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Research Article

The Solution of a Class of Singularly Perturbed Two-Point Boundary Value Problems by the Iterative Reproducing Kernel Method

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In (Wang et al., 2011), we give an iterative reproducing kernel method (IRKM). The main contribution of this paper is to use an IRKM (Wang et al., 2011), in singular perturbation problems with boundary layers. Two numerical examples are studied to demonstrate the accuracy of the present method. Results obtained by the method indicate that the method is simple and effective.

1. Introduction

Singularly perturbed problems (SPPs) arise frequently in applications including geophysical fluid dynamics, oceanic and atmospheric circulation, chemical reactions, and optimal control. In this paper, we consider the following singularly perturbed two-point boundary value problem:

$$\begin{gathered} \varepsilon U_{xx}(x) + a(x)U_x(x) + b(x)U(x) = F(x, U(x)), \quad x \in (0,1), \\ U(0) = \alpha, \qquad U(1) = \beta, \end{gathered} \tag{1.1}$$

where ε is a positive small parameter, $a(x)$, $b(x)$, and $F(x,v)$ are known functions, and $U(x)$ is a unknown function to be determined. In this paper, we assume that (1.1) has a unique solution that belongs to $W_2^3[0,1]$. Like in [1–5], we give reproducing kernel spaces $W_2^1[a,b]$ and $W_2^3[a,b]$. (i) We define the inner product space $W_2^1[a,b] = \{u \mid u$ is one-variable absolutely continuous function, $u' \in L^2[a,b]\}$. The inner product is given by $\langle u(x), v(x)\rangle_{W_2^1} = u(a)v(a) + \int_a^b u'(x)v'(x)dx$. The space $W_2^1[a,b]$ is a reproducing kernel space, and its reproducing kernel is $R_x^{\{1\}}(y)$. (ii) Space $W_2^3[a,b] = \{u \mid u, u', u''$ is one-variable absolutely continuous function, $u(a) = u(b) = 0, u''' \in L^2[a,b]\}$.

The inner product is given by $\langle u(x), v(x)\rangle_{W_2^3} = u'(a)v'(a) + u''(a)v''(a) + \int_a^b u'''(x)v'''(x)dx$. The space $W_2^3[a,b]$ is a reproducing kernel space, and its reproducing kernel is $R_x^{\{2\}}(y)$.

2. Iterative Reproducing Kernel Method (IRKM)

In order to solve (1.1), we first give the analytical and approximate solutions of the following operator equation:

$$(Lu)(x) = F(x, u(x)), \tag{2.1}$$

where $L : H[a,b] \to H_1[a,b]$ is a bounded linear operator and L^{-1} is existent. $H_1[a,b]$ is an RKHS with the reproducing kernel $\overline{K}(x,t)$, $H[a,b]$ is also an RKHS with the reproducing kernel $K(x,t)$.

Theorem 2.1. *If L^{-1} is existent and $\{x_i\}_{i=1}^{\infty}$ are countable dense points in $[a,b]$, Letting $\overline{\varphi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\varphi_k(x)$, where the β_{ik} are the coefficients resulting from the Gram-Schmidt orthonormalization, $\varphi_i(x) = (L_y K(x,y))(x_i)$, $i = 1, 2, \ldots$, then*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k, u(x_k))\overline{\varphi}_i(x) \tag{2.2}$$

is an analytical solution of (2.1).

Proof. $u(x)$ can be expanded to the Fourier series in terms of normal orthogonal basis $\{\overline{\varphi}_i(x)\}_{i=1}^{\infty}$ in $H[a,b]$:

$$\begin{aligned}
u(x) &= \sum_{i=1}^{\infty} \langle u(x), \overline{\varphi}_i(x)\rangle \overline{\varphi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle u(x), \varphi_k(x)\rangle \overline{\varphi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle u(x), (L_s K_x(s))(x_k)\rangle \overline{\varphi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} (L_s \langle u(x), K_x(s)\rangle)(x_k) \overline{\varphi}_i(x)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(L_s u(s))(x_k)\overline{\varphi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik} f(x_k, u(x_k))\overline{\varphi}_i(x).
\end{aligned}
\tag{2.3}
$$

□

(i) Linear Problem

Suppose (2.1) is a linear problem, that is, $f(x,u) = F(x)$. We define an approximate solution $u_n(x)$ by

$$
u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik} F(x_k)\overline{\varphi}_i(x). \tag{2.4}
$$

Theorem 2.2 (convergence analysis). *Let $\varepsilon_n^2 = \|u(x) - u_n(x)\|^2$; then the sequence of real numbers ε_n is monotonously decreasing and $\varepsilon_n \to 0$ and the sequence $u_n(x)$ is convergent uniformly to $u(x)$, $k = 0,1,2$.*

Proof. We have

$$
\varepsilon_n^2 = \|u(x) - u_n(x)\|^2 = \sum_{i=n+1}^{\infty}\langle u(x), \overline{\varphi}_i(x)\rangle\overline{\varphi}_i(x) = \sum_{i=n+1}^{\infty}\left(\langle u(x), \overline{\varphi}_i(x)\rangle\right)^2, \tag{2.5}
$$

and clearly $\varepsilon_{n-1} \ge \varepsilon_n$ and consequently $\{\varepsilon_n\}$ is monotone decreasing in the sense of $\|\cdot\|$. By Theorem 2.1, we know that $\sum_{i=1}^{\infty}\langle u(x), \overline{\varphi}_i(x)\rangle\overline{\varphi}_i(x)$ is convergent in the norm of $\|\cdot\|$, then we have $\varepsilon_n^2 = \|u(x) - u_n(x)\|^2 \to 0$.

For any $x \in [a,b]$, $k = 0,1,2$,

$$
\left|u_n^{(k)}(x) - u^{(k)}(x)\right| = \left|\left\langle u_n(t) - u(t), \frac{\partial^k K(x,t)}{\partial x^k}\right\rangle\right| \le \|u_n(t) - u(t)\| \cdot \left\|\frac{\partial^k K(x,t)}{\partial x^k}\right\|, \tag{2.6}
$$

and by the expression of $K(x,t)$, there exists $C_k > 0$, such that $\|\partial^k K(x,t)/\partial x^k\| < C_k$; thus

$$
\left|u_n^{(k)}(x) - u^{(k)}(x)\right| \le C_k\|u_n(t) - u(t)\| = C_k\varepsilon_n \longrightarrow 0. \tag{2.7}
$$

□

(ii) Nonlinear Problem

Suppose that (1.1) is a nonlinear problem, that is, $f(x,u) = N(u) + F(x)$, where $N : H[a,b] \to H_1[a,b]$ is a nonlinear operator, and we give an iterative sequence $u_n(x)$:

$u_{0,*}(x)$ is the solution of the linear equation $Lu = F(x)$,

$u_{n+1,*}(x)$ is the solution of the linear equation $Lu = N(u_{n,*}) + F(x)$, $n = 0,1,2\ldots$.

Lemma 2.3. *If* $u_{n,*}(x) \to u(x)$, *then* $u(x)$, *is the solution of* (1.1).

Theorem 2.4. *Suppose that the nonlinear operator* $A \triangleq (L^{-1}N) : H_1[a,b] \to H[a,b]$ *satisfies the contractive mapping principle, that is,*

$$\|A(u) - A(v)\| \leq \lambda \|u - v\|, \quad \lambda < 1; \tag{2.8}$$

then $u_{n,*}(x)$ *is convergent.*

3. Solution of Singularly Perturbed Problems

We notice that a small variation in the parameter ε produces a large variation in the solution. In other words, we are treating an ill-posed problem. In this paper, by dividing the domain $[0,1]$ into three subdomains $[0,d]$, $[d,1-d]$, and $[1-d,1]$.

(i) Outer Region

We have

$$\begin{gathered} \varepsilon U_{xx}(x) + a(x)U_x(x) + b(x)U(x) = F(x,U(x)), \quad x \in [d,1-d], \\ u(d) = p, \qquad u(1-d) = q. \end{gathered} \tag{3.1}$$

Letting $\overline{u}(x) = U(x) - ((x-d)/(1-2d))q + ((x+d-1)/(1-2d))$ and $(L_1\overline{u})(x) = \varepsilon\overline{u}_{xx}(x) + a(x)\overline{u}_x + b(x)\overline{u}$, (3.1) can further be converted into

$$(L_1\overline{u})(x) = \overline{f}(x,\overline{u}(x)), \tag{3.2}$$

where $\overline{f}(x,\overline{u}(x)) = F(x,u(x)) - a(x)((1/(1-2d))q - (1/(1-2d))p) - b(x)(((x-d)/(1-2d))q + ((x+d-1)/(1-2d))p)$. Using IRKM, we can get the solution of the outer region problem.

(ii) Left Layer

We have

$$\begin{gathered} \varepsilon U_{xx}(x) + a(x)U_x(x) + b(x)U(x) = F(x,U(x)), \quad x \in [0,d), \\ U(0) = \alpha, \qquad U(d) = p \text{ is known.} \end{gathered} \tag{3.3}$$

Letting $u(x) = U(x) - \alpha - (x/d)(p-\alpha)$, $x/d = x_1$, then $x = dx_1$, $u(x) = u(dx_1) \doteq \overline{u}(x_1)$, $du/dx = (1/d)(d\overline{u}/dx_1)$, and $d^2u/dx^2 = (1/d^2)(d^2\overline{u}/dx_1^2)$. In space $W_2^3[0,1]$, (3.3) can further be converted into following form:

$$(L_\varepsilon\overline{u})(x_1) = \overline{f}(x_1,\overline{u}(x_1)), \tag{3.4}$$

where $\overline{f}(x_1,\overline{u}(x_1)) = d^2(F(x,U(x)) - (a(x)/d)(p-\alpha) - b(x)(\alpha + (x/d)(p-\alpha)))$, $(L_\varepsilon\overline{u})(x_1) = \varepsilon\overline{u}_{x_1x_1}(x_1) + a(x_1)d\overline{u}_{x_1}(x_1) + b(x_1)d^2\overline{u}(x_1)$. Using IRKM, we can get the solution of the inner region (left layer near) problem.

Table 1: Comparison of results, $x \in [0, d]$, $\varepsilon = 2^{-10}$, and $d = 0.1$.

x	$u_T(x)$	$u_{100}(x)$	$\lvert u_T - u_{100}\rvert$	$\lvert u_T - u_{200}\rvert$
0	0	0	0	0
0.01	−0.272864	−0.272866	1.8788×10^{-6}	7.48551×10^{-7}
0.02	−0.468765	−0.46877	5.38727×10^{-6}	1.15911×10^{-7}
0.03	−0.608251	−0.608257	6.10734×10^{-6}	1.17267×10^{-7}
0.04	−0.706254	−0.70626	5.54546×10^{-6}	1.01462×10^{-7}
0.05	−0.773632	−0.773636	4.52354×10^{-6}	8.05672×10^{-7}
0.06	−0.818281	−0.818285	3.45181×10^{-6}	6.04003×10^{-7}
0.07	−0.845955	−0.845958	2.49497×10^{-6}	4.30798×10^{-7}
0.08	−0.860849	−0.86085	1.66887×10^{-6}	2.84958×10^{-7}
0.09	−0.866029	−0.86603	8.9114×10^{-7}	1.50666×10^{-7}
0.1	−0.863746	−0.863746	3.33067×10^{-16}	2.22045×10^{-7}

Table 2: Comparison of results, $x \in [d, 1-d]$, $\varepsilon = 2^{-30}$, and $d = 0.001$.

x	$u_T(x)$	$u_{100}(x)$	$\lvert u_T - u_{10}\rvert$	$\lvert u_T - u_{100}\rvert$
0.001	−0.99999	−0.99999	1.51212×10^{-13}	4.71756×10^{-12}
0.1008	−0.903026	−0.903045	1.89196×10^{-5}	7.69723×10^{-10}
0.2006	−0.652715	−0.652709	5.8384×10^{-6}	1.58393×10^{-10}
0.3004	−0.344297	−0.344299	1.66259×10^{-6}	4.10425×10^{-10}
0.4002	−0.0951225	−0.0951222	3.12214×10^{-7}	2.67338×10^{-10}
0.5998	−0.0951225	−0.0951223	1.7619×10^{-7}	6.92579×10^{-10}
0.6996	−0.344297	−0.344298	1.11205×10^{-6}	2.66992×10^{-10}
0.7994	−0.652715	−0.652711	3.96364×10^{-6}	1.77961×10^{-10}
0.8992	−0.903026	−0.903039	1.27772×10^{-5}	6.22121×10^{-8}
0.999	−0.99999	−0.99999	6.67721×10^{-12}	1.91558×10^{-12}

(iii) Right Layer

We have

$$\begin{gathered} \varepsilon U_{xx}(x) + a(x)U_x(x) + b(x)U(x) = F(x, U(x)), \quad x \in (1-d, 1], \\ U(1) = \beta, \qquad U(1-d) = q. \end{gathered} \tag{3.5}$$

Letting $u(x) = U(x) - \beta + ((x-1)/d)(q-\beta)$, $(x/d) - (1/d) + 1 = x_1$, then $x = dx_1 + 1 - d$, $u(x) = u(dx_1 + 1 - d) \doteq \overline{u}(x_1)$, $du/dx = (1/d)(d\overline{u}/dx_1)$, and $d^2u/dx^2 = (1/d^2)(d^2\overline{u}/dx_1^2)$. In space $W_2^3[0,1]$, (3.5) can further be converted into following form:

$$(L_\varepsilon \overline{u})(x_1) = \overline{f}(x_1, \overline{u}(x_1)), \tag{3.6}$$

where $f(x, u(x)) = F(x, U(x)) + (a(x)/d)(q-\beta) - b(x)(\beta - ((x-1)/d)(q-\beta))$, q is known (the outer solution has been given), $\overline{f}(x_1, \overline{u}(x_1)) = d^2 f(x, u(x))$, and $(L_\varepsilon \overline{u})(x_1) = \varepsilon \overline{u}_{x_1x_1}(x_1) + a(x_1)d\overline{u}_{x_1}(x_1) + b(x_1)d^2\overline{u}(x_1)$. Using IRKM, we can get the solution of the inner region

Table 3: Comparison of results, $x \in [1-d, 1]$, $\varepsilon = 2^{-10}$, and $d = 0.01$.

x	$u_T(x)$	$u_{10}(x)$	$\|u_T - u_{10}\|$	$u_{200}(x)$	$\|u_T - u_{200}\|$
0.991	−0.249439	−0.249431	8.11552×10^{-6}	−0.249439	2.72812×10^{-9}
0.992	−0.225227	−0.225219	7.75753×10^{-6}	−0.225227	3.72799×10^{-9}
0.993	−0.200201	−0.200194	7.17326×10^{-6}	−0.200201	4.18195×10^{-9}
0.994	−0.174338	−0.174331	6.40087×10^{-6}	−0.174338	4.18298×10^{-9}
0.995	−0.147609	−0.147604	5.47897×10^{-6}	−0.147609	3.82517×10^{-9}
0.996	−0.119989	−0.119984	4.44655×10^{-6}	−0.119989	3.20353×10^{-9}
0.997	−0.0914472	−0.0914438	3.34288×10^{-6}	−0.0914472	2.41376×10^{-9}
0.998	−0.0619555	−0.0619533	2.20746×10^{-6}	−0.0619555	1.55216×10^{-9}
0.999	−0.0314835	−0.0314825	1.07993×10^{-6}	−0.0314835	7.1531×10^{-10}
1.	-2.22045×10^{-16}	-3.62743×10^{-17}	1.8577×10^{-16}	-3.62852×10^{-17}	1.85759×10^{-16}

Table 4: Comparison of results, $x \in [0, d]$, and $d = 0.01$

x	$u_T(x)$ $\varepsilon = 2^{-10}$	$u_{3,10}(x)$ $\varepsilon = 2^{-10}$	$\|u_T - u_{3,10}\|$ $\varepsilon = 2^{-10}$	$u_T(x)$ $\varepsilon = 2^{-5}$	$u_{3,10}(x)$ $\varepsilon = 2^{-5}$	$\|u_T - u_{3,10}\|$ $\varepsilon = 2^{-5}$
0.001	0.0324619	0.0324564	5.56236×10^{-6}	0.00628391	0.0062835	-4.09563×10^{-7}
0.002	0.063871	0.0638658	5.18059×10^{-6}	0.0125208	0.0125204	-3.72485×10^{-7}
0.003	0.0942614	0.0942567	4.67451×10^{-6}	0.0187109	0.0187106	-3.57056×10^{-7}
0.004	0.123666	0.123662	4.20082×10^{-6}	0.0248545	0.0248541	-3.30969×10^{-7}
0.005	0.152117	0.152113	3.72089×10^{-6}	0.0309517	0.0309514	-3.0024×10^{-7}
0.006	0.179645	0.179642	3.23223×10^{-6}	0.0370029	0.0370026	-2.61703×10^{-7}
0.007	0.20628	0.206277	2.78314×10^{-6}	0.0430082	0.043008	-2.14219×10^{-7}
0.008	0.232051	0.232049	2.18254×10^{-6}	0.048968	0.0489678	-1.56195×10^{-7}
0.009	0.256986	0.256984	2.14645×10^{-6}	0.0548823	0.0548822	-8.66593×10^{-8}

(right layer near) problem. After solving the inner and outer region problems, we combine their solutions to obtain an approximate solution to the original problem (1.1) over the interval $0 \le x \le 1$.

4. Numerical Examples

Example 4.1. This example is from [6–8]:

$$
\begin{gathered}
-\varepsilon u_{xx} + u = f(x), \quad 0 \le x \le 1, \\
u(0) = 0, \qquad u(1) = 0.
\end{gathered}
\tag{4.1}
$$

We determine $f(x)$ to get the true solution, the true solution $u_T(x) = (e^{(x-1)/\sqrt{\varepsilon}} + e^{-x/\sqrt{\varepsilon}})/(1 + e^{-1/\sqrt{\varepsilon}}) - \cos^2(\pi x)$. The numerical results are given in Tables 1, 2, and 3.

Example 4.2. Considering the following nonlinear singularly perturbed problem with boundary layers

$$\varepsilon u_{xx} + \frac{e^x}{x} u_x + e^u = f(x), \quad 0 < x \le 1,$$
$$u(0) = 0, \qquad u(1) = 0, \tag{4.2}$$

we determine $f(x)$ to get the true solution, the true solution $u_T(x) = 1 + (x-1)e^{-x/\sqrt{\varepsilon}} - xe^{(x-1)/\sqrt{\varepsilon}}$. The numerical results are given in Tables 3 and 4.

5. Conclusions

In this paper, IRKM was employed successfully for solving a class of SPPs with boundary layers. The numerical results show that the present method is an accurate and reliable analytical technique for SPP with boundary layers.

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