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Research Article

A Fictitious Play Algorithm for Matrix Games with Fuzzy Payoffs

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Fuzzy matrix games, specifically two-person zero-sum games with fuzzy payoffs, are considered. In view of the parametric fuzzy max order relation, a fictitious play algorithm for finding the value of the game is presented. A numerical example to demonstrate the presented algorithm is also given.

1. Introduction

Game theory is a mathematical discipline which studies situations of competition and cooperation between several involved parties, and it has many applications in broad areas, such as strategic warfare, economic or social problems, animal behaviour, and political voting systems.

The simplest game is a finite, two-person, zero-sum game. There are only two players, player I and player II and it can be denoted by a matrix. Thus, such a game is called a *matrix game*. More formally, a matrix game is an $m \times n$ matrix G of real numbers. A (mixed) *strategy* of player I is a probability distribution x over the rows of G, that is, an element of the set

$$X_m = \left\{ x = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_i \geq 0 \ \forall i = 1, \ldots, m, \ \sum_{i=1}^{m} x_i = 1 \right\}. \tag{1.1}$$

Similarly, a strategy of player II is a probability distribution y over the columns of G, that is, an element of the set

$$Y_n = \left\{ y = (y_1, \ldots, y_n) \in \mathbb{R}^n : y_i \geq 0 \ \forall i = 1, \ldots, n, \ \sum_{i=1}^{n} y_i = 1 \right\}. \tag{1.2}$$

A strategy x of player I is called *pure* if it does not involve probability, that is, $x_i = 1$ for some $i = 1, \ldots, m$ and it is denoted by I_i. Similarly, pure strategies of player II are denoted by II_j for $j = 1, \ldots, n$.

If player I plays row i (i.e., pure strategy $x = (0, 0, \ldots, x_i = 1, 0, \ldots, 0)$) and player II plays column j (i.e., pure strategy $y = (0, 0, \ldots, y_j = 1, 0, \ldots, 0)$), then player I receives payoff g_{ij} and player II pays g_{ij}, where g_{ij} is the entry in row i and column j of matrix G. If player I plays strategy x and player II plays strategy y, then player I receives the expected payoff

$$g(x, y) = x^{\mathrm{T}} G y, \tag{1.3}$$

where x^{T} denotes the transpose of x.

A strategy x^* is called *maximin strategy* of player I in matrix game G if

$$\min\left\{(x^*)^{\mathrm{T}} G y, y \in Y_n\right\} \geq \min\left\{x^{\mathrm{T}} G y, y \in Y_n\right\}, \tag{1.4}$$

for all $x \in X_m$ and a strategy y^* is called *minimax strategy* of player II in matrix game G if

$$\max\left\{x^{\mathrm{T}} G y^*, x \in X_m\right\} \leq \max\left\{x^{\mathrm{T}} G y, x \in X_m\right\} \tag{1.5}$$

for all $y \in Y_n$. Therefore, a maximin strategy of player I maximizes the minimal payoff of player I, and a minimax strategy of player II minimizes the maximum that player II has to pay to player I.

von Neumann and Morgenstern (see [1]) proved that for every matrix game G there is a real number v with the following properties.

(i) A strategy x of player I guarantees a payoff of at least v to player I (i.e., $x^{\mathrm{T}} G y \geq v$ for all strategies y of player II) if and only if x is a maximin strategy.

(ii) A strategy y of player II guarantees a payment of at most v by player II to player I (i.e., $x^{\mathrm{T}} G y \leq v$ for all strategies x of player I) if and only if y is a minimax strategy.

Hence, player I can obtain a payoff at least v by playing a maximin strategy, and player II can guarantee to pay not more than v by playing a minimax strategy. For these reasons, the number v is also called the *value* of the game G.

A position (i, j) is called a saddle point if $g_{ij} \geq g_{kj}$ for all $k = 1, \ldots, m$ and $g_{ij} \leq g_{il}$ for all $l = 1, \ldots, n$, that is, if g_{ij} is maximal in its column j and minimal in its row i. Evidently, if (i, j) is a saddle point, then g_{ij} must be the value of the game.

2. Fuzzy Numbers and a Two-Person Zero-Sum Game with Fuzzy Payoffs

In the classical theory of zero sum games the payoffs are known with certainty. However, in the real world the certainty assumption is not realistic on many occasions. This lack of precision may be modeled via fuzzy logic. In this case, payoffs are presented by fuzzy numbers.

2.1. Fuzzy Numbers

In this section, we give certain essential concepts of fuzzy numbers and their basic properties. For further information see [2, 3].

A *fuzzy set* $\widetilde{A}$ on a set X is a function $\widetilde{A} : X \to [0,1]$. Generally, the symbol $\mu_{\widetilde{A}}$ is used for the function $\widetilde{A}$ and it is said that the fuzzy set $\widetilde{A}$ is characterized by its *membership function* $\mu_{\widetilde{A}} : X \to [0,1]$ which associates with each $x \in X$, a real number $\mu_{\widetilde{A}}(x) \in [0,1]$. The value of $\mu_{\widetilde{A}}(x)$ is interpreted as the degree to which x belongs to $\widetilde{A}$.

Let $\widetilde{A}$ be a fuzzy set on X. The *support* of $\widetilde{A}$ is given as

$$S\left(\widetilde{A}\right) = \left\{x \in X : \mu_{\widetilde{A}}(x) > 0\right\}, \tag{2.1}$$

and the *height* $\mathrm{h}(\widetilde{A})$ of $\widetilde{A}$ is defined as

$$\mathrm{h}\left(\widetilde{A}\right) = \sup_{x \in X} \mu_{\widetilde{A}}(x). \tag{2.2}$$

If $\mathrm{h}(\widetilde{A}) = 1$, then the fuzzy set $\widetilde{A}$ is called a *normal* fuzzy set.

Let $\widetilde{A}$ be a fuzzy set on X and $\alpha \in [0,1]$. The *α-cut* (α-level set) of the fuzzy set $\widetilde{A}$ is given by

$$\left[\widetilde{A}\right]^{\alpha} = \begin{cases} \left\{x \in X : \mu_{\widetilde{A}}(x) \geq \alpha\right\}, & \text{if } \alpha \in (0,1] \\ \operatorname{cl} S\left(\widetilde{A}\right), & \text{if } \alpha = 0, \end{cases} \tag{2.3}$$

where cl denotes the closure of sets.

The notion of convexity is extended to fuzzy sets on $\mathbb{R}^n$ as follows. A fuzzy set $\widetilde{A}$ on $\mathbb{R}^n$ is called a *convex fuzzy set* if its α-cuts $\widetilde{A}_\alpha$ are convex sets for all $\alpha \in [0,1]$.

Let $\widetilde{A}$ be a fuzzy set in $\mathbb{R}$, then $\widetilde{A}$ is called a *fuzzy number* if

(i) $\widetilde{A}$ is normal,

(ii) $\widetilde{A}$ is convex,

(iii) $\mu_{\widetilde{A}}$ is upper semicontinuous, and

(iv) the support of $\widetilde{A}$ is bounded.

From now on, we will use lowercase letters to denote fuzzy numbers such as $\widetilde{a}$ and we will denote the set of all fuzzy numbers by the symbol $\mathbb{F}$. Generally, some special type of fuzzy numbers, such as trapezoidal and triangular fuzzy numbers, are used for real life applications. We consider here L-fuzzy numbers.

The function $L : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions is called a *shape function*:

(i) L is even function, that is, $L(x) = L(-x)$ for all $x \in \mathbb{R}$,

(ii) $L(x) = 1 \Leftrightarrow x = 0$,

(iii) $L(\cdot)$ is nonincreasing on $[0, +\infty)$,

(iv) if $x_0 = \inf\{x > 0 \mid L(x) = 0\}$, then $0 < x_0 < +\infty$ and x_0 is called the *zero point* of L.

Let a be any number and let δ be any positive number. Let L be any shape function. Then a fuzzy number $\tilde{a}$ is called an *L-fuzzy number* if its membership function is given by

$$\mu_{\tilde{a}}(x) = L\left(\frac{x-a}{\delta}\right) \vee 0, \quad x \in \mathbb{R}. \tag{2.4}$$

Here, $x \vee y = \max\{x, y\}$. Real numbers a and δ are called the *center* and the *deviation parameter* of $\tilde{a}$, respectively. In particular, if $L(x) = 1 - |x|$ we get

$$\mu_{\tilde{a}}(x) = \begin{cases} 1 - \dfrac{1}{\delta}|x-a|, & x \in [a-\delta, a+\delta] \\ 0, & \text{otherwise} \end{cases} \tag{2.5}$$

and $\tilde{a}$ is called a *symmetric triangular fuzzy* number.

It is clear that for any shape function L, an arbitrary L-fuzzy number $\tilde{a}$ can be characterized by the its center a and the deviation parameter δ. Therefore, we denote the L-fuzzy number $\tilde{a}$ by $\tilde{a} \equiv (a, \delta)_L$. In particular, if $\tilde{a}$ is a symmetric triangular fuzzy number, we write $\tilde{a} \equiv (a, \delta)_T$. We also denote the set of all L-fuzzy numbers by $\mathbb{F}_L$.

Let $\tilde{a} \equiv (a, \delta)_L$ be an L-fuzzy number then by (2.4) we see that the graph of $\mu_{\tilde{a}}(x)$ approaching line $x = a$ as δ tends to zero from the right. Therefore, we can write that

$$\mu_{\tilde{a}}(x) = \begin{cases} 1, & x = a \\ 0, & x \neq a. \end{cases} \tag{2.6}$$

The function in (2.6) is just a characteristic function of the real number a. Hence, we get $\mathbb{R} \subset \mathbb{F}_L$. From now on, we will call a fuzzy number $\tilde{a}$ as an L-fuzzy number if its membership function is given by (2.4) or (2.6).

Let $\tilde{a}, \tilde{b} \in \mathbb{F}$ and k be any real number. Then the sum of fuzzy numbers $\tilde{a}$ and $\tilde{b}$ and the scalar product of k and $\tilde{a}$ are defined as

$$\begin{gathered} \mu_{\tilde{a}+\tilde{b}}(z) = \sup_{z=x+y} \min\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\}, \\ \mu_{k\tilde{a}}(z) = \max\left\{0, \sup_{z=kx} \mu_{\tilde{a}}(x)\right\}, \end{gathered} \tag{2.7}$$

respectively. In particular, if $\tilde{a} \equiv (a, \delta_1)_L$ and $\tilde{b} \equiv (b, \delta_2)_L$ are L-fuzzy numbers and k is any real number, then one can verify that

$$\begin{gathered} \tilde{a} + \tilde{b} \equiv (a + b, \delta_1 + \delta_2)_L, \\ k\tilde{a} \equiv (ka, |k|\delta_1)_L. \end{gathered} \tag{2.8}$$

Let $\tilde{a}$ be any L-fuzzy number. By the definition of the α-cut, $[\tilde{a}]^\alpha$ is a closed interval for all $\alpha \in [0, 1]$. Therefore, for all $\alpha \in [0, 1]$ we can denote the α-cut of $\tilde{a}$ by $[a_\alpha^L, a_\alpha^R]$, where a_α^L and a_α^R are end points of the closed interval $[\tilde{a}]^\alpha$.

For any symmetric triangular fuzzy numbers $\tilde{a}, \tilde{b}$ Ramík and Římánek (see [4]) introduced binary relations as follows:

$$\begin{aligned} \tilde{a} \geqq q\tilde{b} &\Longleftrightarrow a_\alpha^L \geqq b_\alpha^L, a_\alpha^R \geqq b_\alpha^R \quad \forall \alpha \in [0,1] \text{ (fuzzy max order)}, \\ \tilde{a} \succeq \tilde{b} &\Longleftrightarrow \tilde{a} \geqq q\tilde{b}, \tilde{a} \neq \tilde{b} \quad \forall \alpha \in [0,1] \text{ (strict fuzzy max order)}, \\ \tilde{a} \succ \tilde{b} &\Longleftrightarrow a_\alpha^L > b_\alpha^L, a_\alpha^R > b_\alpha^R \quad \forall \alpha \in [0,1] \text{ (strong fuzzy max order)}. \end{aligned} \tag{2.9}$$

Following theorem is a useful tool to check fuzzy max order and strong fuzzy max order relations between symmetric triangular fuzzy numbers.

Theorem 2.1 (see [5]). *Let $\tilde{a} \equiv (a, \delta_1)$ and $\tilde{b} \equiv (b, \delta_2)$ be any symmetric triangular fuzzy numbers. Then the statements*

$$\begin{aligned} \tilde{a} \geqq q\tilde{b} &\Longleftrightarrow |\delta_1 - \delta_2| \leqq a - b, \\ \tilde{a} \succ \tilde{b} &\Longleftrightarrow |\delta_1 - \delta_2| < a - b \end{aligned} \tag{2.10}$$

hold.

It is not difficult to check that the fuzzy max order is a partial order. Then we may have many minimal and maximal points with respect to fuzzy max order. Therefore, use of the fuzzy max order is not so efficient in computer algorithms. Furukawa introduced a total order relation which is a modification of the fuzzy max order with a parameter (see [5, 6]).

Let $0 \leq \lambda \leq 1$ be arbitrary but a fixed real number. For any L-fuzzy numbers $\tilde{a} \equiv (a, \delta_1)_L$ and $\tilde{b} \equiv (b, \delta_2)_L$ we define an order relation with parameter λ by

$$\tilde{a} \leq_\lambda \tilde{b} \Longleftrightarrow \begin{cases} \text{(i) } x_0|\delta_1 - \delta_2| \leq b - a, \\ \text{or} \\ \text{(ii) } \lambda x_0|\delta_1 - \delta_2| \leq b - a < x_0|\delta_1 - \delta_2|, \\ \text{or} \\ \text{(ii) } |a - b| < \lambda x_0|\delta_1 - \delta_2|, \delta_2 > \delta_1, \end{cases} \tag{2.11}$$

where x_0 is the zero point of L. The simple expression of (2.11) is as follows:

$$\tilde{a} \leq_\lambda \tilde{b} \Longleftrightarrow \begin{cases} \text{(i) } \lambda x_0\delta_1 + a < \lambda x_0\delta_2 + b, \\ \text{or} \\ \text{(ii) } \lambda x_0\delta_1 + a = \lambda x_0\delta_2 + b,\ \delta_2 \leq \delta_1. \end{cases} \tag{2.12}$$

It is clear that for any L-fuzzy numbers $\tilde{a} \equiv (a, \delta_1)_L$ and $\tilde{b} \equiv (b, \delta_2)_L$, $\tilde{a} \leq_0 \tilde{b}$ if and only if $a \leq b$. Therefore, the relation $\leq_0$ is the order among the centers of L-fuzzy numbers. On the other hand, $\tilde{a} \leq_1 \tilde{b}$ if and only if $\tilde{b} \geqq \tilde{a}$ or they are incomparable and $\delta_2 > \delta_1$. For $0 < \lambda < 1$, the relation $\leq_\lambda$ determines the order with respect to their values of center and their size of

ambiguity. The smaller λ is, the larger the possibility of ordering by the value of center is, and the larger λ is, the larger the possibility of ordering by the size of ambiguity is.

Theorem 2.2 (see [5]). *For every shape function L and for each $\lambda \in [0,1]$, the relation $\leq_\lambda$ is a total order relation on $\mathbb{F}_L$.*

Let $\lambda \in [0,1]$ be fixed arbitrarily and let $\widetilde{V} = (\widetilde{v}_1, \widetilde{v}_2, \ldots, \widetilde{v}_n)$ be any L-fuzzy vector, that is, all components of $\widetilde{V}$ are L-fuzzy numbers and expressed by a common shape function L. Then maximum and minimum of $\widetilde{V}$ in the sense of the total order $\leq_\lambda$ are denoted as

$$\operatorname*{Max}_{\lambda} \widetilde{V}, \qquad \operatorname*{Min}_{\lambda} \widetilde{V}, \tag{2.13}$$

respectively.

Example 2.3. Let $\widetilde{V} = ((-2,0.1)_L, (0,0.1)_L, (-3,0.3)_L, (-1,0.4)_L)$ be L-fuzzy vector. Then

$$\operatorname*{Max}_{\lambda}\left(\widetilde{V}\right) = (0,0.1)_L, \qquad \operatorname*{Min}_{\lambda}\left(\widetilde{V}\right) = (-3,0.1)_L, \tag{2.14}$$

for all $\lambda \in [0,1]$.

Let $\widetilde{a}$ and $\widetilde{b}$ be any L-fuzzy numbers, then the *Hausdorff distance* between $\widetilde{a}$ and $\widetilde{b}$ is defined as

$$d\left(\widetilde{a}, \widetilde{b}\right) = \sup_{\alpha \in [0,1]} \max\left\{\left|a_\alpha^L - b_\alpha^L\right|, \left|a_\alpha^R - b_\alpha^R\right|\right\}, \tag{2.15}$$

that is, $d(\widetilde{a}, \widetilde{b})$ is the maximal distance between α-cuts of $\widetilde{a}$ and $\widetilde{b}$. In particular, if $\widetilde{a} \equiv (a, \delta_1)$ and $\widetilde{b} \equiv (b, \delta_2)$ are any symmetric triangular fuzzy numbers, then $d(\widetilde{a}, \widetilde{b}) = |a - b|$.

2.2. Two-Person Zero-Sum Game with Fuzzy Payoffs and Its Equilibrium Strategy

In this section, we consider zero-sum games with fuzzy payoffs with two players, and we assume that player I tries to maximize the profit and player II tries to minimize the costs.

The two-person zero-sum game with fuzzy payoffs is defined by $m \times n$ matrix $\widetilde{G}$ whose entries are fuzzy numbers. Let $\widetilde{G}$ be a fuzzy matrix game

$$\widetilde{G} = \begin{array}{c} \\ \mathrm{I}_1 \\ \mathrm{I}_2 \\ \vdots \\ \mathrm{I}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{II}_1 & \mathrm{II}_2 & \cdots & \mathrm{II}_n \end{array} \\ \begin{pmatrix} \widetilde{g}_{11} & \widetilde{g}_{12} & \cdots & \widetilde{g}_{1n} \\ \widetilde{g}_{21} & \widetilde{g}_{22} & \cdots & \widetilde{g}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{g}_{m1} & \widetilde{g}_{m2} & \cdots & \widetilde{g}_{mn} \end{pmatrix} \end{array} \tag{2.16}$$

and $x \in X_m$, $y \in Y_n$, that is, x and y are strategies for players I and II. Then the expected payoff for player I is defined by

$$\tilde{g}(x,y) = x^{\mathrm{T}}\tilde{G}y = \sum_i \sum_j x_i y_j \tilde{g}_{ij}. \tag{2.17}$$

Example 2.4. Let

$$\tilde{G} = \begin{array}{c} \\ \mathrm{I}_1 \\ \mathrm{I}_2 \\ \mathrm{I}_3 \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{II}_1 & \mathrm{II}_2 & \mathrm{II}_3 & \mathrm{II}_4 \end{array} \\ \begin{pmatrix} (-10,0.1)_T & (-8,0.7)_T & (-8,0.7)_T & (-6,0.7)_T \\ (-9,0.8)_T & (-1,0.7)_T & (3,0.8)_T & (8,0.9)_T \\ (-3,0.2)_T & (-1,0.5)_T & (-2,0.2)_T & (-3,0.7)_T \end{pmatrix} \end{array} \tag{2.18}$$

be a fuzzy matrix game whose entries are symmetric triangular fuzzy numbers.

For this game, if player I plays second row ($x = (0,1,0)$) and player II plays third column ($y = (0,0,1,0)$), then player I receives and correspondingly player II pays a payoff $\tilde{g}(\mathrm{I}_2, \mathrm{II}_3) \equiv (3,0.8)_T$. On the other hand, for a pair of strategies $x = (1/2,1/2,0)$ and $y = (0,1/3,1/3,1/3)$ the expected payoff for player I is $\tilde{g}(x,y) = (-2,3/4)_T$.

Now, we define three types of *minimax equilibrium strategies* based on the fuzzy max order relation (see [7]). A point $(x^*, y^*) \in X_m \times Y_n$ is said to be a minimax equilibrium strategy to game $\tilde{G}$ if relations

$$\begin{aligned} x^{*\mathrm{T}}\tilde{A}y^* &\geqq qx^{\mathrm{T}}\tilde{A}y^*, \quad \forall x \in X_m, \\ x^{*\mathrm{T}}\tilde{A}y &\geqq qx^{*\mathrm{T}}\tilde{A}y^*, \quad \forall y \in Y_n \end{aligned} \tag{2.19}$$

hold.

If $(x^*, y^*) \in X_m \times Y_n$ is the minimax equilibrium strategy to game $\tilde{G}$, then a point $\tilde{v} = x^{*\mathrm{T}}\tilde{A}y^*$ is said to be the (fuzzy) *value of game* $\tilde{G}$ and the triplet $(x^*, y^*, \tilde{v})$ is said to be a *solution of game* $\tilde{G}$ under the fuzzy max order "$\geqq$".

A point $(x^*, y^*) \in X_m \times Y_n$ is said to be a *nondominated minimax equilibrium strategy* to game $\tilde{G}$ if

(i) there is no $x \in X_m$ such that $x^{\mathrm{T}}\tilde{A}y^* \succeq x^{*\mathrm{T}}\tilde{A}y^*$,

(ii) there is no $y \in Y_n$ such that $x^{*\mathrm{T}}\tilde{A}y^* \succeq x^{*\mathrm{T}}\tilde{A}y$

hold.

A point $(x^*, y^*) \in X_m \times Y_n$ is said to be a *weak nondominated minimax equilibrium strategy* to game $\tilde{G}$ if

(i) there is no $x \in X_m$ such that $x^{\mathrm{T}}\tilde{A}y^* \succ x^{*\mathrm{T}}\tilde{A}y^*$,

(ii) there is no $y \in Y_n$ such that $x^{*\mathrm{T}}\tilde{A}y^* \succ x^{*\mathrm{T}}\tilde{A}y$

hold.

By the above definitions, it is clear that if $(x^*, y^*) \in X_m \times Y_n$ is a minimax equilibrium strategy to game $\widetilde{G}$, it is a nondominated minimax equilibrium strategy, and if $(x^*, y^*) \in X_m \times Y_n$ is a nondominated minimax equilibrium strategy to game $\widetilde{G}$, then it is a weak nondominated minimax strategy.

Furthermore, if $\widetilde{G}$ is crisp, that is, game $\widetilde{G}$ is a two-person zero-sum matrix game, then these definitions coincide and become the definition of the saddle point.

3. The Fictitious Play Algorithm

The solution of matrix games with fuzzy payoffs has been studied by many authors. Most solution techniques are based on linear programming methods (see [3, 8–11] and references therein).

The Fictitious Play Algorithm is a common technique to approximate calculations for the value of a two-person zero-sum game. In this algorithm, the players choose their strategies in each step k assuming that the strategies of the other players in step k correspond to the frequency with which the various strategies were applied in the previous $k-1$ steps. First, Brown (see [12]) conjectured and Robinson (see [13]) proved the convergence of this method for matrix games. This method has also been adapted to interval valued matrix games (see [14]).

Let $G = (g_{ij})$ be $m \times n$ matrix. g_i^r will denote the ith row of G and g_j^c is the jth column.

A system (U, V) consisting of a sequence of n-dimensional vectors $U_0, U_1, \ldots$ and a sequence of m-dimensional vectors $V_0, V_1, \ldots$ is called a vector system for G provided that

(i) $\min U_0 = \max V_0$,

(ii) $U_{k+1} = U_k + g_{i(k)}^r$, $V_{k+1} = V_k + g_{j(k)}^c$,

where

$$v_{i(k)} = \max V_k, \qquad u_{j(k)} = \min U_k. \tag{3.1}$$

Here, $v_{i(k)}$ and $u_{j(k)}$ denote the $i(k)$th and $j(k)$th components of the vectors V and U, respectively.

Theorem 3.1 (see [13]). *If (U, V) is a vector system for G and ν is the value of G, then*

$$\lim_{k\to\infty} \frac{\min U_k}{k} = \lim_{k\to\infty} \frac{\max V_k}{k} = \nu. \tag{3.2}$$

Now, in view of Furukawa's parametric total order relation we will adapt this method for two-person zero-sum games with fuzzy payoffs.

Let

$$\widetilde{G} = \begin{array}{c} \\ \mathrm{I}_1 \\ \mathrm{I}_2 \\ \vdots \\ \mathrm{I}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{II}_1 & \mathrm{II}_2 & \cdots & \mathrm{II}_n \end{array} \\ \begin{pmatrix} \widetilde{g}_{11} & \widetilde{g}_{12} & \cdots & \widetilde{g}_{1n} \\ \widetilde{g}_{21} & \widetilde{g}_{22} & \cdots & \widetilde{g}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{g}_{m1} & \widetilde{g}_{m2} & \cdots & \widetilde{g}_{mn} \end{pmatrix} \end{array} \tag{3.3}$$

be a fuzzy matrix game whose entries are L-fuzzy numbers expressed by a common shape function L. Then a vector system $(\tilde{U}, \tilde{V})$ for fuzzy matrix $\tilde{G}$ is expressed as follows.

Definition 3.2. Let $\lambda \in [0,1]$ be fixed. Then for all $k \in \mathbb{N}$ a pair $(\tilde{U}, \tilde{V})$ consisting of n-dimensional L-fuzzy vectors $\tilde{U}_k = (\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n)$ and m-dimensional L-fuzzy vectors $\tilde{V}_k = (\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_m)$ provided that

$$\begin{gathered} \operatorname*{Min}_{\lambda}\left(\tilde{U}_0\right) = \operatorname*{Max}_{\lambda}\left(\tilde{V}_0\right), \\ \tilde{U}_{k+1} = \tilde{U}_k + \tilde{g}^r_{i(k)}, \qquad \tilde{V}_{k+1} = \tilde{V}_k + \tilde{g}^c_{j(k)}, \end{gathered} \tag{3.4}$$

is called a vector system for fuzzy matrix $\tilde{G}$. Here, $i(k)$ and $j(k)$ satisfy

$$\tilde{v}_{i(k)} = \operatorname*{Max}_{\lambda}\left(\tilde{V}_k\right), \qquad \tilde{u}_{j(k)} = \operatorname*{Min}_{\lambda}\left(\tilde{U}_k\right), \tag{3.5}$$

where $\tilde{g}^r_i$ and $\tilde{g}^c_j$ denote the ith row and jth column of $\tilde{G}$, respectively.

Instead of defining $\tilde{U}_k$ and $\tilde{V}_k$ simultaneously, a new vector system can be obtained by changing the condition on j as $\tilde{u}_{j(k+1)} = \mathrm{Min}_{\lambda}(\tilde{U}_{k+1})$. In numerical calculations, the latter converges more rapidly than the former.

Now, we can state our main theorem, the proof of which resembles the proof of the theorem given in [13, 14].

Theorem 3.3. *Let $\tilde{G}$ be an $m \times n$ fuzzy matrix game whose entries are L-fuzzy numbers expressed by a common shape function L and let $\tilde{v}$ be the value of $\tilde{G}$. If $(\tilde{U}, \tilde{V})$ is a vector system for $\tilde{G}$ and $\lambda \in [0,1]$ fixed, then*

$$\lim_{k\to\infty} \frac{\mathrm{Max}_{\lambda}\left(\tilde{V}_k\right)}{k} = \lim_{k\to\infty} \frac{\mathrm{Min}_{\lambda}\left(\tilde{U}_k\right)}{k} = \tilde{v}. \tag{3.6}$$

Here, the convergence is with respect to the Hausdorff metric on $\mathbb{F}$.

4. A Numerical Example

The best way to demonstrate the Brown-Robinson method for fuzzy matrix games is by means of an example.

Let us consider the modified example of Collins and Hu (see [15]). This shows an investor making a decision as to how to invest a nondivisible sum of money when the economic environment may be categorized into a finite number of states. There is no guarantee that any single value (return on the investment) can adequately model the payoff for any one of the economic states. Hence, it is more realistic to assume that each payoff is a fuzzy number. For this example, it is assumed that the decision of such an investor can be modeled under the assumption that the economic environment (or nature) is, in fact, a rational "player" that will choose an optimal strategy. Suppose that the options for this

player are strong economic growth, moderate economic growth, no growth or shrinkage, and negative growth. For the investor player the options are to invest in bonds, invest in stocks, and invest in a guaranteed fixed return account. In this case, clearly a single value for the payoff of either investment in bonds or stock cannot be realistically modeled by a single value representing the percent of return. Hence, a game matrix with fuzzy payoffs better represents the view of the game from both players' perspectives. Consider then the following fuzzy matrix game for this scenario, where the percentage of return is represented in decimal form:

$$\tilde{G} = \begin{array}{c} \\ \text{Strong} \\ \text{Moderate} \\ \text{None} \\ \text{Negative} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Bonds} & \text{Stocks} & \text{Fixed} \end{array} \\ \begin{pmatrix} (0.1230, 0.1300)_T & (0.1420, 0.1700)_T & (0.0450, 0)_T \\ (0.1025, 0.1950)_T & (0.0310, 0.0110)_T & (0.0450, 0)_T \\ (0.0555, 0.0065)_T & (0.0310, 0.0110)_T & (0.0450, 0)_T \\ (0.0260, 0.0040)_T & (-0.1250, 0.0275)_T & (0.0450, 0)_T \end{pmatrix} \end{array}. \tag{4.1}$$

We choose $\lambda = 0.5$ and we first assume that $\tilde{U}_0 = ((0, 0.1)_T, (0, 0.1)_T, (0, 0.1)_T)$ and $\tilde{V}_0 = ((0, 0.1)_T, (0, 0.1)_T, (0, 0.1)_T, (0, 0.1)_T)$. Then $\mathrm{Min}_\lambda(\tilde{U}_0) = \mathrm{Max}_\lambda(\tilde{V}_0) = (0, 0.1)_T$.

In the next step ($k = 1$), since all components are the same, we can choose $i(1)$ and $j(1)$ as any integer from 1 to 3 and from 1 to 4, respectively. If we choose $i(1) = 1$ and $j(1) = 1$, then we find

$$\begin{aligned}
\tilde{U}_1 = \tilde{U}_0 + \tilde{g}^r_{i(1)} &= ((0, 0.1)_T, (0, 0.1)_T, (0, 0.1)_T) \\
&\quad + ((0.1230, 0.1300)_T, (0.1420, 0.1700)_T, (0.0450, 0)_T) \\
&= ((0.1230, 0.2300)_T, (0.1420, 0.2700)_T, (0.0450, 0.1)_T), \\
\tilde{V}_1 = \tilde{V}_0 + \tilde{g}^c_{j(1)} &= ((0, 0.1)_T, (0, 0.1)_T, (0, 0.1)_T, (0, 0.1)_T) \\
&\quad + ((0.1230, 0.1300)_T, (0.1025, 0.1950)_T, (0.0555, 0.0065)_T, (0.0260, 0.0040)_T) \\
&= ((0.1230, 0.2300)_T, (0.1025, 0.2950)_T, (0.0555, 0.1065)_T, (0.0260, 0.1040)_T).
\end{aligned} \tag{4.2}$$

In the second step, we get

$$\begin{aligned}
\underset{\lambda}{\mathrm{Min}}\left(\tilde{U}_1\right) &= \underset{\lambda}{\mathrm{Min}}((0.1230, 0.2300)_T, (0.1420, 0.2700)_T, (0.0450, 0.1)_T) \\
&= (0.0450, 0.1)_T, \\
\underset{\lambda}{\mathrm{Max}}\left(\tilde{V}_1\right) &= \underset{\lambda}{\mathrm{Max}}((0.1230, 0.2300)_T, (0.1025, 0.2950)_T, (0.0555, 0.1065)_T, (0.0260, 0.1040)_T) \\
&= (0.1230, 0.2300)_T.
\end{aligned} \tag{4.3}$$

Table 1: The Brown-Robinson method for solving the example.

k	$i(k)$	$j(k)$	$\mathrm{Min}_\lambda(\tilde{U}_k)/k$	$\mathrm{Max}_\lambda(\tilde{V}_k)/k$
1	1	1	$(.450e-1,.1)_T$	$(.1230,.2300)_T$
2	1	3	$(.4500000000e-1,.5000000000e-1)_T$	$(.8400000000e-1,.1150000000)_T$
3	1	3	$(.4500000000e-1,.3333333333e-1)_T$	$(.7099999999e-1,.7666666666e-1)_T$
4	1	3	$(.4500000000e-1,.2500000000e-1)_T$	$(.6450000000e-1,.5750000000e-1)_T$
5	1	3	$(.4500000000e-1,.2000000000e-1)_T$	$(.6060000000e-1,.4600000000e-1)_T$
6	1	3	$(.4500000001e-1,.1666666667e-1)_T$	$(.5800000001e-1,.3833333334e-1)_T$
7	1	3	$(.4500000001e-1,.1428571429e-1)_T$	$(.5614285716e-1,.3285714287e-1)_T$
8	1	3	$(.4500000000e-1,.1250000000e-1)_T$	$(.5475000000e-1,.2875000000e-1)_T$
9	1	3	$(.4500000000e-1,.1111111111e-1)_T$	$(.5366666666e-1,.2555555555e-1)_T$
10	1	3	$(.4500000000e-1,.1000000000e-1)_T$	$(.5280000000e-1,.2300000000e-1)_T$
$\vdots$			$\vdots$	$\vdots$
10^2	1	3	$(.4500000000e-1,.1000000000e-2)_T$	$(.4578000000e-1,.2300000000e-2)_T$
$\vdots$			$\vdots$	$\vdots$
10^3	1	3	$(.4500000000e-1,.1000000000e-3)_T$	$(.4507800000e-1,.2300000000e-3)_T$
$\vdots$			$\vdots$	$\vdots$
10^4	1	3	$(.4500000000e-1,.1000000000e-4)_T$	$(.4500780000e-1,.2300000000e-4)_T$
$\vdots$			$\vdots$	$\vdots$
10^5	1	3	$(.4500000000e-1,.1000000000e-5)_T$	$(.4500078000e-1,.2300000000e-5)_T$
$\vdots$			$\vdots$	$\vdots$
10^6	1	3	$(.4500000000e-1,.1000000000e-6)_T$	$(.4500007800e-1,.2300000000e-6)_T$

Therefore, we obtain $i(2) = 1$, $j(2) = 3$ and

$$\begin{aligned}
\tilde{U}_2 &= \tilde{U}_1 + \tilde{g}^r_{i(2)} \\
&= ((0.1230, 0.2300)_T, (0.1420, 0.2700)_T, (0.0450, 0.1)_T) \\
&\quad + ((0.1230, 0.1300)_T, (0.1420, 0.1700)_T, (0.0450, 0)_T) \\
&= ((0.2460, 0.3600)_T, (0.2840, 0.4400)_T, (0.0900, 0.1000)_T), \\
\tilde{V}_2 &= \tilde{V}_1 + \tilde{g}^c_{j(2)} \\
&= ((0.1230, 0.2300)_T, (0.1025, 0.2950)_T, (0.0555, 0.1065)_T, (0.0260, 0.1040)_T) \\
&\quad + ((0.1420, 0.1700)_T, (0.0310, 0.0110)_T, (0.0310, 0.0110)_T, (-0.1250, 0.0275)_T) \\
&= ((0.1680, 0.2300)_T, (0.1485, 0.2950)_T, (0.1025, 0.1065)_T, (0.0740, 0.1040)_T).
\end{aligned} \tag{4.4}$$

Continuing in this way, and using the Maple computer algebra system, we build up Table 1. By virtue of Table 1, we see that the value of the game approaches the crisp number $(0.045, 0)_T$.

5. Conclusion

In this paper, we have adapted the Brown-Robinson method to fuzzy matrix games. It is shown that by means of this method, the value of fuzzy matrix games can be easily calculated. Although the method is no way as useful as linear programming for calculating the solution of the game exactly, it is an interesting result and the method can be easily programmed by novice programmers. In addition, linear programming methods are not efficient enough for high dimensional fuzzy matrix games, but the Brown-Robinson method can be used even if the matrix dimension is too high.

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