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Research Article

Asymptotic Behaviour of a Two-Dimensional Differential System with a Finite Number of Nonconstant Delays under the Conditions of Instability

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The asymptotic behaviour of a real two-dimensional differential system $x'(t) = \mathsf{A}(t)x(t) + \sum_{k=1}^{m} \mathsf{B}_k(t)x(\theta_k(t)) + h(t, x(t), x(\theta_1(t)), \ldots, x(\theta_m(t)))$ with unbounded nonconstant delays $t-\theta_k(t) \geq 0$ satisfying $\lim_{t\to\infty}\theta_k(t) = \infty$ is studied under the assumption of instability. Here, A, $\mathsf{B}_{k,}$ and h are supposed to be matrix functions and a vector function. The conditions for the instable properties of solutions and the conditions for the existence of bounded solutions are given. The methods are based on the transformation of the considered real system to one equation with complex-valued coefficients. Asymptotic properties are studied by means of a Lyapunov-Krasovskii functional and the suitable Ważewski topological principle. The results generalize some previous ones, where the asymptotic properties for two-dimensional systems with one constant or nonconstant delay were studied.

1. Introduction

Consider the real two-dimensional system

$$x'(t) = \mathsf{A}(t)x(t) + \sum_{k=1}^{m} \mathsf{B}_k(t)x(\theta_k(t)) + h(t, x(t), x(\theta_1(t)), \ldots, x(\theta_m(t))), \tag{1.1}$$

where $\theta_k(t)$ are real functions, $\mathsf{A}(t) = (a_{ij}(t))$, $\mathsf{B}_k(t) = (b_{ijk}(t))(i, j = 1, 2; k = 1, \ldots, m)$ are real square matrices, and $h(t, x, y) = (h_1(t, x, y_1, \ldots, y_m), h_2(t, x, y_1, \ldots, y_m))$ is a real vector function, $x = (x_1, x_2)$, $y_k = (y_{1k}, y_{2k})$. It is supposed that the functions θ_k, a_{ij} are locally absolutely continuous on $[t_0, \infty)$, b_{ijk} are locally Lebesgue integrable on $[t_0, \infty)$, and the function h satisfies Carathéodory conditions on $[t_0, \infty) \times \mathbb{R}^{2(m+1)}$.

There are a lot of papers dealing with the stability and asymptotic behaviour of n-dimensional real vector equations with delay. Among others we should mention the recent results [1–13]. Since the plane has special topological properties different from those of n-dimensional space, where $n \geq 3$ or $n = 1$, it is interesting to study the asymptotic behaviour of two-dimensional systems by using tools that are typical and effective for two-dimensional systems. The convenient tool is the combination of the method of complexification and the method of Lyapunov-Krasovskii functional. For the case of instability, it is useful to add to this combination the version of Ważewski topological principle formulated by Rybakowski in the papers [14, 15]. Using these techniques, we obtain new and easy applicable results on stability, asymptotic stability, instability, or boundedness of solutions of the system (1.1).

The main idea of the investigation, the combination of the method of complexification and the method of Lyapunov-Krasovskii functional, was introduced for ordinary differential equations in the paper by Ráb and Kalas [16] in 1990. The principle was transferred to differential equations with delay by Kalas and Baráková [17] in 2002. The results in the case of instability were obtained for ODEs by Kalas and Osička [18] in 1994 and for delayed differential equations by Kalas [19] in 2005.

We extend such type of results to differential equations with a finite number of nonconstant delays. We introduce the transformation of the considered real system to one equation with complex-valued coefficients. We present sufficient conditions for the instability of a solution and for the existence of a bounded solution. The applicability of the results is demonstrated with several examples.

At the end of this introduction we append a brief overview of notation used in the paper and the transformation of the real system to one equation with complex-valued coefficients.

$\mathbb{R}$ is the set of all real numbers,

$\mathbb{R}_+$ the set of all positive real numbers,

$\mathbb{R}_+^0$ the set of all nonnegative real numbers,

$\mathbb{R}_-$ the set of all negative real numbers,

$\mathbb{R}_-^0$ the set of all nonpositive real numbers,

$\mathbb{C}$ the set of all complex numbers,

$\mathcal{C}$ the class of all continuous functions $[-r, 0] \to \mathbb{C}$,

$AC_{\text{loc}}(I, M)$ the class of all locally absolutely continuous functions $I \to M$,

$L_{\text{loc}}(I, M)$ the class of all locally Lebesgue integrable functions $I \to M$,

$K(I \times \Omega, M)$ the class of all functions $I \times \Omega \to M$ satisfying Carathéodory conditions on $I \times \Omega$, Rez the real part of z,

Imz the imaginary part of z, and

$\overline{z}$ the complex conjugate of z.

Introducing complex variables $z = x_1 + ix_2, w_1 = y_{11} + iy_{12},\ldots, w_m = y_{m1} + iy_{m2}$, we can rewrite the system (1.1) into an equivalent equation with complex-valued coefficients

$$\begin{aligned} z'(t) &= a(t)z(t) + b(t)\overline{z}(t) + \sum_{k=1}^{m}[A_k(t)z(\theta_k(t)) + B_k(t)\overline{z}(\theta_k(t))] \\ &\quad + g(t, z(t), z(\theta_1(t)), \ldots, z(\theta_m(t))), \end{aligned} \tag{1.2}$$

where $\theta_k \in AC_{\mathrm{loc}}(J,\mathbb{R})$ for $k = 1,\ldots,m$, $A_k, B_k \in L_{\mathrm{loc}}(J,\mathbb{C})$, $a, b \in AC_{\mathrm{loc}}(J,\mathbb{C})$, $g \in K(J \times \mathbb{C}^{m+1},\mathbb{C})$, $J = [t_0,\infty)$.

The relations between the functions are as follows:

$$\begin{aligned} a(t) &= \frac{1}{2}(a_{11}(t) + a_{22}(t)) + \frac{i}{2}(a_{21}(t) - a_{12}(t)), \\ b(t) &= \frac{1}{2}(a_{11}(t) - a_{22}(t)) + \frac{i}{2}(a_{21}(t) + a_{12}(t)), \\ A_k(t) &= \frac{1}{2}(b_{11k}(t) + b_{22k}(t)) + \frac{i}{2}(b_{21k}(t) - b_{12k}(t)), \\ B_k(t) &= \frac{1}{2}(b_{11k}(t) - b_{22k}(t)) + \frac{i}{2}(b_{21k}(t) + b_{12k}(t)), \\ g(t,z,w_1,\ldots,w_m) &= h_1\left(t, \frac{1}{2}(z+\overline{z}), \frac{1}{2i}(z-\overline{z}), \frac{1}{2}(w_1+\overline{w}_1), \ldots, \frac{1}{2i}(w_m-\overline{w}_m)\right) \\ &\quad + ih_2\left(t, \frac{1}{2}(z+\overline{z}), \frac{1}{2i}(z-\overline{z}), \frac{1}{2}(w_1+\overline{w}_1), \frac{1}{2i}(w_1-\overline{w}_1), \ldots, \frac{1}{2i}(w_m-\overline{w}_m)\right). \end{aligned} \tag{1.3}$$

Conversely, putting

$$\begin{aligned} a_{11}(t) &= \mathrm{Re}[a(t)+b(t)], \qquad a_{12}(t) = \mathrm{Im}[b(t)-a(t)], \\ a_{21}(t) &= \mathrm{Im}[a(t)+b(t)], \qquad a_{22}(t) = \mathrm{Re}[a(t)-b(t)], \\ b_{11k}(t) &= \mathrm{Re}[A_k(t)+B_k(t)], \qquad b_{12k}(t) = \mathrm{Im}[B_k(t)-A_k(t)], \\ b_{21k}(t) &= \mathrm{Im}[A_k(t)+B_k(t)], \qquad b_{22k}(t) = \mathrm{Re}[A_k(t)-B_k(t)], \\ h_1(t,x,y_1,\ldots,y_m) &= \mathrm{Re}\, g(t, x_1+ix_2, y_{11}+iy_{12}, \ldots, y_{m1}+iy_{m2}), \\ h_2(t,x,y_1,\ldots,y_m) &= \mathrm{Im}\, g(t, x_1+ix_2, y_{11}+iy_{12}, \ldots, y_{m1}+iy_{m2}), \end{aligned} \tag{1.4}$$

the equation (1.2) can be written in the real form (1.1) as well.

2. Preliminaries

We consider (1.2) in the case when

$$\liminf_{t\to\infty}\left(\left|\operatorname{Im} a(t)\right| - |b(t)|\right) > 0 \tag{2.1}$$

and study the behavior of solutions of (1.2) under this assumption. This situation corresponds to the case when the equilibrium 0 of the autonomous homogeneous system

$$x' = \mathbf{A}x, \tag{2.2}$$

where $\mathbf{A}$ is supposed to be regular constant matrix, is a centre or a focus. See [16] for more details.

Regarding (2.1) and since the delay functions θ_k satisfy $\lim_{t\to\infty}\theta_k(t) = \infty$, there are numbers $T_1 \geq t_0$, $T \geq T_1$, and $\mu > 0$ such that

$$\left|\operatorname{Im} a(t)\right| > |b(t)| + \mu \quad \text{for } t \geq T_1, \qquad t \geq \theta_k(t) \geq T_1 \quad \text{for } t \geq T\,(k = 1,\ldots,m). \tag{2.3}$$

Denote

$$\widetilde{\gamma}(t) = \operatorname{Im} a(t) + \sqrt{\left(\operatorname{Im} a(t)\right)^2 - |b(t)|^2}\operatorname{sgn}\left(\operatorname{Im} a(t)\right), \quad \widetilde{c}(t) = -ib(t). \tag{2.4}$$

Notice that the above-defined function $\widetilde{\gamma}$ need not be positive.

Since $|\widetilde{\gamma}(t)| > |\operatorname{Im} a(t)|$ and $|\widetilde{c}(t)| = |b(t)|$, the inequality

$$\left|\widetilde{\gamma}(t)\right| > |\widetilde{c}(t)| + \mu \tag{2.5}$$

is valid for $t \geq T_1$. It can be easily verified that $\widetilde{\gamma}, \widetilde{c} \in AC_{\text{loc}}([T_1,\infty),\mathbb{C})$.

For the rest of this section we will denote

$$\widetilde{\vartheta}(t) = \frac{\operatorname{Re}\left(\widetilde{\gamma}(t)\widetilde{\gamma}'(t) - \overline{\widetilde{c}}(t)\widetilde{c}'(t)\right) - \left|\widetilde{\gamma}(t)\widetilde{c}'(t) - \widetilde{\gamma}'(t)\widetilde{c}(t)\right|}{\widetilde{\gamma}^2(t) - |\widetilde{c}(t)|^2}. \tag{2.6}$$

The instability and boundedness of solutions are studied subject to suitable subsets of the following assumptions.

(i) The numbers $T_1 \geq t_0$, $T \geq T_1$, and $\mu > 0$ are such that (2.3) holds.

(ii) There exist functions $\widetilde{\varkappa}, \widetilde{\kappa}_k, \varrho : [T,\infty) \to \mathbb{R}$ such that

$$\begin{aligned}
&\left|\widetilde{\gamma}(t)g(t,z,w_1,\dots,w_m)+\widetilde{c}(t)\overline{g}(t,z,w_1,\dots,w_m)\right| \\
&\quad\le \widetilde{\varkappa}(t)\left|\widetilde{\gamma}(t)z+\widetilde{c}(t)\overline{z}\right|+\sum_{k=1}^{m}\widetilde{\kappa}_k(t)\left|\widetilde{\gamma}(\theta_k(t))w_k+\widetilde{c}(\theta_k(t))\overline{w}_k\right|+\varrho(t)
\end{aligned} \tag{2.7}$$

for $t \ge T$, $z, w_k \in \mathbb{C}(k = 1,\dots,m)$, where ϱ is continuous on $[T,\infty)$.

($\mathrm{ii_n}$) There exist numbers $R_n \ge 0$ and functions $\widetilde{\varkappa}_n, \widetilde{\kappa}_{nk} : [T,\infty) \to \mathbb{R}$ such that

$$\begin{aligned}
&\left|\ \widetilde{\gamma}(t)g(t,z,w_1,\dots,\ w_m)+\widetilde{c}(t)g(t,z,w_1,\dots,w_m)\right| \\
&\quad\le \widetilde{\varkappa}_n(t)\left|\widetilde{\gamma}(t)z+\widetilde{c}(t)z\right|+\sum_{k=1}^{m}\widetilde{\kappa}_{nk}(t)\ \left|\widetilde{\gamma}(\theta_k(t))w_k+\widetilde{c}(\theta_k(t))\overline{w}_k\right|
\end{aligned} \tag{2.8}$$

for $t \ge \tau_n \ge T, |z| + \sum_{k=1}^{m} |w_k| > R_n$.

(iii) $\widetilde{\beta} \in AC_{\mathrm{loc}}([T,\infty),\mathbb{R}_-^0)$ is a function satisfying

$$\theta_k'(t)\widetilde{\beta}(t) \le -\widetilde{\lambda}_k(t) \quad \text{a.e. on } [T,\infty), \tag{2.9}$$

where $\widetilde{\lambda}_k$ is defined for $t \ge T$ by

$$\widetilde{\lambda}_k(t) = \widetilde{\kappa}_k(t) + (|A_k(t)| + |B_k(t)|)\frac{\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|}{\left|\widetilde{\gamma}(\theta_k(t))\right| - |\widetilde{c}(\theta_k(t))|}. \tag{2.10}$$

($\mathrm{iii_n}$) $\widetilde{\beta}_n \in AC_{\mathrm{loc}}[T,\infty),\mathbb{R}_-^0)$ is a function satisfying

$$\theta_k'(t)\widetilde{\beta}_n(t) \le -\widetilde{\lambda}_{nk}(t) \qquad \text{a.e. on } [\tau_n,\infty), \tag{2.11}$$

where $\widetilde{\lambda}_{nk}$ is defined for $t \ge T$ by

$$\widetilde{\lambda}_{nk}(t) = \widetilde{\kappa}_{nk}(t) + (|A_k(t)| + |B_k(t)|)\frac{\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|}{\left|\widetilde{\gamma}(\theta_k(t))\right| - |\widetilde{c}(\theta_k(t))|}. \tag{2.12}$$

($\mathrm{iv_n}$) $\widetilde{\Lambda}_n$ is a real locally Lebesgue integrable function satisfying the inequalities $\widetilde{\beta}_n'(t) \ge \widetilde{\Lambda}_n(t)\widetilde{\beta}_n(t)$, $\widetilde{\Theta}_n(t) \ge \widetilde{\Lambda}_n(t)$ for almost all $t \in [\tau_n,\infty)$, where $\widetilde{\Theta}_n$ is defined by

$$\widetilde{\Theta}_n(t) = \mathrm{Re}\,a(t) + \widetilde{\vartheta}(t) - \widetilde{\varkappa}_n(t) + m\widetilde{\beta}_n(t). \tag{2.13}$$

Obviously, if A_k, B_k, $\widetilde{\kappa}_k$, and θ_k' are locally absolutely continuous on $[T,\infty)$ and $\widetilde{\lambda}_k(t) \ge 0$, $\theta_k'(t) > 0$, the choice $\widetilde{\beta}(t) = -\max_{k=1,\dots,m}[\widetilde{\lambda}_k(t)(\theta_k'(t))^{-1}]$ is admissible in (iii). Similarly, if A_k,

B_k, $\widetilde{\kappa}_{nk}$, and θ_k' are locally absolutely continuous on $[T,\infty)$ and $\widetilde{\lambda}_{nk}(t)\geq 0$, $\theta_k'(t)>0$, the choice $\widetilde{\beta}_n(t)=-\max_{k=1,\dots,m}[\widetilde{\lambda}_{nk}(t)(\theta_k'(t))^{-1}]$ is admissible in (iii$_\mathrm{n}$).

Denote

$$\widetilde{\Theta}(t)=\operatorname{Re}a(t)+\widetilde{\vartheta}(t)-\widetilde{\varkappa}(t). \tag{2.14}$$

From assumption (i) it follows that

$$\begin{aligned}\left|\widetilde{\vartheta}\right| &\leq \frac{\left|\operatorname{Re}\left(\widetilde{\gamma}\widetilde{\gamma}'-\overline{\widetilde{c}}\widetilde{c}'\right)\right|+\left|\widetilde{\gamma}c'-\widetilde{\gamma}'c\right|}{\widetilde{\gamma}^2-|\widetilde{c}|^2}\leq\frac{\left(\left|\widetilde{\gamma}'\right|+|\widetilde{c}'|\right)\left(\left|\widetilde{\gamma}\right|+|\widetilde{c}|\right)}{\widetilde{\gamma}^2-|\widetilde{c}|^2}\\ &=\frac{\left|\widetilde{\gamma}'\right|+|\widetilde{c}'|}{\left|\widetilde{\gamma}\right|-|\widetilde{c}|}\leq\frac{1}{\mu}\left(\left|\widetilde{\gamma}'\right|+|\widetilde{c}'|\right);\end{aligned} \tag{2.15}$$

therefore the function $\widetilde{\vartheta}$ is locally Lebesgue integrable on $[T,\infty)$, assuming that (i) holds true. If the relations $\widetilde{\beta}_n\in AC_{\mathrm{loc}}([T,\infty),\mathbb{R}_-)$, $\widetilde{\varkappa}_n\in L_{\mathrm{loc}}([T,\infty),\mathbb{R})$, and $\widetilde{\beta}_n'(t)/\widetilde{\beta}_n(t)\leq\widetilde{\Theta}_n(t)$ for almost all $t\geq\tau_n$ together with conditions (i) and (ii$_\mathrm{n}$) are fulfilled, then we can choose $\widetilde{\Lambda}_n(t)=\widetilde{\Theta}_n(t)$ for $t\in[T,\infty)$ in (iv$_\mathrm{n}$).

3. Results

Theorem 3.1. *Let assumptions (i), (ii$_0$), (iii$_0$), and (iv$_0$) be fulfilled for some $\tau_0\geq T$. Suppose there exist $t_1\geq\tau_0$ and $\nu\in(-\infty,\infty)$ such that*

$$\inf_{t\geq t_1}\left[\int_{t_1}^{t}\widetilde{\Lambda}_0(s)ds-\ln\left(\left|\widetilde{\gamma}(t)\right|+|\widetilde{c}(t)|\right)\right]\geq\nu. \tag{3.1}$$

If $z(t)$ is any solution of (1.2) *satisfying*

$$\min_{\theta(t_1)\leq s\leq t_1}|z(s)|>R_0,\qquad \Delta(t_1)>R_0e^{-\nu}, \tag{3.2}$$

where

$$\begin{aligned}\theta(t)&=\min_{k=1,\dots,m}\theta_k(t),\\ \Delta(t)&=\left(\left|\widetilde{\gamma}(t)\right|-|\widetilde{c}(t)|\right)|z(t)|+\widetilde{\beta}_0(t)\max_{\theta(t)\leq s\leq t}|z(s)|\sum_{k=1}^{m}\int_{\theta_k(t)}^{t}\left(\left|\widetilde{\gamma}(s)\right|+|\widetilde{c}(s)|\right)ds,\end{aligned} \tag{3.3}$$

then

$$|z(t)| \geq \frac{\Delta(t_1)}{|\widetilde{\gamma}(t)| + |\widetilde{c}(t)|} \exp\left[\int_{t_1}^{t} \widetilde{\Lambda}_0(s)ds\right] \tag{3.4}$$

for all $t \geq t_1$, for which $z(t)$ is defined.

In the proof we use the following Lemma.

Lemma 3.2. *Let a_1, a_2, b_1, $b_2 \in \mathbb{C}$, $|a_2| > |b_2|$. Then,*

$$\operatorname{Re}\frac{a_1 z + b_1 \overline{z}}{a_2 z + b_2 \overline{z}} \geq \frac{\operatorname{Re}\left(a_1 \overline{a}_2 - b_1 \overline{b}_2\right) - |a_1 b_2 - a_2 b_1|}{|a_2|^2 - |b_2|^2} \tag{3.5}$$

for $z \in \mathbb{C}$, $z \neq 0$.

The proof is analogous to that of Lemma 1 in [20, page 101] or to the proof of Lemma in [16, page 131].

Proof of Theorem 3.1. Let $z(t)$ be any solution of (1.2) satisfying (3.2). Consider the Lyapunov functional

$$V(t) = U(t) + \widetilde{\beta}_0(t) \sum_{k=1}^{m} \int_{\theta_k(t)}^{t} U(s)ds, \tag{3.6}$$

where

$$U(t) = \left|\widetilde{\gamma}(t) z(t) + \widetilde{c}(t) \overline{z}(t)\right|. \tag{3.7}$$

For brevity we shall denote $w_k(t) = z(\theta_k(t))$ and we shall write the function of variable t simply without indicating the variable t, for example, $\widetilde{\gamma}$ instead of $\widetilde{\gamma}(t)$.

In view of (3.6), we have

$$\begin{aligned} V' = U' + \widetilde{\beta}_0' \sum_{k=1}^{m} \int_{\theta_k(t)}^{t} U(s)ds + m\widetilde{\beta}_0 \left|\widetilde{\gamma} z + \widetilde{c}\overline{z}\right| \\ - \sum_{k=1}^{m} \theta_k' \widetilde{\beta}_0 \left|\widetilde{\gamma}(\theta_k(t)) w_k + \widetilde{c}(\theta_k(t)) \overline{w}_k\right| \end{aligned} \tag{3.8}$$

for almost all $t \geq t_1$ for which $z(t)$ is defined and $U'(t)$ exists. Put $\mathcal{K} = \{t \geq t_1 : z(t)$ exists, $|z(t)| > R_0\}$. Clearly $U(t) \neq 0$ for $t \in \mathcal{K}$. The derivative $U'(t)$ exists for almost all $t \in \mathcal{K}$.

Since $z(t)$ is a solution of (1.2), we obtain

$$
\begin{aligned}
UU' &= \operatorname{Re}\left[\left(\tilde{\gamma}\overline{z} + \overline{\tilde{c}}z\right)\left(\tilde{\gamma}'z + \tilde{\gamma}z' + \tilde{c}'\overline{z} + \tilde{c}\overline{z}'\right)\right] \\
&= \operatorname{Re}\left\{\left(\tilde{\gamma}\overline{z} + \overline{\tilde{c}}z\right)\left[\tilde{\gamma}'z + \tilde{c}'\overline{z} + \tilde{\gamma}\left(az + b\overline{z} + \sum_{k=1}^{m}(A_k w_k + B_k\overline{w}_k) + g\right)\right.\right. \\
&\qquad\left.\left. + \tilde{c}\left(\overline{az} + \overline{b}z\right) + \sum_{k=1}^{m}\left(\overline{A}_k\overline{w}_k + \overline{B}_k w_k + \overline{g}\right)\right]\right\} \\
&= \operatorname{Re}\left\{\left(\tilde{\gamma}\overline{z} + \overline{\tilde{c}}z\right)\left[\tilde{\gamma}'z + \tilde{c}'\overline{z} + \left(\tilde{\gamma}a + \tilde{c}\overline{b}\right)z + \left(\tilde{\gamma}b + \tilde{c}\overline{a}\right)\overline{z}\right.\right. \\
&\qquad\left.\left. + \tilde{\gamma}\left(\sum_{k=1}^{m}(A_k w_k + B_k\overline{w}_k) + g\right) + \tilde{c}\left(\sum_{k=1}^{m}\left(\overline{A}_k\overline{w}_k + \overline{B}_k w_k\right) + \overline{g}\right)\right]\right\}
\end{aligned}
\tag{3.9}
$$

for almost all $t \in \mathcal{K}$. Taking into account

$$
\left(\tilde{\gamma}a + \tilde{c}\overline{b}\right)\tilde{c} = \left(\tilde{\gamma}b + \tilde{c}\overline{a}\right)\tilde{\gamma},
\tag{3.10}
$$

we get

$$
\begin{aligned}
UU' &\ge \operatorname{Re}\left\{\left(\tilde{\gamma}\overline{z} + \overline{\tilde{c}}z\right)\left(\tilde{\gamma}a + \tilde{c}\overline{b}\right)\left(z + \frac{\tilde{c}}{\overline{\tilde{\gamma}}}\overline{z}\right)\right\} \\
&\quad + \operatorname{Re}\left\{\left(\tilde{\gamma}\overline{z} + \overline{\tilde{c}}z\right)\left[\tilde{\gamma}\sum_{k=1}^{m}(A_k w_k + B_k\overline{w}_k) + \tilde{c}\sum_{k=1}^{m}\left(\overline{A}_k\overline{w}_k + \overline{B}_k w_k\right)\right]\right\} \\
&\quad + \operatorname{Re}\left\{\left(\tilde{\gamma}\overline{z} + \overline{\tilde{c}}z\right)\left(\tilde{\gamma}g + \tilde{c}\overline{g}\right)\right\} + \operatorname{Re}\left\{\left(\tilde{\gamma}\overline{z} + \overline{\tilde{c}}z\right)\left(\tilde{\gamma}'z + \tilde{c}'\overline{z}\right)\right\} \\
&\ge U^2\operatorname{Re}\left(a + \frac{\tilde{c}}{\overline{\tilde{\gamma}}}\overline{b}\right) - U\left(|\tilde{\gamma}| + |\tilde{c}|\right)\left(\sum_{k=1}^{m}|A_k w_k + B_k\overline{w}_k|\right) \\
&\quad - U\left|\tilde{\gamma}g + \tilde{c}\overline{g}\right| + U^2\operatorname{Re}\frac{\tilde{\gamma}'z + \tilde{c}'\overline{z}}{\tilde{\gamma}z + \tilde{c}\overline{z}}.
\end{aligned}
\tag{3.11}
$$

By the use of Lemma 3.2, we get

$$
\operatorname{Re}\frac{\tilde{\gamma}'z + \tilde{c}'\overline{z}}{\tilde{\gamma}z + \tilde{c}\overline{z}} \ge \tilde{\vartheta}.
\tag{3.12}
$$

The last inequality together with (2.12), taken for $n = 0$, assumption (ii_0), and the relation

$$
\operatorname{Re}\left(a + \frac{\tilde{c}}{\overline{\tilde{\gamma}}}\overline{b}\right) = \operatorname{Re} a
\tag{3.13}
$$

yields

$$
\begin{aligned}
UU' &\geq U^2\left(\operatorname{Re} a + \widetilde{\vartheta} - \widetilde{\varkappa}_0\right) - U\sum_{k=1}^{m}\left(\widetilde{\kappa}_{0k}\left|\widetilde{\gamma}(\theta_k)w_k + \widetilde{c}(\theta_k)\overline{w}_k\right|\right) \\
&\quad - U\left(\left|\widetilde{\gamma}\right| + |\widetilde{c}|\right)\left(\sum_{k=1}^{m}\frac{|A_k||w_k| + |B_k||\overline{w}_k|}{\left|\widetilde{\gamma}(\theta_k)\right| - |\widetilde{c}(\theta_k)|}\left(\left|\widetilde{\gamma}(\theta_k)\right| - |\widetilde{c}(\theta_k)|\right)\right) \\
&\geq U^2\left(\operatorname{Re} a + \widetilde{\vartheta} - \widetilde{\varkappa}_0\right) \\
&\quad - U\left\{\sum_{k=1}^{m}\left[\widetilde{\kappa}_{0k} + (|A_k| + |B_k|)\frac{\left|\widetilde{\gamma}\right| + |\widetilde{c}|}{\left|\widetilde{\gamma}(\theta_k)\right| - |\widetilde{c}(\theta_k)|}\right]\left|\widetilde{\gamma}(\theta_k)w_k + \widetilde{c}(\theta_k)\overline{w}_k\right|\right\} \\
&\geq U^2\left(\operatorname{Re} a + \widetilde{\vartheta} - \widetilde{\varkappa}_0\right) - U\sum_{k=1}^{m}\widetilde{\lambda}_{0k}\left|\widetilde{\gamma}(\theta_k)w_k + \widetilde{c}(\theta_k)\overline{w}_k\right|
\end{aligned}
\tag{3.14}
$$

for almost all $t \in \mathcal{K}$.

Consequently,

$$
U' \geq U\left(\operatorname{Re} a + \widetilde{\vartheta} - \widetilde{\varkappa}_0\right) - \sum_{k=1}^{m}\widetilde{\lambda}_{0k}\left|\widetilde{\gamma}(\theta_k)w_k + \widetilde{c}(\theta_k)\overline{w}_k\right| \tag{3.15}
$$

for almost all $t \in \mathcal{K}$. Inequality (3.15) together with relation (3.8) gives

$$
\begin{aligned}
V' &\geq U\left(\operatorname{Re} a + \widetilde{\vartheta} - \widetilde{\varkappa}_0 + m\widetilde{\beta}_0\right) - \sum_{k=1}^{m}\left(\widetilde{\lambda}_{0k} + \theta_k'\widetilde{\beta}_0\right)\left|\widetilde{\gamma}(\theta_k)w_k + \widetilde{c}(\theta_k)\overline{w}_k\right| \\
&\quad + \widetilde{\beta}_0'\sum_{k=1}^{m}\int_{\theta_k(t)}^{t}\left|\widetilde{\gamma}(s)z(s) + \widetilde{c}(s)\overline{z}(s)\right|ds.
\end{aligned}
\tag{3.16}
$$

Using (2.11) and (2.13) for $n = 0$, we obtain

$$
V'(t) \geq U(t)\widetilde{\Theta}_0(t) + \widetilde{\beta}_0'(t)\sum_{k=1}^{m}\int_{\theta_k(t)}^{t}U(s)ds. \tag{3.17}
$$

Hence, in view of (iv_0),

$$
V'(t) - \widetilde{\Lambda}_0(t)V(t) \geq 0 \tag{3.18}
$$

for almost all $t \in \mathcal{K}$.

Multiplying (3.18) by $\exp[-\int_{t_1}^{t}\widetilde{\Lambda}_0(s)ds]$ and integrating over $[t_1, t]$, we get

$$
V(t)\exp\left[-\int_{t_1}^{t}\widetilde{\Lambda}_0(s)ds\right] - V(t_1) \geq 0 \tag{3.19}
$$

on any interval $[t_1, \omega)$, where the solution $z(t)$ exists and satisfies the inequality $|z(t)| > R_0$. Now, with respect to (3.6), (3.7), and $\widetilde{\beta}_0 \le 0$, we have

$$\left(\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|\right)|z(t)| \ge V(t) \ge V(t_1)\exp\left[\int_{t_1}^{t} \widetilde{\Lambda}_0(s)ds\right] \ge \Delta(t_1)\exp\left[\int_{t_1}^{t} \widetilde{\Lambda}_0(s)ds\right]. \tag{3.20}$$

If (3.2) is fulfilled, there is $R > R_0$ such that $\Delta(t_1) > Re^{-\nu}$. By virtue of (3.1), and (3.2), we can easily see that

$$|z(t)| \ge \frac{\Delta(t_1)}{\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|}\exp\left[\int_{t_1}^{t} \widetilde{\Lambda}_0(s)ds\right] \ge Re^{-\nu}e^{\nu} = R \tag{3.21}$$

for all $t \ge t_1$, for which $z(t)$ is defined. □

To obtain results on the existence of bounded solutions, we shall suppose that (1.2) satisfies the uniqueness property of solutions. Moreover, we suppose that the delays are bounded, that is, that the functions θ_k satisfy the condition

$$t - r \le \theta_k(t) \le t \quad \text{for } t \ge t_0 + r, \tag{3.22}$$

where $r > 0$ is a constant. Our assumptions imply the existence of numbers $T_1 = t_0 + r$, $T \ge T_1$, and $\mu > 0$ such that

$$\left|\operatorname{Im} a(t)\right| > |b(t)| + \mu \quad \text{for } t \ge T_1, \quad t \ge \theta_k(t) \ge t - r \quad \text{for } t \ge T (k = 1, \ldots, m). \tag{5$'$}$$

In view of this, we replace (2.3) in assumption (i) with (5′). All other assumptions we keep in validity.

In the proof of the following theorem we shall utilize Ważewski topological principle for retarded functional differential equations of Carathéodory type. Details of this theory can be found in the paper of Rybakowski [15].

Theorem 3.3. *Let conditions (i), (ii), and (iii) be fulfilled, and let $\widetilde{\Lambda}$, θ_k' $(k = 1, \ldots, m)$ be continuous functions such that the inequality $\widetilde{\Lambda}(t) \le \widetilde{\Theta}(t)$ holds a.e. on $[T, \infty)$, where $\widetilde{\Theta}$ is defined by* (2.14). *Suppose that $\xi : [T - r, \infty) \to \mathbb{R}$ is a continuous function such that*

$$\widetilde{\Lambda}(t) + \widetilde{\beta}(t)\sum_{k=1}^{m}\theta_k'(t)\exp\left[-\int_{\theta_k(t)}^{t} \xi(s)ds\right] - \xi(t) > \varrho(t)C^{-1}\exp\left[-\int_{T}^{t} \xi(s)ds\right] \tag{3.23}$$

for $t \in [T, \infty]$ *and some constant* $C > 0$. *Then, there exist* $t_2 > T$ *and a solution* $z_0(t)$ *of* (1.2) *satisfying*

$$|z_0(t)| \le \frac{C}{|\widetilde{\gamma}(t)| - |\widetilde{c}(t)|} \exp\left[\int_T^t \xi(s)ds\right] \tag{3.24}$$

for $t \ge t_2$.

Proof. Write (1.2) in the form

$$z' = F(t, z_t), \tag{2'}$$

where $F : J \times \mathcal{C} \to \mathbb{C}$ is defined by

$$\begin{aligned} F(t, \psi) = {} & a(t)\psi(0) + b(t)\overline{\psi}(0) + \sum_{k=1}^{m} \left[A_k(t)\psi(\theta_k(t) - t) + B_k(t)\overline{\psi}(\theta_k(t) - t)\right] \\ & + g(t, \psi(0), \psi(\theta_1(t) - t), \dots, \psi(\theta_m(t) - t)) \end{aligned} \tag{3.25}$$

and z_t is the element of $\mathcal{C}$ defined by the relation $z_t(\widetilde{\theta}) = z(t + \widetilde{\theta})$, $\widetilde{\theta} \in [-r, 0]$. Let $\tau > T$. Put

$$\begin{aligned} \widetilde{U}(t, z, \overline{z}) &= \left|\widetilde{\gamma}(t)z + \widetilde{c}(t)\overline{z}\right| - \varphi(t), \\ \varphi(t) &= C \exp\left[\int_T^t \xi(s)ds\right], \\ \Omega^0 &= \left\{(t, z) \in (\tau, \infty) \times \mathbb{C} : \widetilde{U}(t, z, \overline{z}) < 0\right\}, \\ \Omega_{\widetilde{U}} &= \left\{(t, z) \in (\tau, \infty) \times \mathbb{C} : \widetilde{U}(t, z, \overline{z}) = 0\right\}. \end{aligned} \tag{3.26}$$

It can be easily verified that Ω^0 is a polyfacial set generated by the functions $\widehat{U}(t) = \tau - t$, $\widetilde{U}(t, z, \overline{z})$ (see Rybakowski [15, page 134]). It holds that $\Omega_{\widetilde{U}} \subset \partial\Omega^0$. As $(|\widetilde{\gamma}(t)| + |\widetilde{c}(t)|)|z(t)| \ge |\widetilde{\gamma}(t)z + \widetilde{c}(t)\overline{z}|$, we have

$$|z| \ge \frac{\varphi(t)}{|\widetilde{\gamma}(t)| + |\widetilde{c}(t)|} = \frac{C}{|\widetilde{\gamma}(t)| + |\widetilde{c}(t)|} \exp\left[\int_T^t \xi(s)ds\right] > 0 \tag{3.27}$$

for $(t, z) \in \Omega_{\widetilde{U}}$. It holds that

$$D^+\widehat{U}(t) = \frac{\partial}{\partial t}(\tau - t) = -1 < 0. \tag{3.28}$$

Let $(t^*, \zeta) \in \Omega_{\widetilde{U}}$ and $\phi \in \mathcal{C}$ be such that $\phi(0) = \zeta$ and $(t^* + \widetilde{\theta}, \phi(\widetilde{\theta})) \in \Omega^0$ for all $\widetilde{\theta} \in [-r, 0)$. If $(t, \psi) \in (\tau, \infty) \times \mathcal{C}$, then

$$\begin{aligned}
D^+\widetilde{U}(t, \psi(0), \overline{\psi}(0)) :&= \limsup_{h\to 0+}\left(\frac{1}{h}\right)\Big[\widetilde{U}\Big(t+h, \psi(0) + hF(t,\psi), \overline{\psi}(0) + h\overline{F}(t,\psi)\Big) \\
&\qquad - \widetilde{U}(t, \psi(0), \overline{\psi}(0))\Big] \\
&= \frac{\partial \widetilde{U}(t, \psi(0), \overline{\psi}(0))}{\partial t} + \frac{\partial \widetilde{U}(t, \psi(0), \overline{\psi}(0))}{\partial z} F(t,\psi) \\
&\quad + \frac{\partial \widetilde{U}(t, \psi(0), \overline{\psi}(0))}{\partial \overline{z}} \overline{F}(t,\psi).
\end{aligned} \tag{3.29}$$

Therefore,

$$\begin{aligned}
D^+\widetilde{U}(t, \psi(0), \overline{\psi}(0)) &= \left|\widetilde{\gamma}(t)\psi(0) + \widetilde{c}(t)\overline{\psi}(0)\right| \operatorname{Re} \frac{\widetilde{\gamma}'(t)\psi(0) + \widetilde{c}'(t)\overline{\psi}(0)}{\widetilde{\gamma}(t)\psi(0) + \widetilde{c}(t)\overline{\psi}(0)} - \varphi'(t) \\
&\quad + \frac{1}{2}\left|\widetilde{\gamma}(t)\psi(0) + \widetilde{c}(t)\overline{\psi}(0)\right|^{-1} \\
&\quad \times \Big\{ \Big[\overline{\widetilde{\gamma}}(t)\Big(\widetilde{\gamma}(t)\overline{\psi}(0) + \overline{\widetilde{c}}(t)\psi(0)\Big) + \big(\widetilde{\gamma}(t)\psi(0) + \widetilde{c}(t)\overline{\psi}(0)\big)\overline{\widetilde{c}}(t)\Big] F(t,\psi) \\
&\qquad + \Big[\widetilde{c}(t)\Big(\widetilde{\gamma}(t)\overline{\psi}(0) + \overline{\widetilde{c}}(t)\psi(0)\Big) + \widetilde{\gamma}(t)\big(\widetilde{\gamma}(t)\psi(0) + \widetilde{c}(t)\overline{\psi}(0)\big)\Big]\overline{F}(t,\psi)\Big\}
\end{aligned} \tag{3.30}$$

provided that the derivatives $\widetilde{\gamma}'(t)$, $\widetilde{c}'(t)$ exist and that $\psi(0) \neq 0$. Thus,

$$\begin{aligned}
D^+\widetilde{U}(t, \psi(0), \overline{\psi}(0)) &= \left|\widetilde{\gamma}(t)\psi(0) + \widetilde{c}(t)\overline{\psi}(0)\right| \operatorname{Re} \frac{\widetilde{\gamma}'(t)\psi(0) + \widetilde{c}'(t)\overline{\psi}(0)}{\widetilde{\gamma}(t)\psi(0) + \widetilde{c}(t)\overline{\psi}(0)} - \varphi'(t) \\
&\quad + \left|\widetilde{\gamma}(t)\psi(0) + \widetilde{c}(t)\overline{\psi}(0)\right|^{-1} \operatorname{Re}\Big\{\widetilde{\gamma}(t)\Big(\widetilde{\gamma}(t)\overline{\psi}(0) + \overline{\widetilde{c}}(t)\psi(0)\Big)F(t,\psi) \\
&\qquad + \widetilde{c}(t)\Big(\widetilde{\gamma}(t)\overline{\psi}(0) + \overline{\widetilde{c}}(t)\psi(0)\Big)\overline{F}(t,\psi)\Big\} \\
&= \left|\widetilde{\gamma}(t)\psi(0) + \widetilde{c}(t)\overline{\psi}(0)\right| \operatorname{Re} \frac{\widetilde{\gamma}'(t)\psi(0) + \widetilde{c}'(t)\overline{\psi}(0)}{\widetilde{\gamma}(t)\psi(0) + \widetilde{c}(t)\overline{\psi}(0)} - \varphi'(t) \\
&\quad + \left|\widetilde{\gamma}(t)\psi(0) + \widetilde{c}(t)\overline{\psi}(0)\right|^{-1} \operatorname{Re}\Big\{\Big(\widetilde{\gamma}(t)\overline{\psi}(0) + \overline{\widetilde{c}}(t)\psi(0)\Big) \\
&\qquad \times \Big(\widetilde{\gamma}(t)F(t,\psi) + \widetilde{c}(t)\overline{F}(t,\psi)\Big)\Big\}.
\end{aligned} \tag{3.31}$$

Using (3.10), (3.13), and (ii), similarly to the proof of Theorem 3.1, we obtain

$$\begin{aligned}
D^+\widetilde{U}(t, \psi(0), \overline{\psi}(0)) &\geq \left|\widetilde{\gamma}(t)\psi(0) + \widetilde{c}(t)\overline{\psi}(0)\right| \operatorname{Re} a(t) \\
&\quad - \sum_{k=1}^{m}\left|A_k(t)\psi(\theta_k(t) - t) + B_k(t)\overline{\psi}(\theta_k(t) - t)\right|\big(\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|\big) \\
&\quad - \widetilde{\varkappa}(t)\left|\widetilde{\gamma}(t)\psi(0) + \widetilde{c}(t)\overline{\psi}(0)\right|
\end{aligned}$$

$$- \sum_{k=1}^{m} \widetilde{\kappa}_k(t) \left| \widetilde{\gamma}(\theta_k(t)) \varphi(\theta_k(t) - t) + \widetilde{c}(\theta_k(t)) \overline{\varphi}(\theta_k(t) - t) \right|$$
$$+ \widetilde{\vartheta}(t) \left| \widetilde{\gamma}(t) \varphi(0) + \widetilde{c}(t) \overline{\varphi}(0) \right| - \varrho(t) - \varphi'(t) \tag{3.32}$$

and consequently, with respect to (iii),

$$\begin{aligned} D^+ \widetilde{U}(t, \varphi(0), \overline{\varphi}(0)) &\geq \left(\mathrm{Re}\, a(t) + \widetilde{\vartheta}(t) - \widetilde{\varkappa}(t) \right) \left| \widetilde{\gamma}(t) \varphi(0) + \widetilde{c}(t) \overline{\varphi}(0) \right| \\ &\quad - \sum_{k=1}^{m} \widetilde{\lambda}_k(t) \left| \widetilde{\gamma}(\theta_k(t)) \varphi(\theta_k(t) - t) + \widetilde{c}(\theta_k(t)) \overline{\varphi}(\theta_k(t) - t) \right| - \varrho(t) - \varphi'(t) \\ &\geq \widetilde{\Theta}(t) \left| \widetilde{\gamma}(t) \varphi(0) + \widetilde{c}(t) \overline{\varphi}(0) \right| \\ &\quad + \widetilde{\beta}(t) \sum_{k=1}^{m} \theta_k'(t) \left| \widetilde{\gamma}(\theta_k(t)) \varphi(\theta_k(t) - t) + \widetilde{c}(\theta_k(t)) \overline{\varphi}(\theta_k(t) - t) \right| - \varrho(t) - \varphi'(t) \\ &\geq \widetilde{\Lambda}(t) \left| \widetilde{\gamma}(t) \varphi(0) + \widetilde{c}(t) \overline{\varphi}(0) \right| \\ &\quad + \widetilde{\beta}(t) \sum_{k=1}^{m} \theta_k'(t) \left| \widetilde{\gamma}(\theta_k(t)) \varphi(\theta_k(t) - t) + \widetilde{c}(\theta_k(t)) \overline{\varphi}(\theta_k(t) - t) \right| - \varrho(t) - \varphi'(t) \end{aligned} \tag{3.33}$$

for almost all $t \in (\tau, \infty)$ and for $\varphi \in C$ sufficiently close to ϕ. Replacing t and φ by t^* and ϕ, respectively, in the last expression, we get

$$\begin{aligned} &\widetilde{\Lambda}(t^*) \left| \widetilde{\gamma}(t^*) \phi(0) + \widetilde{c}(t^*) \overline{\phi}(0) \right| + \widetilde{\beta}(t^*) \sum_{k=1}^{m} \theta_k'(t^*) \left| \widetilde{\gamma}(\theta_k(t^*)) \phi(\theta_k(t^*) - t^*) + \widetilde{c}(\theta_k(t^*)) \overline{\phi}(\theta_k(t^*) - t^*) \right| \\ &\quad - \varrho(t^*) - \varphi'(t^*) \\ &\quad \geq \widetilde{\Lambda}(t^*) \left| \widetilde{\gamma}(t^*) \zeta + \widetilde{c}(t^*) \overline{\zeta} \right| + \widetilde{\beta}(t^*) \sum_{k=1}^{m} \theta_k'(t^*) \varphi(\theta_k(t^*)) - \varrho(t^*) - \varphi'(t^*) \\ &\quad \geq \widetilde{\Lambda}(t^*) \varphi(t^*) + \widetilde{\beta}(t^*) \sum_{k=1}^{m} \theta_k'(t^*) \varphi(\theta_k(t^*)) - \varrho(t^*) - \varphi'(t^*) \\ &\quad = \widetilde{\Lambda}(t^*) C \exp\left[\int_T^{t^*} \xi(s) ds \right] + \widetilde{\beta}(t^*) \sum_{k=1}^{m} \theta_k'(t^*) C \exp\left[\int_T^{\theta_k(t^*)} \xi(s) ds \right] \\ &\quad\quad - \varrho(t^*) - C \xi(t^*) \exp\left[\int_T^{t^*} \xi(s) ds \right] \\ &\quad = \left\{ \widetilde{\Lambda}(t^*) + \widetilde{\beta}(t^*) \sum_{k=1}^{m} \theta_k'(t^*) \exp\left[- \int_{\theta_k(t^*)}^{t^*} \xi(s) ds \right] - \xi(t^*) \right\} C \exp\left[\int_T^{t^*} \xi(s) ds \right] \\ &\quad\quad - \varrho(t^*) > 0. \end{aligned} \tag{3.34}$$

Therefore, in view of the continuity, $D^+\widetilde{U}(t,\psi(0),\overline{\psi}(0)) > 0$ holds for ψ sufficiently close to ϕ and almost all t sufficiently close to t^*. Hence, Ω^0 is a regular polyfacial set with respect to (2′).

Choose $Z = \{(t_2, z) \in \Omega^0 \cup \Omega_{\widetilde{U}}\}$, where $t_2 > \tau + r$ is fixed. It can be easily verified that $Z \cap \Omega_{\widetilde{U}}$ is a retract of $\Omega_{\widetilde{U}}$, but $Z \cap \Omega_{\widetilde{U}}$ is not a retract of Z. Let $\eta \in C$ be such that $\eta(0) = 1$ and $0 \le \eta(\theta) < 1$ for $\theta \in [-r, 0)$. Define the mapping $p : Z \to C$ for $(t_2, z) \in Z$ by the relation

$$p(t_2,z)(\theta) = \frac{\varphi(t_2+\theta)\eta(\theta)}{\left(\widetilde{\gamma}^2(t_2+\theta) - |\widetilde{c}(t_2+\theta)|^2\right)\varphi(t_2)}\Big[\left(\widetilde{\gamma}(t_2)\widetilde{\gamma}(t_2+\theta) - \overline{\widetilde{c}}(t_2)\widetilde{c}(t_2+\theta)\right)z + \left(\widetilde{\gamma}(t_2+\theta)\widetilde{c}(t_2) - \widetilde{\gamma}(t_2)\widetilde{c}(t_2+\theta)\right)\overline{z}\Big] \tag{3.35}$$

The mapping p is continuous, and it holds that

$$p(t_2,z)(0) = z \quad \text{for } (t_2,z) \in Z, \qquad p(t_2,0)(\theta) = 0 \quad \text{for } \theta \in [-r,0]. \tag{3.36}$$

Since

$$\widetilde{\gamma}(t_2+\theta)p(t_2,z)(\theta) + \widetilde{c}(t_2+\theta)\overline{p(t_2,z)(\theta)} = \frac{\varphi(t_2+\theta)\eta(\theta)}{\varphi(t_2)}\left(\widetilde{\gamma}(t_2)z + \widetilde{c}(t_2)\overline{z}\right), \tag{3.37}$$

we have

$$\left|\widetilde{\gamma}(t_2)z + \widetilde{c}(t_2)\overline{z}\right| < \varphi(t_2), \tag{3.38}$$

$$\left|\widetilde{\gamma}(t_2+\theta)p(t_2,z)(\theta) + \widetilde{c}(t_2+\theta)\overline{p(t_2,z)(\theta)}\right| < \varphi(t_2+\theta) \tag{3.39}$$

for $(t_2, z) \in Z \cap \Omega^0$ and $\theta \in [-r, 0]$. Clearly, inequality (3.39) holds also for $(t_2, z) \in Z \cap \Omega_{\widetilde{U}}$ and $\theta \in [-r, 0)$.

Using a topological principle for retarded functional differential equations (see Rybakowski [15, Theorem 2.1]), we see that there is a solution $z_0(t)$ of (1.2) such that $(t, z_0(t)) \in \Omega^0$ for all $t \ge t_2$ for which the solution $z_0(t)$ exists. Obviously, $z_0(t)$ exists for all $t \ge t_2$ and

$$\left(\left|\widetilde{\gamma}(t)\right| - |\widetilde{c}(t)|\right)|z_0(t)| \le \left|\widetilde{\gamma}(t)z_0(t) + \widetilde{c}(t)\overline{z}_0(t)\right| \le \varphi(t) \quad \text{for } t \ge t_2. \tag{3.40}$$

Hence

$$|z_0(t)| \le \frac{\varphi(t)}{\left|\widetilde{\gamma}(t)\right| - |\widetilde{c}(t)|} \quad \text{for } t \ge t_2. \tag{3.41}$$

□

Theorem 3.4. *Suppose that hypotheses (i), (ii), (ii$_n$), (iii), (iii$_n$), and (iv$_n$) are fulfilled for $\tau_n \ge T$ and $n \in \mathbb{N}$, where $R_n > 0$, $\inf_{n\in\mathbb{N}} R_n = 0$. Let $\widetilde{\Lambda}$, θ'_k be continuous functions satisfying the inequality*

$\widetilde{\Lambda}(t) \leq \widetilde{\Theta}(t)$ a.e. on $[T, \infty)$, where $\widetilde{\Theta}$ is defined by (2.14). *Assume that $\xi : [T - r, \infty) \to \mathbb{R}$ is a continuous function such that*

$$\widetilde{\Lambda}(t) + \widetilde{\beta}(t) \sum_{k=1}^{m} \theta_k'(t) \exp\left[-\int_{\theta_k(t)}^{t} \xi(s) ds\right] - \xi(t) > \rho(t) C^{-1} \exp\left(-\int_{T}^{t} \xi(s) ds\right) \tag{3.42}$$

for $t \in [T, \infty)$ and some constant $C > 0$. Suppose that

$$\limsup_{t \to \infty} \left[\int_{T}^{t} \left(\widetilde{\Lambda}_n(s) - \xi(s)\right) ds + \ln \frac{\left|\widetilde{\gamma}(t)\right| - |\widetilde{c}(t)|}{\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|}\right] = \infty, \tag{3.43}$$

$$\lim_{t \to \infty} \left[\widetilde{\beta}_n(t) \max_{\theta(t) \leq s \leq t} \frac{\exp\left[\int_T^s \xi(\sigma)\ d\sigma\right]}{\left|\widetilde{\gamma}(s)\right| - |\widetilde{c}(s)|} \sum_{k=1}^{m} \int_{\theta_k(t)}^{t} \left(\left|\widetilde{\gamma}(s)\right| + |\widetilde{c}(s)|\right) ds\right] = 0, \tag{3.44}$$

$$\inf_{\tau_n \leq s \leq t < \infty} \left[\int_{s}^{t} \widetilde{\Lambda}_n(\sigma) d\sigma - \ln\left(\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|\right)\right] \geq \nu \tag{3.45}$$

for $n \in \mathbb{N}$, where $\theta(t) = \min_{k=1,\ldots,m} \theta_k(t)$ and $\nu \in (-\infty, \infty)$. Then, there exists a solution $z_0(t)$ of (1.2) *such that*

$$\lim_{t \to \infty} \min_{\theta(t) \leq s \leq t} |z_0(s)| = 0. \tag{3.46}$$

Proof. By the use of Theorem 3.3 we observe that there is a $t_2 \geq T$ and a solution $z_0(t)$ of (1.2) with property

$$|z_0(t)| \leq \frac{C}{\left|\widetilde{\gamma}(t)\right| - |\widetilde{c}(t)|} \exp\left[\int_{T}^{t} \xi(s) ds\right] \tag{3.47}$$

for $t \geq t_2$. Suppose that (3.46) is not satisfied. Then, there is $\varepsilon_0 > 0$ such that

$$\limsup_{t \to \infty} \min_{\theta(t) \leq s \leq t} |z_0(s)| > \varepsilon_0. \tag{3.48}$$

Choose $N \in \mathbb{N}$ such that

$$\max\left\{R_N, \frac{2}{\mu} R_N e^{-\nu}\right\} < \varepsilon_0. \tag{3.49}$$

It holds that

$$\min_{\theta(\tau) \leq s \leq \tau} |z_0(s)| > \max\left\{R_N, \frac{2}{\mu} R_N e^{-\nu}\right\} \tag{3.50}$$

for some $\tau > \max\{T, \tau_N, t_2\}$. In view of (3.44), we can suppose that

$$\left|\widetilde{\beta}_N(\tau)\right| C \max_{\theta(\tau)\le s\le\tau} \frac{\exp\left[\int_T^s \xi(\sigma)\, d\sigma\right]}{\left|\widetilde{\gamma}(s)\right| - |\widetilde{c}(s)|} \sum_{k=1}^{m} \int_{\theta_k(\tau)}^{\tau} \left(\left|\widetilde{\gamma}(s)\right| + |\widetilde{c}(s)|\right) ds < \frac{1}{2} R_N e^{-\nu}. \tag{3.51}$$

Therefore, taking into account (2.5), (3.47), (3.50), and (3.51) and the nonpositiveness of β_N, we have

$$\begin{aligned}
&\left(\left|\widetilde{\gamma}(\tau)\right| - |\widetilde{c}(\tau)|\right)|z_0(\tau)| + \widetilde{\beta}_N(\tau) \max_{\theta(\tau)\le s\le\tau} |z_0(s)| \sum_{k=1}^{m} \int_{\theta_k(\tau)}^{\tau} \left(\left|\widetilde{\gamma}(s)\right| + |\widetilde{c}(s)|\right) ds \\
&\quad \ge \left(\left|\widetilde{\gamma}(\tau)\right| - |\widetilde{c}(\tau)|\right)|z_0(\tau)| \\
&\qquad + \widetilde{\beta}_N(\tau) C \max_{\theta(\tau)\le s\le\tau} \frac{\exp\left[\int_T^s \xi(\sigma)\, d\sigma\right]}{\left|\widetilde{\gamma}(s)\right| - |\widetilde{c}(s)|} \sum_{k=1}^{m} \int_{\theta_k(\tau)}^{\tau} \left(\left|\widetilde{\gamma}(s)\right| + |\widetilde{c}(s)|\right) ds \\
&\quad \ge \mu \frac{2}{\mu} R_N e^{-\nu} - \frac{1}{2} R_N e^{-\nu} > R_N e^{-\nu}.
\end{aligned} \tag{3.52}$$

Moreover, (3.45) implies that

$$\inf_{\tau\le t<\infty} \left[\int_\tau^t \widetilde{\Lambda}_N(s) ds - \ln\left(\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|\right)\right] \ge \nu > -\infty. \tag{3.53}$$

By Theorem 3.1, we obtain an estimation

$$|z_0(t)| \ge \frac{\Psi(\tau)}{\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|} \exp\left[\int_\tau^t \widetilde{\Lambda}_N(s) ds\right] \tag{3.54}$$

for all $t \ge \tau$, Ψ being defined by

$$\Psi(\tau) = \left(\left|\widetilde{\gamma}(\tau)\right| - |\widetilde{c}(\tau)|\right)|z_0(\tau)| + \widetilde{\beta}_N(\tau) \max_{\theta(\tau)\le s\le\tau} |z_0(s)| \sum_{k=1}^{m} \int_{\theta_k(\tau)}^{\tau} \left(\left|\widetilde{\gamma}(s)\right| + |\widetilde{c}(s)|\right) ds. \tag{3.55}$$

Relation (3.47) together with (3.54) yields

$$\frac{\Psi(\tau)}{\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|} \exp\left[\int_\tau^t \widetilde{\Lambda}_N(s) ds\right] \le \frac{C}{\left|\widetilde{\gamma}(t)\right| - |\widetilde{c}(t)|} \exp\left[\int_T^t \xi(s) ds\right], \tag{3.56}$$

that is

$$\int_T^t \left[\widetilde{\Lambda}_N(s) - \xi(s)\right] ds + \ln \frac{\left|\widetilde{\gamma}(t)\right| - |\widetilde{c}(t)|}{\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|} \le \int_T^\tau \widetilde{\Lambda}_N(s) ds - \ln\left[C^{-1}\Psi(\tau)\right] \tag{3.57}$$

for $t \ge \tau$. However, the last inequality contradicts (3.43) and Theorem 3.4 is proved. □

From Theorem 3.1 we easily obtain several corollaries.

Corollary 3.5. *Let the assumptions of Theorem 3.1 be fulfilled with* $R_0 > 0$*. If*

$$\liminf_{t\to\infty}\left[\int_{t_1}^{t}\widetilde{\Lambda}_0(s)ds - \ln\left(\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|\right)\right] = \varsigma > \nu, \tag{3.58}$$

then for any ε, $0 < \varepsilon < R_0 e^{\varsigma-\nu}$, *there is* $t_2 \geq t_1$ *such that*

$$|z(t)| > \varepsilon \tag{3.59}$$

for all $t \geq t_2$, *for which* $z(t)$ *is defined.*

Proof. Without loss of generality we can assume that $\varepsilon > R_0$. Choose χ, $0 < \chi < 1$ such that $R_0 < \varepsilon < \chi R_0 e^{\varsigma-\nu}$. In view of (3.58), there is $t_2 \geq t_1$ such that

$$\int_{t_1}^{t}\widetilde{\Lambda}_0(s)ds - \ln\left(\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|\right) > \varsigma + \ln\chi \tag{3.60}$$

for $t \geq t_2$. Hence,

$$\int_{t_1}^{t}\widetilde{\Lambda}_0(s)ds - \ln\left(\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|\right) > \nu + \ln\frac{\varepsilon}{R_0} \tag{3.61}$$

for $t \geq t_2$. Estimation (3.4) together with (3.2) now yields

$$|z(t)| > R_0 e^{-\nu} e^{\nu}\frac{\varepsilon}{R_0} = \varepsilon \tag{3.62}$$

for all $t \geq t_2$, for which $z(t)$ is defined. □

Corollary 3.6. *Let the assumptions of Theorem 3.1 be fulfilled with* $R_0 > 0$*. If*

$$\lim_{t\to\infty}\left[\int_{t_1}^{t}\widetilde{\Lambda}_0(s)ds - \ln\left(\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|\right)\right] = \infty, \tag{3.63}$$

then for any $\varepsilon > 0$ *there exists* $t_2 \geq t_1$ *such that* (3.59) *holds for all* $t \geq t_2$, *for which* $z(t)$ *is defined.*

The efficiency of Theorem 3.1 and Corollary 3.6 is demonstrated in the following example.

Example 3.7. Consider (1.2) where $a(t) \equiv 4 + 3i$, $b(t) \equiv 2$, $A_k(t) \equiv 0$, $B_k(t) \equiv 0$ for $k = 1, \dots, m$, $\theta_k(t) = t + (1/2k)(\cos kt - 1)$, $g(t, z, w_1, \dots, w_m) = 3z + \sum_{k=1}^{m}(1/2m)e^{-t}w_k$.

Obviously, $t-(1/k) \le \theta_k(t) \le t$ and $1/2 \le \theta_k'(t) \le (3/2)$. Suppose that $t_0 = 1$ and $T \ge 2$. Then, $\widetilde{\gamma} \equiv 3+\sqrt{5}$, $\widetilde{c} \equiv -2i$. Further,

$$\begin{aligned} \left|\widetilde{\gamma}(t)g(t,z,w_1,\dots,w_m)+\widetilde{c}(t)\overline{g}(t,z,w_1,\dots,w_m)\right| &\le 3\left|\widetilde{\gamma}(t)z+\widetilde{c}(t)\overline{z}\right| \\ &+\sum_{k=1}^{m}\frac{1}{2m}\overset{-t}{\mathrm{e}}\left|\widetilde{\gamma}(\theta_k(t))w_k+\widetilde{c}(\theta_k(t))\overline{w}_k\right|. \end{aligned} \tag{3.64}$$

Taking $\widetilde{\varkappa}_0(t) \equiv 3$, $\widetilde{\kappa}_{0k}(t) = (1/2m)\mathrm{e}^{-t}$, $\tau_0 = T$, $R_0 = 0$, $\widetilde{\vartheta}(t) \equiv 0$, $\widetilde{\beta}_0(t) = -(1/m)\mathrm{e}^{-t}$, $\widetilde{\Lambda}_0(t) = \widetilde{\Theta}_0(t) = 1-\mathrm{e}^{-t}$ (>0) in Theorem 3.1, we have

$$\theta_k'(t)\widetilde{\beta}_0(t) \le -\widetilde{\lambda}_{0k}(t), \qquad \widetilde{\beta}_0'(t) \ge \widetilde{\Theta}_0(t)\widetilde{\beta}_0(t) \tag{3.65}$$

for $t \in [T,\infty)$ and Theorem 3.1 and Corollary 3.6 are applicable to the considered equation.

As a corollary of Theorem 3.3 we obtain sufficient conditions for the existence of a bounded solution of (1.2) or the existence of a solution $z_0(t)$ of (1.2) satisfying $\lim_{t\to\infty} z_0(t) = 0$.

Corollary 3.8. *Let the assumptions of Theorem 3.3 be satisfied. If*

$$\limsup_{t\to\infty}\left[\frac{1}{\left|\widetilde{\gamma}(t)\right|-\left|\widetilde{c}(t)\right|}\exp\left(\int_T^t \xi(s)ds\right)\right] < \infty, \tag{3.66}$$

then there is a bounded solution $z_0(t)$ of (1.2)*. If*

$$\lim_{t\to\infty}\left[\frac{1}{\left|\widetilde{\gamma}(t)\right|-\left|\widetilde{c}(t)\right|}\exp\left(\int_T^t \xi(s)ds\right)\right] = 0, \tag{3.67}$$

then there is a solution $z_0(t)$ of (1.2) *such that*

$$\lim_{t\to\infty} z_0(t) = 0. \tag{3.68}$$

The next example shows how Theorem 3.3 and Corollary 3.8 (namely the first part) can be used.

Example 3.9. Consider (1.2) where $a(t) \equiv 4+3i$, $b(t) \equiv i$, $A_k(t) \equiv 0$, $B_k(t) \equiv 0$, $\theta_k(t) = t-e^{-kt}$ for $k = 1,\dots,m$, $g(t,z,w_1,\dots,w_m) = (1/2)z + \sum_{k=1}^{m}(1/4m)w_k + e^{-t}$.

Obviously $t-1 \le \theta_k(t) \le t$ and $\theta_k'(t) = 1+k\mathrm{e}^{-kt} \ge 1 > 0$ for $t \ge 0$. Suppose that $t_0 = 1$ and $T \ge 2$. Then,

$$\begin{gathered} \widetilde{\gamma}(t) = \operatorname{Im}a(t) + \sqrt{\left(\operatorname{Im}a(t)\right)^2 - |b(t)|^2\operatorname{sgn}\left(\operatorname{Im}a(t)\right)} \equiv 3+2\sqrt{2}, \\ \widetilde{c}(t) = -ib(t) \equiv 1. \end{gathered} \tag{3.69}$$

Further,

$$\begin{aligned}
&\left|\tilde{\gamma}(t)g(t,z,w_1,\dots,w_m)+\tilde{c}(t)\overline{g}(t,z,w_1,\dots,w_m)\right| \\
&\quad\le \frac{|\tilde{\gamma}|+|\tilde{c}|}{|\tilde{\gamma}|-|\tilde{c}|}\frac{1}{2}\left|\tilde{\gamma}(t)z+\tilde{c}(t)\overline{z}\right| \\
&\qquad+\frac{|\tilde{\gamma}|+|\tilde{c}|}{|\tilde{\gamma}|-|\tilde{c}|}\sum_{k=1}^{m}\left[\frac{1}{4m}\left|\tilde{\gamma}(\theta_k(t))w_k+\tilde{c}(\theta_k(t))\overline{w}_k\right|\right]+\overset{-t}{\mathrm{e}} \\
&\quad=\frac{\sqrt{2}}{2}\left|\tilde{\gamma}(t)z+\tilde{c}(t)\overline{z}\right|+\sqrt{2}\sum_{k=1}^{m}\left[\frac{1}{4m}\left|\tilde{\gamma}(\theta_k(t))w_k+\tilde{c}(\theta_k(t))\overline{w}_k\right|\right]+\overset{-t}{\mathrm{e}}.
\end{aligned}\tag{3.70}$$

If we take $\tilde{\varkappa}(t)\equiv\sqrt{2}/2$, $\tilde{\kappa}_k(t)=\sqrt{2}/4m$, $\tilde{\vartheta}(t)\equiv 0$, $\tilde{\beta}(t)=-\sqrt{2}/4m$, $\tilde{\Lambda}(t)=\tilde{\Theta}(t)=4-(\sqrt{2}/2)$ in Theorem 3.3, we observe that

$$\theta_k'(t)\tilde{\beta}(t)=-\left(1+k\overset{-kt}{\mathrm{e}}\right)\frac{\sqrt{2}}{4m}\le-\frac{\sqrt{2}}{4m}=-\tilde{\lambda}_k(t)\tag{3.71}$$

for $t\in[T,\infty)$. Then, for $\xi\equiv 0$ and $C=1$, we have

$$\begin{aligned}
&\tilde{\Lambda}(t)+\tilde{\beta}(t)\sum_{k=1}^{m}\theta_k'(t)\exp\left[-\int_{\theta_k(t)}^{t}\xi(s)ds\right]-\xi(t) \\
&\quad=4-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{4m}\sum_{k=1}^{m}\left(1+k\overset{-kt}{\mathrm{e}}\right)\ge 4-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{4m}\cdot 2m=4-\sqrt{2}>1 \\
&\quad>\overset{-t}{\mathrm{e}}=\rho(t)C^{-1}\exp\left(-\int_{T}^{t}\xi(s)ds\right)
\end{aligned}\tag{3.72}$$

for $t\in[T,\infty)$ and

$$\limsup_{t\to\infty}\left[\frac{1}{|\tilde{\gamma}(t)|-|\tilde{c}(t)|}\exp\left(\int_T^t\xi(s)ds\right)\right]=\frac{1}{2+2\sqrt{2}}<\infty,\tag{3.73}$$

and hence the assertions of Theorem 3.3 and the first part of Corollary 3.8 hold true.

4. Summary

We investigated the problem of instability and asymptotic behaviour of real two-dimensional differential system with a finite number of nonconstant delays. We focused on the case corresponding to the situation when the equilibrium 0 of the autonomous system (2.2) is a focus or a centre and it is unstable. We obtained several criteria for instability properties of the solutions as well as conditions for the existence of bounded solutions. We used the methods of complexification, the method of Lyapunov-Krasovskii functional and a Ważewski topological principle for retarded functional differential equations of Carathéodory type. At the end we supplied several corollaries and explanatory examples.

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