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Research Article

Existence of Three Solutions for a Nonlinear Fractional Boundary Value Problem via a Critical Points Theorem

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This paper is concerned with the existence of three solutions to a nonlinear fractional boundary value problem as follows: $(d/dt)((1/2)_0D_t^{\alpha-1}({}_0^C D_t^{\alpha}u(t)) - (1/2)_tD_T^{\alpha-1}({}_t^C D_T^{\alpha}u(t))) + \lambda a(t)f(u(t)) = 0$, a.e. $t \in [0,T], u(0) = u(T) = 0$, where $\alpha \in (1/2, 1]$, and λ is a positive real parameter. The approach is based on a critical-points theorem established by G. Bonanno.

1. Introduction

Differential equations with fractional order have recently proved to be strong tools in the modeling of many physical phenomena in various fields of physical, chemical, biology, engineering, and economics. There has been significant development in fractional differential equations, one can see the monographs [1–5] and the papers [6–20] and the references therein.

Critical-point theory, which proved to be very useful in determining the existence of solution for integer-order differential equation with some boundary conditions, for example, one can refer to [21–25]. But till now, there are few results on the solution to fractional boundary value problem which were established by the critical-point theory, since it is often very difficult to establish a suitable space and variational functional for fractional boundary value problem. Recently, Jiao and Zhou [26] investigated the following fractional boundary value problem:

$$\frac{d}{dt}\left(\frac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}\,{}_tD_T^{-\beta}\;(u'(t))\right) + \nabla F(t, u(t)) = 0, \quad \text{a.e. } t \in [0, T],$$
$$u(0) = u(T) = 0 \tag{1.1}$$

by using the critical point theory, where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \leq \beta < 1$, respectively, $F : [0,T] \times \mathbf{R}^N \to \mathbf{R}$ is a given function and $\nabla F(t,x)$ is the gradient of F at x.

In this paper, by using the critical-points theorem established by Bonanno in [27], a new approach is provided to investigate the existence of three solutions to the following fractional boundary value problems:

$$\frac{d}{dt}\left(\frac{1}{2}\,{}_0D_t^{\alpha-1}\left({}_0^CD_t^{a}u(t)\right) - \frac{1}{2}\,{}_tD_T^{\alpha-1}\left({}_t^CD_T^{a}u(t)\right)\right) + \lambda a(t) f(u(t)) = 0, \quad \text{a.e. } t \in [0,T],$$
$$u(0) = u(T) = 0, \tag{1.2}$$

where $\alpha \in (1/2, 1]$, ${}_0D_t^{\alpha-1}$ and ${}_tD_T^{\alpha-1}$ are the left and right Riemann-Liouville fractional integrals of order $1-\alpha$ respectively, ${}_0^cD_t^{\alpha}$ and ${}_t^cD_T^{\alpha}$ are the left and right Caputo fractional derivatives of order α respectively, λ is a positive real parameter, $f : \mathbf{R} \to \mathbf{R}$ is a continuous function, and $a : \mathbf{R} \to \mathbf{R}$ is a nonnegative continuous function with $a(t) \not\equiv 0$.

2. Preliminaries

In this section, we first introduce some necessary definitions and properties of the fractional calculus which are used in this paper.

Definition 2.1 (see [5]). Let f be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional integrals of order α for function f denoted by ${}_aD_t^{-\alpha}f(t)$ and ${}_tD_b^{-\alpha}f(t)$, respectively, are defined by

$$\begin{aligned} {}_aD_t^{-\alpha}f(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds, \quad t \in [a,b],\ \alpha > 0, \\ {}_tD_b^{-\alpha}f(t) &= \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}f(s)ds, \quad t \in [a,b],\ \alpha > 0, \end{aligned} \tag{2.1}$$

provided the right-hand sides are pointwise defined on $[a,b]$, where $\Gamma(\alpha)$ is the gamma function.

Definition 2.2 (see [5]). Let $\gamma \geq 0$ and $n \in \mathbf{N}$.

(i) If $\gamma \in (n-1, n)$ and $f \in AC^n([a,b], \mathbf{R}^N)$, then the left and right Caputo fractional derivatives of order γ for function f denoted by ${}_a^CD_t^{\gamma}f(t)$ and ${}_t^CD_b^{\gamma}f(t)$, respectively, exist almost everywhere on $[a,b]$, ${}_a^CD_t^{\gamma}f(t)$ and ${}_t^CD_b^{\gamma}f(t)$ are represented by

$$\begin{aligned} {}_a^CD_t^{\gamma}f(t) &= \frac{1}{\Gamma(n-\gamma)}\int_a^t (t-s)^{n-\gamma-1}f^{(n)}(s)ds, \quad t \in [a,b], \\ {}_t^CD_b^{\gamma}f(t) &= \frac{(-1)^n}{\Gamma(n-\gamma)}\int_t^b (s-t)^{n-\gamma-1}f^{(n)}(s)ds, \quad t \in [a,b], \end{aligned} \tag{2.2}$$

respectively.

(ii) If $\gamma = n-1$ and $f \in AC^{n-1}([a,b],\mathbf{R}^N)$, then ${}_a^C D_t^{n-1} f(t)$ and ${}_t^C D_b^{n-1} f(t)$ are represented by

$$ {}_a^C D_t^{n-1} f(t) = f^{(n-1)}(t), \qquad {}_t^C D_b^{n-1} f(t) = (-1)^{(n-1)} f^{(n-1)}(t), \quad t \in [a,b]. \tag{2.3}$$

With these definitions, we have the rule for fractional integration by parts, and the composition of the Riemann-Liouville fractional integration operator with the Caputo fractional differentiation operator, which were proved in [2, 5].

Property 1 (see [2, 5]). we have the following property of fractional integration:

$$\int_a^b \left[{}_a D_t^{-\gamma} f(t) \right] g(t) dt = \int_a^b \left[{}_t D_b^{-\gamma} g(t) \right] f(t) dt, \quad \gamma > 0 \tag{2.4}$$

provided that $f \in L^p([a,b],\mathbf{R}^N)$, $g \in L^q([a,b],\mathbf{R}^N)$, and $p \geq 1$, $q \geq 1$, $1/p + 1/q \leq 1 + \gamma$ or $p \neq 1$, $q \neq 1$, $1/p + 1/q = 1 + \gamma$.

Property 2 (see [5]). Let $n \in \mathbf{N}$ and $n-1 < \gamma \leq n$. If $f \in AC^n([a,b],\mathbf{R}^N)$ or $f \in C^n([a,b],\mathbf{R}^N)$, then

$$\begin{aligned} {}_a D_t^{-\gamma} \left({}_a^C D_t^{\gamma} f(t) \right) &= f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!} (t-a)^j, \\ {}_t D_b^{-\gamma} \left({}_t^C D_b^{\gamma} f(t) \right) &= f(t) - \sum_{j=0}^{n-1} \frac{(-1)^j f^{(j)}(b)}{j!} (b-t)^j, \end{aligned} \tag{2.5}$$

for $t \in [a,b]$. In particular, if $0 < \gamma \leq 1$ and $f \in AC([a,b],\mathbf{R}^N)$ or $f \in C^1([a,b],\mathbf{R}^N)$, then

$$ {}_a D_t^{-\gamma} \left({}_a^C D_t^{\gamma} f(t) \right) = f(t) - f(a), \qquad {}_t D_b^{-\gamma} \left({}_t^C D_b^{\gamma} f(t) \right) = f(t) - f(b). \tag{2.6}$$

Remark 2.3. In view of Property 1 and Definition 2.2, it is obvious that $u \in AC([0,T])$ is a solution of BVP (1.2) if and only if u is a solution of the following problem:

$$\begin{gathered} \frac{d}{dt}\left(\frac{1}{2} {}_0 D_t^{-\beta}(u'(t)) + \frac{1}{2} {}_t D_T^{-\beta}(u'(t)) \right) + \lambda a(t) f(u(t)) = 0, \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{gathered} \tag{2.7}$$

where $\beta = 2(1-\alpha) \in [0,1)$.

In order to establish a variational structure for BVP (1.2), it is necessary to construct appropriate function spaces.

Denote by $C_0^{\infty}[0,T]$ the set of all functions $g \in C^{\infty}[0,T]$ with $g(0) = g(T) = 0$.

Definition 2.4 (see [26]). Let $0 < \alpha \leq 1$. The fractional derivative space E_0^α is defined by the closure of $C_0^\infty[0,T]$ with respect to the norm

$$\|u\|_\alpha = \left(\int_0^T \left|{}_0^C D_t^\alpha u(t)\right|^2 dt + \int_0^T |u(t)|^2 dt\right)^{1/2}, \quad \forall u \in E_0^\alpha. \tag{2.8}$$

Remark 2.5. It is obvious that the fractional derivative space E_0^α is the space of functions $u \in L^2[0,T]$ having an α-order Caputo fractional derivative ${}_0^C D_t^\alpha u \in L^2[0,T]$ and $u(0) = u(T) = 0$.

Proposition 2.6 (see [26]). *Let $0 < \alpha \leq 1$. The fractional derivative space E_0^α is reflexive and separable Banach space.*

Lemma 2.7 (see [26]). *Let $1/2 < \alpha \leq 1$. For all $u \in E_0^\alpha$, one has the following:*

(i)

$$\|u\|_{L^2} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \left\|{}_0^C D_t^\alpha u\right\|_{L^2}. \tag{2.9}$$

(ii)

$$\|u\|_\infty \leq \frac{T^{\alpha-1/2}}{\Gamma(\alpha)(2(\alpha-1)+1)^{1/2}} \left\|{}_0^C D_t^\alpha u\right\|_{L^2}. \tag{2.10}$$

By (2.9), we can consider E_0^α with respect to the norm

$$\|u\|_\alpha = \left(\int_0^T \left|{}_0^C D_t^\alpha u(t)\right|^2 dt\right)^{1/2} = \left\|{}_0^C D_t^\alpha u\right\|_{L^2}, \quad \forall u \in E_0^\alpha \tag{2.11}$$

in the following analysis.

Lemma 2.8 (see [26]). *Let $1/2 < \alpha \leq 1$, then for all any $u \in E_0^\alpha$, one has*

$$|\cos(\pi\alpha)| \|u\|_\alpha^2 \leq -\int_0^T {}_0^C D_t^\alpha u(t) \cdot {}_t^C D_T^\alpha u(t) dt \leq \frac{1}{|\cos(\pi\alpha)|} \|u\|_\alpha^2. \tag{2.12}$$

Our main tool is the critical-points theorem [27] which is recalled below.

Theorem 2.9 (see [27]). *Let X be a separable and reflexive real Banach space; $\Phi : X \to \mathbf{R}$ be a nonnegative continuously Gateaux differentiable and sequentially weakly lower semicontinuous functional whose Gateaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbf{R}$ be a continuously Gateaux differentiable function whose Gateaux derivative is compact. Assume that there exists $x_0 \in X$ such that $\Phi(x_0) = \Psi(x_0) = 0$, and that*

(i) $\lim_{\|x\| \to +\infty} (\Phi(x) - \lambda \Psi(x)) = +\infty, forall\lambda \in [0, +\infty]$. *Further, assume that there are $r > 0$, $x_1 \in X$ such that*

(ii) $r < \Phi(x_1)$;

(iii) $\sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^{w}} \Psi(x) < (r/(r + \Phi(x_1)))\Psi(x_1)$.

Then, for each

$$\lambda \in \Lambda_1 = \left] \frac{\Phi(x_1)}{\Psi(x_1) - \sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^{w}} \Psi(x)}, \frac{r}{\sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^{w}} \Psi(x)} \right[, \tag{2.13}$$

the equation

$$\Phi'(x) - \lambda \Psi'(x) = 0 \tag{2.14}$$

has at least three solutions in X *and, moreover, for each* $h > 1$, *there exists an open interval*

$$\Lambda_2 \subset \left[0, \frac{hr}{(r(\Psi(x_1)/\Phi(x_1))) - \sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^{w}} \Psi(x)}\right] \tag{2.15}$$

and a positive real number σ *such that, for each* $\lambda \in \Lambda_2$, (2.14) *has at least three solutions in* X *whose norms are less than* σ.

3. Main Result

For given $u \in E_0^{\alpha}$, we define functionals $\Phi, \Psi : E^{\alpha} \to \mathbf{R}$ as follows:

$$\begin{gathered}
\Phi(u) := -\frac{1}{2} \int_0^T {}_0^C D_t^{\alpha} u(t) \cdot {}_t^C D_T^{\alpha} u(t) dt, \\
\Psi(u) := \int_0^T a(t) F(u(t)) dt,
\end{gathered} \tag{3.1}$$

where $F(u) = \int_0^u f(s) ds$. Clearly, Φ and Ψ are Gateaux differentiable functional whose Gateaux derivative at the point $u \in E_0^{\alpha}$ are given by

$$\begin{gathered}
\Phi'(u)v = -\frac{1}{2} \int_0^T \left({}_0^C D_t^{\alpha} u(t) \cdot {}_t^C D_T^{\alpha} v(t) + {}_t^C D_T^{\alpha} u(t) \cdot {}_0^C D_t^{\alpha} v(t) \right) dt, \\
\Psi'(u)v = \int_0^T a(t) f(u(t)) v(t) dt = -\int_0^T \int_0^t a(s) f(u(s)) ds \cdot v'(t) dt,
\end{gathered} \tag{3.2}$$

for every $v \in E_0^\alpha$. By Definition 2.2 and Property 2, we have

$$\Phi'(u)v = \int_0^T \left(\frac{1}{2}{}_0D_t^{\alpha-1}\left({}_0^C D_t^\alpha u(t)\right) - \frac{1}{2}{}_tD_T^{\alpha-1}\left({}_t^C D_T^\alpha u(t)\right)\right) \cdot v'(t)dt. \tag{3.3}$$

Hence, $I_\lambda = \Phi - \lambda\Psi \in C^1(E_0^\alpha, \mathbf{R})$. If $u_* \in E_0^\alpha$ is a critical point of I_λ, then

$$\begin{aligned} 0 &= I_\lambda'(u_*)v \\ &= \int_0^T \left(\frac{1}{2}{}_0D_t^{\alpha-1}\left({}_0^C D_t^\alpha u_*(t)\right) - \frac{1}{2}{}_tD_T^{\alpha-1}\left({}_t^C D_T^\alpha u_*(t)\right)\right. \\ &\qquad \left. +\lambda \int_0^t a(s) f(u_*(s))ds\right) \cdot v'(t)dt, \end{aligned} \tag{3.4}$$

for $v \in E_0^\alpha$. We can choose $v \in E_0^\alpha$ such that

$$v(t) = \sin\frac{2k\pi t}{T} \quad \text{or} \quad v(t) = 1 - \cos\frac{2k\pi t}{T}, \quad k = 1, 2, \ldots. \tag{3.5}$$

The theory of Fourier series and (3.4) imply that

$$\frac{1}{2}{}_0D_t^{\alpha-1}\left({}_0^C D_t^\alpha u_*(t)\right) - \frac{1}{2}{}_tD_T^{\alpha-1}\left({}_t^C D_T^\alpha u_*(t)\right) + \lambda \int_0^t a(s) f(u_*(s))ds = C \tag{3.6}$$

a.e. on $[0, T]$ for some $C \in \mathbf{R}$. By (3.6), it is easy to know that $u_* \in E_0^\alpha$ is a solution of BVP (1.2).

By Lemma 2.7, if $\alpha > 1/2$, we have for each $u \in E_0^\alpha$ that

$$\|u\|_\infty \leq \Omega\left(\int_0^T \left|{}_0^C D_t^\alpha u(t)\right|^2 dt\right)^{1/2} = \Omega\|u\|_\alpha, \tag{3.7}$$

where

$$\Omega = \frac{T^{\alpha-1/2}}{\Gamma(\alpha)\sqrt{2(\alpha-1)+1}}. \tag{3.8}$$

Given two constants $c \geq 0$ and $d \neq 0$, with $c \neq \sqrt{(2A(\alpha)/|\cos(\pi\alpha)|)}\Omega \cdot d$, where Ω as in (3.8).

For convenience, set

$$A(\alpha) := \frac{8\Gamma^2(2-\alpha)}{\Gamma(4-2\alpha)} T^{1-2\alpha}\left(\left(1 + 3^{3-2\alpha}\right)2^{4\alpha-5} - 2^{2\alpha-3} - 1\right). \tag{3.9}$$

Theorem 3.1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function,* $a : \mathbf{R} \to \mathbf{R}$ *be a nonnegative continuous function with* $a(t) \not\equiv 0$, *and* $1/2 < \alpha \leq 1$. *Put* $F(x) = \int_0^x f(s)ds$ *for every* $x \in \mathbb{R}$, *and assume that there exist four positive constants* c, d, μ, *and* p, *with* $c < \sqrt{(2A(\alpha)/|\cos(\pi\alpha)|)}\Omega \cdot d$ *and* $p < 2$, *such that*

(H1) $F(x) \leq \mu(1 + |x|^p)$, *for all* $x \in \mathbb{R}$;

(H2) $F(x) \geq 0$ *for all* $x \in [0, \Gamma(2-\alpha)d]$, *and*

$$\begin{aligned} F(x) < {} & \frac{|\cos(\pi\alpha)|c^2}{(|\cos(\pi\alpha)|c^2 + 2\Omega^2 A(\alpha)d^2)\int_0^T a(t)dt} \\ & \times \Bigg[F(\Gamma(2-\alpha)d)\int_{T/4}^{3T/4} a(t)dt \\ & \quad + \frac{T}{4\Gamma(2-\alpha)d}\int_0^{\Gamma(2-\alpha)d} b(s)F(s)ds \Bigg], \quad \forall x \in [-c, c], \end{aligned} \tag{3.10}$$

where $b(s) = a((T/4\Gamma(2-\alpha)d)s) + a(T - (T/4\Gamma(2-\alpha)d)s)$. *Then, for each*

$$\begin{aligned} & \lambda \in \Lambda_1 \\ & = \left] \frac{A(\alpha)d^2}{\Re_a + \Re\int_0^{\Gamma(2-\alpha)d} b(x)F(x)dx - \int_0^T a(t)dt \cdot \max_{|x|\leq c} F(x)}, \frac{c^2|\cos(\pi\alpha)|}{2\Omega^2\int_0^T a(t)dt \cdot \max_{|x|\leq c} F(x)} \right[, \end{aligned} \tag{3.11}$$

where $\Re_a$ *and* $\Re$ *denote* $F(\Gamma(2-\alpha)d)\int_{T/4}^{3T/4} a(t)dt$ *and* $T/(4\Gamma(2-\alpha)d)$ *respectively, the problem* (1.2) *admits at least three solutions in* E_0^α *and, moreover, for each* $h > 1$, *there exists an open interval*

$$\Lambda_2 \subset \left[0, \frac{hA(\alpha)d^2}{\Re_a + \Re\int_0^{2\Gamma(2-\alpha)} b(x)F(x)dx - (2\Omega^2 A(\alpha)d^2/c^2|\cos(\pi\alpha)|)\int_0^T a(t)dt \cdot \max_{|x|\leq c} F(x)} \right] \tag{3.12}$$

such that, for each $\lambda \in \Lambda_2$, *the problem* (1.2) *admits at least three solutions in* E_0^α *whose norms are less that* σ.

Proof. Let Φ, Ψ be the functionals defined in the above. By the Lemma 5.1 in [26], Φ is continuous and convex, hence it is weakly sequentially lower semicontinuous. Moreover, Φ is coercive, continuously Gateaux differentiable functional whose Gateaux derivative admits a continuous inverse on E_0^α. The functional Ψ is well defined, continuously Gateaux differentiable and with compact derivative. It is well known that the critical point of the functional $\Phi - \lambda\Psi$ in E_0^α is exactly the solution of BVP (1.2). □

From (H1) and (2.12), we get

$$\lim_{\|u\|_\alpha \to +\infty} (\Phi(u) - \lambda\Psi(u)) = +\infty, \tag{3.13}$$

for all $\lambda \in [0, +\infty[$. Put

$$u_1(t) = \begin{cases} \dfrac{4\Gamma(2-\alpha)d}{T}t, & t \in \left[0, \dfrac{T}{4}\right[, \\ \Gamma(2-\alpha)d, & t \in \left[\dfrac{T}{4}, \dfrac{3T}{4}\right], \\ \dfrac{4\Gamma(2-\alpha)d}{T}(T-t), & t \in \left]\dfrac{T}{4}, T\right]. \end{cases} \tag{3.14}$$

It is easy to check that $u_1(0) = u_1(T) = 0$ and $u_1 \in L^2[0,T]$. The direct calculation shows

$$\begin{aligned}
{}_0^CD_t^\alpha u_1(t) &= \begin{cases} \dfrac{4d}{T}t^{1-\alpha}, & t \in \left[0, \dfrac{T}{4}\right[, \\ \dfrac{4d}{T}\left(t^{1-\alpha} - \left(t - \dfrac{T}{4}\right)^{1-\alpha}\right), & t \in \left[\dfrac{T}{4}, \dfrac{3T}{4}\right], \\ \dfrac{4d}{T}\left(t^{1-\alpha} - \left(t - \dfrac{T}{4}\right)^{1-\alpha} - \left(t - \dfrac{3T}{4}\right)^{1-\alpha}\right), & t \in \left]\dfrac{3T}{4}, T\right], \end{cases} \\
\|u_1\|_\alpha^2 &= \int_0^T \left({}_0^CD_t^\alpha u_1(t)\right)^2 dt = \int_0^{T/4} + \int_{T/4}^{3T/4} + \int_{3T/4}^T \left({}_0^CD_t^\alpha u_1(t)\right)^2 dt \\
&= \frac{16d^2}{T^2}\left[\int_0^T t^{2(1-\alpha)}dt + \int_{T/4}^T \left(t - \frac{T}{4}\right)^{2(1-\alpha)} dt + \int_{3T/4}^T \left(t - \frac{3T}{4}\right)^{2(1-\alpha)} dt \right. \\
&\qquad - 2\int_{T/4}^T t^{1-\alpha}\left(t - \frac{T}{4}\right)^{1-\alpha} dt - 2\int_{3T/4}^T t^{1-\alpha}\left(t - \frac{3T}{4}\right)^{1-\alpha} dt \\
&\qquad \left. + 2\int_{3T/4}^T \left(t - \frac{T}{4}\right)^{1-\alpha}\left(t - \frac{3T}{4}\right)^{1-\alpha} dt\right] \\
&= \frac{16d^2}{T^2}\left[\left(1 + \left(\frac{3}{4}\right)^{3-2\alpha} + \left(\frac{1}{4}\right)^{3-2\alpha}\right)\frac{T^{3-2\alpha}}{3-2\alpha} - 2\int_{T/4}^T t^{1-\alpha}\left(t - \frac{T}{4}\right)^{1-\alpha} dt \right. \\
&\qquad \left. - 2\int_{3T/4}^T t^{1-\alpha}\left(t - \frac{3T}{4}\right)^{1-\alpha} dt + 2\int_{3T/4}^T \left(t - \frac{T}{4}\right)^{1-\alpha}\left(t - \frac{3T}{4}\right)^{1-\alpha} dt\right] < \infty .
\end{aligned} \tag{3.15}$$

That is, ${}_{0}^{C}D_{t}^{\alpha}u_1 \in L^2[0,T]$. Thus, $u_1 \in E_0^{\alpha}$. Moreover, the direct calculation shows

$$
\begin{gathered}
{}_{t}^{C}D_{T}^{\alpha}u_1(t) = \begin{cases} \dfrac{4d}{T}\left((T-t)^{1-\alpha} - \left(\dfrac{3T}{4}-t\right)^{1-\alpha} - \left(\dfrac{T}{4}-t\right)^{1-\alpha}\right), & t \in \left[0, \dfrac{T}{4}\right[, \\ \dfrac{4d}{T}\left((T-t)^{1-\alpha} - \left(\dfrac{3T}{4}-t\right)^{1-\alpha}\right), & t \in \left[\dfrac{T}{4}, \dfrac{3T}{4}\right], \\ \dfrac{4d}{T}(T-t)^{1-\alpha}, & t \in \left]\dfrac{3T}{4}, T\right], \end{cases} \\
\begin{aligned}
\Phi(u_1) &= -\frac{1}{2}\int_0^T {}_{0}^{C}D_{t}^{\alpha}u_1(t) \cdot {}_{t}^{C}D_{T}^{\alpha}u_1(t)\,dt \\
&= -\frac{8d^2}{T^2}\Bigg[\int_0^{T/4} t^{1-\alpha}\left((T-t)^{1-\alpha} - \left(\frac{3T}{4}-t\right)^{1-\alpha} - \left(\frac{T}{4}-t\right)^{1-\alpha}\right)dt \\
&\qquad + \int_{T/4}^{3T/4}\left(t^{1-\alpha} - \left(t-\frac{T}{4}\right)^{1-\alpha}\right)\left((T-t)^{1-\alpha} - \left(\frac{3T}{4}-t\right)^{1-\alpha}\right)dt \\
&\qquad + \int_{3T/4}^{T}\left(t^{1-\alpha} - \left(t-\frac{T}{4}\right)^{1-\alpha} - \left(t-\frac{3T}{4}\right)^{1-\alpha}\right)(T-t)^{1-\alpha}dt\Bigg] \\
&= -\frac{8d^2}{T^2}\Bigg[\int_0^T t^{1-\alpha}(T-t)^{1-\alpha}dt - \int_0^{T/4} t^{1-\alpha}\left(\frac{T}{4}-t\right)^{1-\alpha}dt \\
&\qquad + \int_{T/4}^{3T/4}\left(t-\frac{T}{4}\right)^{1-\alpha}\left(\frac{3T}{4}-t\right)^{1-\alpha}dt - \int_{3T/4}^{T}\left(t-\frac{3T}{4}\right)^{1-\alpha}(T-t)^{1-\alpha}dt \\
&\qquad - \int_0^{3T/4} t^{1-\alpha}\left(\frac{3T}{4}-t\right)^{1-\alpha} - \int_{T/4}^{T}\left(t-\frac{T}{4}\right)^{1-\alpha}(T-t)^{1-\alpha}dt\Bigg] \\
&= \frac{8\Gamma^2(2-\alpha)}{\Gamma(4-2\alpha)}T^{1-2\alpha}d^2\left(\left(1+3^{3-2\alpha}\right)2^{4\alpha-5} - 2^{2\alpha-3} - 1\right) = A(\alpha)d^2,
\end{aligned} \\
\begin{aligned}
\Psi(u_1) &= \int_0^T a(t)F(u_1(t))\,dt \\
&= \int_0^{T/4} a(t)F\left(\frac{4\Gamma(2-\alpha)d}{T}t\right)dt + \int_{T/4}^{3T/4} a(t)F(\Gamma(2-\alpha)d)\,dt \\
&\quad + \int_{3T/4}^{T} a(t)F\left(\frac{4\Gamma(2-\alpha)d}{T}(T-t)\right)dt \\
&= F(\Gamma(2-\alpha)d)\int_{T/4}^{3T/4} a(t)\,dt + \frac{T}{4\Gamma(2-\alpha)d}\int_0^{\Gamma(2-\alpha)d} b(x)F(x)\,dx.
\end{aligned}
\end{gathered}
\tag{3.16}
$$

Let $r = (|\cos(\pi\alpha)|/2\Omega^2)c^2$. Since $c < \sqrt{(2A(\alpha)/|\cos(\pi\alpha)|)}\Omega \cdot d$, we obtain $r < \Phi(u_1)$.

By (2.12) and (3.7), one has $\Phi(u) \le r \Rightarrow \|u\|_\infty \le c$. Thus,

$$\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}} \Psi(u) = \sup_{u\in\Phi^{-1}(]-\infty,r\,])} \Psi(u) \le \max_{|x|\le c} F(x) \int_0^T a(t)dt. \tag{3.17}$$

Moreover, we have

$$\begin{aligned}
&\frac{r}{r+\Phi(u_1)}\Psi(u_1)\\
&\quad= \frac{\left(|\cos(\pi\alpha)|/2\Omega^2\right)c^2}{\left(|\cos(\pi\alpha)|/2\Omega^2\right)c^2 + A(\alpha)d^2}\\
&\qquad\times\left[F(\Gamma(2-\alpha)d)\int_{T/4}^{3T/4} a(t)dt + \frac{T}{4\Gamma(2-\alpha)d}\int_0^{\Gamma(2-\alpha)d} b(x)F(x)dx\right]\\
&\quad= \frac{|\cos(\pi\alpha)|c^2}{|\cos(\pi\alpha)|c^2 + 2\Omega^2 A(\alpha)d^2}\\
&\qquad\times\left[F(\Gamma(2-\alpha)d)\int_{T/4}^{3T/4} a(t)dt + \frac{T}{4\Gamma(2-\alpha)d}\int_0^{\Gamma(2-\alpha)d} b(x)F(x)dx\right].
\end{aligned} \tag{3.18}$$

Hence, from (H2) one has

$$\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}} \Psi(u) < \frac{r}{r+\Phi(u_1)}\Psi(u_1). \tag{3.19}$$

Now, taking into account that

$$\begin{aligned}
&\frac{\Phi(u_1)}{\Psi(u_1) - \sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}} \Psi(u)}\\
&\quad\le \frac{A(\alpha)d^2}{\Re_a + \Re\int_0^{\Gamma(2-\alpha)d} b(x)F(x)dx - \int_0^T a(t)dt\cdot\max_{|x|\le c} F(x)},\\
&\frac{r}{\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}} \Psi(u)} \ge \frac{c^2|\cos(\pi\alpha)|}{2\Omega^2\int_0^T a(t)dt\cdot\max_{|x|\le c} F(x)},\\
&\frac{hr}{r(\Psi(u_1)/\Phi(u_1)) - \sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}} \Psi(u)}\\
&\quad\le \frac{hA(\alpha)d^2}{\Re_a + \Re\int_0^{2\Gamma(2-\alpha)} b(x)F(x)dx - \left(2\Omega^2 A(\alpha)d^2/c^2|\cos(\pi\alpha)|\right)\int_0^T a(t)dt\cdot\max_{|x|\le c} F(x)}\\
&\quad= m.
\end{aligned} \tag{3.20}$$

Thus, by Theorem 2.9 it follows that, for each $\lambda \in \Lambda_1$, BVP (1.2) admits at least three solutions, and there exists an open interval $\Lambda_2 \subset [0, m]$ and a real positive number σ such that, for each $\lambda \in \Lambda_2$, BVP (1.2) admits at least three solutions in E_0^α whose norms are less than σ.

Finally, we give an example to show the effectiveness of the results obtained here.

Let $\alpha = 0.8$, $T = 1$, $a(t) \equiv 1$, and $f(u) = e^{-u}u^8(9-u) + \sqrt{u}$. Then BVP (1.2) reduces to the following boundary value problem:

$$\begin{gathered}\frac{d}{dt}\left(\frac{1}{2}{}_0D_t^{-0.2}\left({}_0^CD_t^{0.8}u(t)\right) - \frac{1}{2}{}_tD_1^{-0.2}\left({}_t^CD_1^{0.8}u(t)\right)\right) + \lambda\left(e^{-u}u^8(9-u) + \sqrt{u}\right) \\ = 0, \quad \text{a.e. } t \in [0,1], \\ u(0) = u(1) = 0.\end{gathered} \tag{3.21}$$

Example 3.2. Owing to Theorem 3.1, for each $\lambda \in]0.291, 0.318[$, BVP (3.21) admits at least three solutions. In fact, put $c = 1$ and $d = 2$, it is easy to calculate that $\Omega = 1.1089$, $A(0.8) = 1.3313$, and

$$\sqrt{\frac{2A(0.8)}{|\cos(0.8\pi)|}}\Omega \cdot d = 4.0235 > 1 = c. \tag{3.22}$$

Since

$$F(x) = \int_0^x f(s)ds = e^{-x}x^9 + \frac{2}{3}x^{3/2}, \tag{3.23}$$

we have that condition (H1) holds. Moreover, $F(x) \geq 0$ for each $x \in [0, 2\Gamma(1.2)]$, and

$$\begin{gathered}\frac{|\cos(0.8\pi)|}{|\cos(0.8\pi)| + 2\Omega^2A(0.8)\cdot 2^2}\left[\frac{1}{2}F(2\Gamma(1.2)) + \frac{1}{4\Gamma(1.2)}\int_0^{2\Gamma(1.2)}F(s)ds\right] \\ > 1.064 > 1.0345 = e^{-1} + \frac{2}{3} \geq F(x), \quad |x| \leq 1,\end{gathered} \tag{3.24}$$

which implies that condition (H2) holds. Thus, by Theorem 3.1, for each $\lambda \in]0.291, 0.318[$, the problem (3.21) admits at least three nontrivial solutions in $E_0^{0.8}$. Moreover, for each $h > 1$, there exists an open interval $\Lambda \subset]0, 3.4674h[$ and a real positive number σ such that, for each $\lambda \in \Lambda$, the problem (3.21) admits at least three solutions in $E_0^{0.8}$ whose norms are less than σ.

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