmatrix} i\gamma + \gamma \\ t \end{bmatrix}. \tag{5.13}$$

Example 5.4. Let $\gamma = 1/q$, $q \in N$, we call

$$\nabla^{\gamma} y(t) - a\nabla^{0} y(t) = 0, \quad t \in N_0, \tag{5.14}$$

the fractional difference equation of order $(1, q)$.

In order to solve this equation, we need to introduce some special functions.

Definition 5.5. Define function

$$\Lambda(t, \gamma, \lambda) = {}_a\nabla^{-\gamma} \lambda^t, \quad \gamma \in R, \tag{5.15}$$

where $t = a$ mod (1). Sometimes denote it $\Lambda(\gamma, \lambda)$ or $\Lambda(t, \gamma, \lambda; a)$.

In view of Theorems 4.2 and 4.3, we can establish the following theorem.

Theorem 5.6. *Assume the following function is well defined; then*

(1) $\Lambda(t, \gamma, \lambda) = (1 - 1/\lambda)\Lambda(t, \gamma + 1, \lambda) + (t - a + 1)^{\overline{\gamma}} / \Gamma(\gamma + 1)$,

(2) $\nabla\Lambda(t, \gamma + 1, \lambda) = \Lambda(t, \gamma, \lambda)$,

(3) $\nabla^{p}\Lambda(t, \gamma + t, \lambda) = \Lambda(t, \gamma, \lambda)$, *where* $p = 0, 1, 2, \ldots$,

(4) $\nabla^{\mu}\Lambda(t,\gamma,\lambda) = \Lambda(t,\gamma-\mu,\lambda)$, *where* $p-1<\mu\le p$,

(5) $\nabla^{-\mu}\Lambda(t,\gamma,\lambda) = \Lambda(t,\gamma+\mu,\lambda)$.

Now we will use the method of undetermined coefficients to solve Example 5.4. By Theorem 5.6, we notice that

$$\begin{gathered}
\nabla^{\gamma}\Lambda(t,0,\lambda) = \Lambda(t,-\gamma,\lambda), \\
\nabla^{\gamma}\Lambda(t,-\gamma,\lambda) = \Lambda(t,-2\gamma,\lambda), \\
\vdots \\
\nabla^{\gamma}\Lambda(t,-(q-2)\gamma,\lambda) = \Lambda(t,-(q-1)\gamma,\lambda), \\
\nabla^{\gamma}\Lambda(t,-(q-1)\gamma,\lambda) = \Lambda(t,-1,\lambda) = \left(1-\frac{1}{\lambda}\right)\Lambda(t,0,\lambda).
\end{gathered} \tag{5.16}$$

The significance of these applications is that if we apply the operator ∇^{γ} to

$$\Lambda(t,0,\lambda), \Lambda(t,-\gamma,\lambda), \dots, \Lambda(t,-(q-1)\gamma,\lambda), \tag{5.17}$$

then we get a cyclic permutation of the same functions. That is, no new functions are introduced. Therefore, we will choose a linear combination of these functions as a candidate for a solution of (5.14). Say

$$\begin{aligned}
y(t) = {} & b_0\Lambda(t,0,\lambda) + b_1\Lambda(t,-\gamma,\lambda) \\
& + \dots + b_{q-2}\Lambda(t,-(q-2)\gamma,\lambda) + b_{q-1}\Lambda(t,-(q-1)\gamma,\lambda).
\end{aligned} \tag{5.18}$$

Then

$$\begin{aligned}
\nabla^{\gamma}y(t) = {} & b_0\Lambda(t,-\gamma,\lambda) + b_1\Lambda(t,-2\gamma,\lambda) \\
& + \dots + b_{q-2}\Lambda(t,-(q-1)\gamma,\lambda) + b_{q-1}\left(1-\frac{1}{\lambda}\right)\Lambda(t,0,\lambda).
\end{aligned} \tag{5.19}$$

Taking $y(t)$, $\nabla^{\gamma}y(t)$ into the left side of (5.14), we obtain

$$\begin{aligned}
\nabla^{\gamma}y(t) - ay(t) = {} & \left[b_{q-1}\left(1-\frac{1}{\lambda}\right) - ab_0\right]\Lambda(t,0,\lambda) \\
& + (b_0 - ab_1)\Lambda(t,-\gamma,\lambda) + \cdots + (b_{q-2} - ab_{q-1})\Lambda(t,-(q-1)\gamma,\lambda).
\end{aligned} \tag{5.20}$$

In order to make the right side equate zero, set

$$b_k = c\alpha^{-k}, \quad (k = 1,2,\dots,q-1). \tag{5.21}$$

Then

$$b_{k-1} - ab_k = c\left(\alpha^{-k+1} - a\alpha^{-k}\right) = c\alpha^{-k}(\alpha - a). \tag{5.22}$$

If we let α be a root of the indicial equation

$$P(x) = x - a = 0, \tag{5.23}$$

or $\alpha = a$, then we have

$$b_{k-1} - ab_k = ca^{-k}P(a) = 0 \quad (k = 1, 2, \dots, q-1). \tag{5.24}$$

Since we also need

$$0 = b_{q-1}\left(1 - \frac{1}{\lambda}\right) - ab_0 = ac\left[\left(1 - \frac{1}{\lambda}\right)a^{-q} - 1\right], \tag{5.25}$$

so let us set

$$\left(1 - \frac{1}{\lambda}\right) = a^q, \quad \lambda = \frac{1}{1 - \alpha^q}. \tag{5.26}$$

Since c is an arbitrary number, set $c = a^{q-1}$, then

$$b_k = a^{q-1-k}. \tag{5.27}$$

Therefore, we obtain a solution of fractional difference of order $(1, q)$ as

$$\begin{aligned} y(t) &= \sum_{k=0}^{q-1} b_k \Lambda(t, -k\gamma, \lambda) \\ &= \sum_{k=0}^{q-1} a^{q-1-k} \Lambda\left(t, -k\gamma, \frac{1}{1 - a^q}\right) \triangleq \lambda_a(t). \end{aligned} \tag{5.28}$$

The fractional difference equation of order $(1, q)$ in Example 5.4 can be solved by the method of N_0 transform. Make N_1 transform to the following equation:

$$\nabla^\gamma y(t) - a\nabla^0 y(t) = 0. \tag{5.29}$$

We have

$$\begin{aligned} s^{\gamma} N_0(f(t)) - (1-s)^{-1} f(0) + a N_1(f(t)) &= 0, \\ N_1(f(t)) &= \sum_{t=1}^{\infty} (1-s)^{t-1} f(t) \\ &= \sum_{t=0}^{\infty} (1-s)^{t-1} f(t) - (1-s)^{-1} f(0). \end{aligned} \tag{5.30}$$

Taking them into previos equation, we get

$$s^{\gamma} N_0(f(t)) - (1-a)(1-s)^{-1} f(0) - a N_0(f(t)) = 0, \tag{5.31}$$

and we have

$$\begin{aligned} N_0(f(t)) &= (1-a) y(0) \frac{1}{(1-s)(s^{\gamma}-a)} \\ &= (1-a) y(0) \frac{\sum_{k=0}^{q-1} a^{q-1-k} s^{k\gamma}}{(1-s)(s^{\gamma}-a) \sum_{k=0}^{q-1} a^{q-1-k} s^{k\gamma}} \\ &= (1-a) y(0) \frac{\sum_{k=0}^{q-1} a^{q-1-k} s^{k\gamma}}{(1-s)(s-a^{q})}. \end{aligned} \tag{5.32}$$

In [23], we have the following

Theorem 5.7. *The following equality holds:*

(1) $N_0(\Lambda(t,0,\lambda)) = N_0(\lambda^t) = 1/(1-s) \cdot 1/(1-(1-s)\lambda)$,

(2) $N_0(\Lambda(t,-k\gamma,\lambda)) = N_0(\nabla^{k\gamma}\lambda^t) = 1/(1-s) \cdot s^{k\gamma}/(1-(1-s)\lambda) \cdot \;\; (k = 1,2,\ldots,q-1)$.

Set $\lambda = 1/(1-a^q)$, *then*

$$\begin{aligned} N_0 \Lambda\left(t, 0, \frac{1}{1-a^q}\right) &= \frac{1}{1-s} \cdot \frac{1-a^q}{s-a^q}, \\ N_0 \Lambda\left(t, -k\gamma, \frac{1}{1-a^q}\right) &= \frac{s^{k\gamma}}{1-s} \cdot \frac{1-a^q}{s-a^q}. \quad (k = 1,2,\ldots,q-1). \end{aligned} \tag{5.33}$$

By Theorem 5.7 and (5.33), *we know that*

$$y(t) = (1-a) y(0) \sum_{k=0}^{q-1} a^{q-1-k} \Lambda\left(t, -k\gamma, \frac{1}{1-a^q}\right) \tag{5.34}$$

is a solution of (5.14). *Except a constant, the solution* $y(t)$ *is the same as the solution* (5.28), *where*

$$y(t) = \lambda_a(t), \tag{5.35}$$

which is solved by the method of undetermined coefficients before.

6. Relationship between the Fractional Difference Equations and the Fractional Differential Equations

In this section, we only give an example to demonstrate the relationship between integers order difference equations and integral order differential equation.

Let us recall the definition of fractional sum when step $h = 1$

$$ {}_a\nabla_t^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \sum_{s=a}^{t} (t-\rho(s))^{\overline{\gamma-1}} f(s), \tag{6.1} $$

where $t \in N_a = \{a, a+1, a+2, \ldots\}$. If we set

$$ \begin{gathered} t-a = n \in N_0, \qquad s-a = r \in N_0, \\ f_r^{\{a\}} = f(r+a) = f(s), \qquad f_n^{\{a\}} = f(n+a) = f(t), \end{gathered} \tag{6.2} $$

then

$$ \begin{aligned} \frac{1}{\Gamma(\gamma)} \sum_{s=a}^{t} (t-\rho(s))^{\overline{\gamma-1}} f(s) &= \frac{1}{\Gamma(\gamma)} \sum_{s=a}^{n+a} (n+a-\rho(s))^{\overline{\gamma-1}} f(s) \\ &= \frac{1}{\Gamma(\gamma)} \sum_{r=0}^{n} (n+a-(r+a+1))^{\overline{\gamma-1}} f(r+a) \\ &= \frac{1}{\Gamma(\gamma)} \sum_{r=0}^{n} (n-r+1)^{\overline{\gamma-1}} f_r^{\{a\}} = {}_0\nabla_n^{-\gamma} f_n^{\{a\}}. \end{aligned} \tag{6.3} $$

And it is easy to prove that

$$ {}_a\nabla_t^{\mu} f(t) = {}_0\nabla_n^{\mu} f_n^{\{a\}}, \quad (\mu > 0). \tag{6.4} $$

Therefore, we have the following.

Theorem 6.1. *Let $t \in N_a$, and set $t-a = n \in N_0$, $f_n^{\{a\}} = f(n+a) = f(t)$, then*

$$ {}_a\nabla_t^{-\gamma} f(t) = {}_0\nabla_n^{-\gamma} f_n^{\{a\}}; \quad {}_a\nabla_t^{\mu} f(t) = {}_0\nabla_n^{\mu} f_n^{\{a\}}, \quad (\mu, \gamma > 0). \tag{6.5} $$

Example 6.2. (1) Set $\gamma = 1/q$, $q \in N, n \in N$, and solve the fractional difference equation of order $(1, q)$,

$$ \nabla^{\gamma} x(n) - \alpha x(n) = 0. \tag{6.6} $$

(2) Let $t \in R$, and solve the equation

$$ \nabla^{\gamma} x(t) - \alpha x(t) = 0. \tag{6.7} $$

(3) Let $h \in R^{+}, t \in R$, and solve the equation

$$\nabla_h^{\gamma} x(t) - \alpha x(t) = 0. \tag{6.8}$$

(4) If we let $h \to 0$, we ask whether the limit solution of (6.8) is equivalent to that of the following fractional differential equation? Consider

$$D^{\gamma} x(t) - \alpha x(t) = 0, \quad (t \in R). \tag{6.9}$$

Solution 1. (1) By a result in Chapter 7 of book [23], the solution of (6.6) is

$$x(n) = \lambda_{\alpha}(n) = \sum_{k=0}^{q-1} \alpha^{q-k-1} \Lambda_n \left[-k\gamma, \left(\frac{1}{1-\alpha^q} \right)^n \right]. \tag{6.10}$$

(2) Set $t - t_0 = n \in N_0$, and define

$$\begin{gathered} x_n^{\{t_0\}} = x(n + t_0) = x(t), \qquad x_{n-1}^{\{t_0\}} = x(n - 1 + t_0) = x(t-1), \dots, \\ x_0^{\{t_0\}} = x(0 + t_0) = x(t_0). \end{gathered} \tag{6.11}$$

Hence, we can regard the following $x(t_0), x(t_0 + 1), \dots, x(t), \dots$ as a sequence

$$x_0^{\{t_0\}}, x_1^{\{t_0\}}, \dots x_n^{\{t_0\}}, \dots. \tag{6.12}$$

Under this definition, (6.7) is actually equivalent to the following integer variable difference equation:

$$\nabla^{\gamma} x_n^{\{t_0\}} - \alpha x_n^{\{t_0\}} = 0. \tag{6.13}$$

By (1), we know that its solution is

$$\begin{aligned} x_n^{\{t_0\}} &= \sum_{k=0}^{q-1} \alpha^{q-k-1} \Lambda_n \left[-k\gamma, \left(\frac{1}{1-\alpha^q} \right)^n \right] \\ &= \sum_{k=0}^{q-1} \alpha^{q-k-1} \Lambda \left[t, -k\gamma, \left(\frac{1}{1-\alpha^q} \right)^t \right]. \end{aligned} \tag{6.14}$$

That is

$$x(t) = \sum_{k=0}^{q-1} \alpha^{q-k-1} \Lambda \left[t, -k\gamma, \left(\frac{1}{1-\alpha^q} \right)^t \right] \triangleq \lambda_{\alpha}(t). \tag{6.15}$$

(3) Set $t = sh, x(t) = x(sh) = y(s)$, then (6.8) is equivalent to

$$h^{-\gamma}\nabla^{\gamma}y(s) - \alpha y(s) = 0. \tag{6.16}$$

By (2), we obtain that the solution of (6.16) is

$$\begin{aligned} x(t) &= y(s) = \lambda_{\alpha h^{\gamma}}(s) \\ &= \sum_{k=0}^{q-1}(\alpha h^{\gamma})^{q-k-1}\Lambda\left[s, -k\gamma, \left(\frac{1}{1-(\alpha h^{\gamma})^{q}}\right)^{s}\right]. \end{aligned} \tag{6.17}$$

Since

$$\Lambda\left[s, -k\gamma, \left(\frac{1}{1-(\alpha h^{\gamma})^{q}}\right)^{s}\right] = h^{k\gamma}\Lambda_{h}\left[t, -k\gamma, \left(\frac{1}{1-(\alpha h^{\gamma})^{q}}\right)^{t/h}\right], \tag{6.18}$$

hence we have

$$x(t) = \sum_{k=0}^{q-1}(\alpha h^{\gamma})^{q-k-1}h^{k\gamma}\Lambda_{h}\left[t, -k\gamma, \left(\frac{1}{1-(\alpha h^{\gamma})^{q}}\right)^{t/h}\right]. \tag{6.19}$$

(4) Let $h \longrightarrow 0$, and since

$$\begin{gathered} \left(\frac{1}{1-\alpha^{q}h}\right)^{t/h} \longrightarrow e^{\alpha^{q}}, \\ h^{k\gamma}\Lambda_{h}\left[t, -k\gamma, \left(\frac{1}{1-\alpha^{q}h}\right)^{t/h}\right] = h^{k\gamma}\nabla_{h}^{k\gamma}\left(\frac{1}{1-\alpha^{q}h}\right)^{t/h} \longrightarrow D^{k\gamma}e^{\alpha^{q}} = E(-k\gamma, \alpha^{q}). \end{gathered} \tag{6.20}$$

We then obtain

$$x(t) = e_{\alpha}(t) = \sum_{k=0}^{q-1}\alpha^{q-k-1}E(-k\gamma, \alpha^{q}), \tag{6.21}$$

and this is exactly the solution of (6.9). (See Chapter 5 in monographer [2]).

Remark 6.3. If we take $\gamma = 1/2$, $q = 2$, then the followong occurs.

(1) The solution of (6.19) reduces to

$$\begin{aligned} x(t) &= \alpha\left(\frac{1}{1-\alpha^{2}}\right)^{t} + \nabla^{1/2}\left(\frac{1}{1-\alpha^{2}}\right)^{t} \\ &= \alpha F\left(t, 0, \frac{1}{1-\alpha^{2}}\right) + F\left(t, -\frac{1}{2}, \frac{1}{1-\alpha^{2}}\right), \end{aligned} \tag{6.22}$$

and this result is consistent with the solution (5.28) or (5.34) in Example 5.4 in Section 5.

(2) The solution (6.21) reduces to

$$x(t) = \alpha h^{1/2}\left(\frac{1}{1-\alpha^q h}\right)^{t/h} + h^{1/2}\nabla_h^{1/2}\left(\frac{1}{1-\alpha^q h}\right)^{t/h}$$
$$= h^{1/2}\left[\alpha\left(\frac{1}{1-\alpha^q h}\right)^{t/h} + \nabla_h^{1/2}\left(\frac{1}{1-\alpha^q h}\right)^{t/h}\right]. \tag{6.23}$$

Let $h \to 0$, then

$$\left[\alpha\left(\frac{1}{1-\alpha^q h}\right)^{t/h} + \nabla_h^{1/2}\left(\frac{1}{1-\alpha^q h}\right)^{t/h}\right] \tag{6.24}$$

tend to

$$\alpha e^{\alpha^q t} + D^{1/2}e^{\alpha^q t} = e_\alpha(t). \tag{6.25}$$

The results perfectly coincide with the monographer [2].

From Theorem 6.1, we see that if we take t as $a, a+1, a+2, \ldots$, it is only a sequence with step 1, but the initial time is not zero but a. If we make a translation variable transformation, set $t = n + a$, $n \in N_0$, then we can change the definition of fractional sum and fractional difference with real variable into the definition of fractional sum and difference with integer variable. But, no doubt, it will be more convenient for us to study fractional sum and difference with integer variable.

7. Conclusion

This work reveals some results in discrete fractional calculus and fractional h-difference equations. This study also provides a reference for researchers in this area. First, this paper gives the definition of the fractional h-difference from the difference of integer order. Then some integral transforms are proposed, that is, Z transform, N transform, and R transform. These integral transforms are applied to linear fractional h-difference equations, and approximate solutions are obtained. At last, the study explains the relationship between the fractional difference equations and the fractional differential equations.

Acknowledgment

The authors wish to thank the referee for the very careful reading of the manuscript and fruitful comments and suggestions.

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