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Research Article

Some Operator Inequalities on Chaotic Order and Monotonicity of Related Operator Function

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We will discuss some operator inequalities on chaotic order about several operators, which are generalization of Furuta inequality and show monotonicity of related Furuta type operator function.

1. Introduction

An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all vectors x in a Hilbert space, and T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

Theorem LH (Löwner-Heinz inequality, denoted by (LH) briefly). *If $A \geq B \geq 0$ holds, then $A^{\alpha} \geq B^{\alpha}$ for any $\alpha \in [0, 1]$.*

This was originally proved in [1, 2] and then in [3]. Although (LH) asserts that $A \geq B \geq 0$ ensures $A^{\alpha} \geq B^{\alpha}$ for any $\alpha \in [0, 1]$, unfortunately $A^{\alpha} \geq B^{\alpha}$ does not always hold for $\alpha > 1$. The following result has been obtained from this point of view.

Theorem F (Furuta inequality). *If $A \geq B \geq 0$, then for each $r \geq 0$,*

(i) $(B^{r/2} A^p B^{r/2})^{1/q} \geq (B^{r/2} B^p B^{r/2})^{1/q}$,

(ii) $(A^{r/2} A^p A^{r/2})^{1/q} \geq (A^{r/2} B^p A^{r/2})^{1/q}$

hold for $p \geq 0$ and $q \geq 1$ with $(1 + r)q \geq p + r$.

The original proof of Theorem F is shown in [4], an elementary one-page proof is in [5], and alternative ones are in [6, 7]. We remark that the domain of the parameters p, q, and r in Theorem F is the best possible for the inequalities (i) and (ii) under the assumption $A \geq B \geq 0$; see [8].

We write $A \gg B$ if $\log A \geq \log B$ for $A, B > 0$, which is called the chaotic order.

Theorem A. *For $A, B > 0$, the following (i) and (ii) hold:*

(i) *$A \gg B$ holds if and only if $A^r \geq (A^{r/2} B^p A^{r/2})^{r/(p+r)}$ for $p, r \geq 0$;*

(ii) *$A \gg B$ holds if and only if for any fixed $\delta \geq 0$, $F_{A,B}(p, r) = A^{-r/2}(A^{r/2} B^p A^{r/2})^{(\delta+r)/(p+r)} A^{-r/2}$ is a decreasing function of $p \geq \delta$ and $r \geq 0$.*

(i) in Theorem A is shown in [9, 10], an excellent proof in [11], a proof in the case $p = r$ in [12], (ii) in [9, 10], and so forth.

Lemma B (see [11]). *Let A be a positive invertible operator, and let B be an invertible operator. For any real number λ,*

$$(BAB^*) = BA^{1/2}\left(A^{1/2}B^*BA^{1/2}\right)^{\lambda-1}A^{1/2}B^*. \tag{1}$$

Definition 1. Let $A_n, A_{n-1}, \ldots, A_2, A_1,\ B \geq 0, r_1, r_2, \ldots, r_n \geq 0$, and $p_1, p_2, \ldots, p_n \geq 0$ for a natural number n.

Let $C_{A_i,B}[n]$ be defined by

$$\begin{aligned} C_{A_i,B}[n] &= A_n^{r_n/2}\left\{A_{n-1}^{r_{n-1}/2}\left[\cdots A_3^{r_3/2}\left\{A_2^{r_2/2}\left(A_1^{r_1/2}B^{p_1}A_1^{r_1/2}\right)^{p_2}A_2^{r_2/2}\right\}^{p_3}\right.\right. \\ &\qquad \left.\left.\times A_3^{r_3/2}\cdots\right]A_{n-1}^{r_{n-1}/2}\right\}^{p_n}A_n^{r_n/2}. \end{aligned} \tag{2}$$

For example,

$$C_{A_i,B}[2] = A_2^{r_2/2}\left(A_1^{r_1/2}B^{p_1}A_1^{r_1/2}\right)^{p_2}A_2^{r_2/2},$$
$$C_{A_i,B}[4] = A_4^{r_4/2}\left\{A_3^{r_3/2}\left[A_2^{r_2/2}\left(A_1^{r_1/2}B^{p_1}A_1^{r_1/2}\right)^{p_2}A_2^{r_2/2}\right]^{p_3}\times A_3^{r_3/2}\right\}^{p_4}A_4^{r_4/2}. \tag{3}$$

Let $q[n]$ be defined by

$$q[n] = \{\cdots[(p_1 + r_1)p_2 + r_2]p_3 + \cdots + r_{n-1}\}p_n + r_n. \tag{4}$$

For example,

$$q[1] = p_1 + r_1, \qquad q[2] = (p_1 + r_1)p_2 + r_2,$$
$$q[4] = \{[(p_1 + r_1)p_2 + r_2]p_3 + r_3\}p_4 + r_4. \tag{5}$$

For the sake of convenience, we define

$$C_{A_i,B}[0] = B, \qquad q[0] = 1, \tag{6}$$

and these definitions in (6) may be reasonable by (2) and (4).

Lemma 2. *For $A_n, A_{n-1}, \ldots, A_2, A_1, B \geq 0$ and any natural number n, we have*

(i) $C_{A_i,B}[n] = A_n^{r_n/2}C_{A_i,B}[n-1]^{p_n}A_n^{r_n/2}$,

(ii) $q[n] = q[n-1]p_n + r_n$.

Proof. (i) and (ii) can be easily obtained by definitions (2) and (4). □

2. Basic Results Associated with $C_{A_i,B}[n]$ and $q[n]$

We will give some operator inequalities on chaotic order, and Theorem 5 is further extension of Theorem 3.1 in [13].

Lemma 3. *If $A \gg B$, for $p \geq 0$ and $r \geq 0$, then $A \gg (A^{r/2}B^pA^{r/2})^{1/(p+r)}$.*

Proof. Since $A \gg B$, we can obtain the following inequality.

$A^r \geq (A^{r/2}B^pA^{r/2})^{r/(p+r)}$ holds for $p \geq 0$ and $r \geq 0$ by (i) of Theorem A.

Take the logarithm on both sides of the previous inequality; that is,

$$\log A^r \geq \log\left(A^{r/2}B^pA^{r/2}\right)^{r/(p+r)}, \tag{7}$$

therefor we have

$$A \gg \left(A^{r/2}B^pA^{r/2}\right)^{1/(p+r)}. \tag{8}$$

□

Theorem 4. *If $A_n \gg A_{n-1} \gg \cdots \gg A_2 \gg A_1 \gg B$ and $r_1, r_2, \ldots, r_n \geq 0$, $p_1, p_2, \ldots, p_n \geq 0$ for a natural number n. Then the following inequality holds:*

$$A_n \gg C_{A_i,B}[n]^{1/q[n]}, \tag{9}$$

where $C_{A_i,B}[n]$ and $q[n]$ are defined in (2) *and* (4).

Proof. We will show (9) by mathematical induction. In the case $n = 1$.

Since $A_1 \gg B$ implies

$$A_1 \gg \left(A_1^{r_1/2}B^{p_1}A_1^{r_1/2}\right)^{r_1/(p_1+r_1)} \tag{10}$$

holds for any $p_1 \geq 0$ and $r_1 \geq 0$ by Lemma 3, whence (9) for $n = 1$.

Assume that (9) holds for a natural number k ($1 \leq k < n$). We will show that (9) holds $r_1, r_2, \ldots, r_k, r_{k+1} \geq 0$ and $p_1, p_2, \ldots, p_k, p_{k+1} \geq 0$ for $k + 1$.

Put $D = A_{k+1}$, $E = A_k$, and $F = C_{A_i,B}[k]^{1/q[k]}$, and (9) holds for $n = k$ implying

$$D \gg E \gg F > 0. \tag{11}$$

Equation (11) yields the following by Lemma 3, for $r \geq 0$ and $p \geq 0$:

$$D \gg \left(D^{r/2}F^pD^{r/2}\right)^{1/(p+r)}, \tag{12}$$

that is,

$$A_{k+1} \gg \left(A_{k+1}^{r/2}C_{A_i,B}[k]^{p/q[k]}A_{k+1}^{r/2}\right)^{1/(p+r)}. \tag{13}$$

Put $r = r_{k+1}$, $p = q[k]p_{k+1}$ in (13), then by (ii) of Lemma 2, the exponential power $1/(p + r)$ of the right hand side of (13) can be written as follows:

$$\frac{1}{p+r} = \frac{1}{q[k]p_{k+1} + r_{k+1}} = \frac{1}{q[k+1]}, \tag{14}$$

and we have the following desired (15) by (12) and (13):

$$A_{k+1} \gg \left\{A_{k+1}^{r_{k+1}/2}\left(C_{A_i,B}[k]\right)^{p_{k+1}}A_{k+1}^{r_{k+1}/2}\right\}^{1/q[k+1]}$$
$$= C_{A_i,B}[k+1]^{1/q[k+1]}, \tag{15}$$

so that (15) shows that (9) holds for $k + 1$. □

Theorem 5. *If $A_n \gg A_{n-1} \gg \cdots \gg A_2 \gg A_1 \gg B$ and $r_1, r_2, \ldots, r_n \geq 0$ for a natural number n. For any fixed $\delta \geq 0$, let $p_1, p_2, \ldots, p_n$ be satisfied by*

$$\begin{aligned} p_1 &\geq \delta, \\ p_2 &\geq \frac{\delta + r_1}{p_1 + r_1}, \\ &\vdots \\ p_k &\geq \frac{\delta + r_1 + r_2 + \cdots + r_{k-1}}{q[k-1]}, \\ &\vdots \\ p_n &\geq \frac{\delta + r_1 + r_2 + \cdots + r_{n-1}}{q[n-1]}. \end{aligned} \tag{16}$$

The operator function $I_k(p_k, r_k)$ for any natural number k such that $1 \leq k \leq n$ is defined by

$$I_k(p_k, r_k) = A_k^{-r_k/2} C_{A_i,B}[k]^{(\delta + r_1 + r_2 + \cdots + r_k)/q[k]} A_k^{-r_k/2}. \tag{17}$$

Then the following inequality holds:

$$A_{k-1}^{r_{k-1}/2} I_{k-1}(p_{k-1}, r_{k-1}) A_{k-1}^{r_{k-1}/2} \geq I_k(p_k, r_k) \tag{18}$$

for every natural number k such that $1 \leq k \leq n$, where $C_{A_i,B}[n]$ and $q[n]$ are defined in (2) and (4).

Proof. Since $C_{A_i,B}[0] = B$, $q[0] = 1$ in (6), we may define $I_0(p_0, r_0) = B^\delta$ for $p_0 = r_0 = 0$.

Because $A_1 \gg B$, then for any fixed $\delta \geq 0$,

$$B^\delta \geq A_1^{-r_1/2}\left(A_1^{r_1/2} B^{p_1} A_1^{r_1/2}\right)^{(\delta + r_1)/(p_1 + r_1)} A_1^{-r_1/2} \quad \text{for } p_1 \geq \delta,\ r_1 \geq 0, \tag{19}$$

since $F_{A_1,B}(\delta, r_0) \geq F_{A_1,B}(p_1, r_1)$ holds by (ii) of Theorem A. And (19) can be expressed as

$$B^\delta = A_0^{r_0/2} I_0(p_0, r_0) A_0^{r_0/2} \geq I_1(p_1, r_1). \tag{20}$$

We can apply Theorem 4, and we have the following (21) for any natural number k such that $1 \leq k \leq n$:

$$A_{k+1} \gg A_k \gg C_{A_i,B}[k]^{1/q[k]}. \tag{21}$$

Since $X \gg Y$ implies that $X^t \gg Y^t$ holds for any $t \geq 0$, (21) ensures

$$A_{k+1}^{\delta + r_1 + r_2 + \cdots + r_k} \gg C_{A_i,B}[k]^{(\delta + r_1 + r_2 + \cdots + r_k)/q[k]}. \tag{22}$$

Putting $A = A_{k+1}^{\delta + r_1 + r_2 + \cdots + r_k}$, $B_1 = C_{A_i,B}[k]^{(\delta + r_1 + r_2 + \cdots + r_k)/q[k]}$ and applying (19) for $\delta = 1$ and $A \gg B_1$, we have

$$B_1 \geq A^{-r/2}\left(A^{r/2} B_1^p A^{r/2}\right)^{(1+r)/(p+r)} A^{-r/2} \tag{23}$$

holds for $p \geq 1$ and $r \geq 0$.

Putting $r_{k+1} = r(\delta + r_1 + r_2 + \cdots + r_k)$ in (23), then (23) can be rewritten by

$$\begin{aligned} B_1 \geq A_{k+1}^{-r_{k+1}/2} &\left(A_{k+1}^{r_{k+1}/2} C_{A_i,B}[k]^{((\delta + r_1 + r_2 + \cdots + r_k)/q[k])p} \right. \\ &\left. \times A_{k+1}^{r_{k+1}/2}\right)^{(1+r)/(p+r)} A_{k+1}^{-r_{k+1}/2}. \end{aligned} \tag{24}$$

Putting $p = (q[k]p_{k+1})/(\delta + r_1 + r_2 + \cdots + r_k) \geq 1$, since $p_{k+1} \geq (\delta + r_1 + r_2 + \cdots + r_k)/q[k]$ in (16), then we have

$$\begin{aligned} &A_k^{r_k/2} I_k(p_k, r_k) A_k^{r_k/2} \\ &= B_1 = C_{A_i,B}[k]^{(\delta + r_1 + r_2 + \cdots + r_k)/q[k]} \\ &\geq A_{k+1}^{-r_{k+1}/2} \\ &\quad \times \left(A_{k+1}^{r_{k+1}/2} C_{A_i,B}[k]^{((\delta + r_1 + r_2 + \cdots + r_k)/q[k])p} A_{k+1}^{r_{k+1}/2}\right)^{(1+r)/(p+r)} \\ &\quad \times A_{k+1}^{-r_{k+1}/2} \\ &= A_{k+1}^{-r_{k+1}/2} C_{A_i,B}[k+1]^{(\delta + r_1 + r_2 + \cdots + r_k + r_{k+1})/(q[k+1])} A_{k+1}^{-r_{k+1}/2} \\ &= I_{k+1}(p_{k+1}, r_{k+1}), \end{aligned} \tag{25}$$

and we have (18) for k such that $1 \leq k \leq n$ by (25) and (20) since (20) means (18) for $k = 1$. □

Corollary 6. *If $A_n \gg A_{n-1} \gg \cdots \gg A_2 \gg A_1 \gg B$ and $r_1, r_2, \ldots, r_n \geq 0$ for a natural number n. For any fixed $\delta \geq 0$, let $p_1, p_2, \ldots, p_n$ be satisfied by (16).*

Then the following inequalities hold:

$$\begin{aligned} B^\delta &\geq A_1^{-r_1/2}\left(A_1^{r_1/2} B^{p_1} A_1^{r_1/2}\right)^{(\delta + r_1)/(p_1 + r_1)} A_1^{-r_1/2} \\ &\geq A_1^{-r_1/2} A_2^{-r_2/2} \\ &\quad \times \left[A_2^{r_2/2}\left(A_1^{r_1/2} B^{p_1} A_1^{r_1/2}\right)^{p_2} A_2^{r_2/2}\right]^{(\delta + r_1 + r_2)/((p_1 + r_1)p_2 + r_2)} \\ &\quad \times A_2^{-r_2/2} A_1^{-r_1/2} \\ &\ \ \vdots \\ &\geq A_1^{-r_1/2} A_2^{-r_2/2} A_3^{-r_3/3} \cdots A_{n-1}^{-r_{n-1}/2} A_n^{-r_n/2} \\ &\quad \times C_{A_i,B}[n]^{(\delta + r_1 + r_2 + \cdots + r_n)/q[n]} \\ &\quad \times A_n^{-r_n/2} A_{n-1}^{-r_{n-1}/2} \cdots A_3^{-r_3/3} A_2^{-r_2/2} A_1^{-r_1/2}, \end{aligned} \tag{26}$$

where $C_{A_i,B}[n]$, $q[n]$, and $I_k(p_k, r_k)$ $(1 \leq k \leq n)$ are defined in (2), (4), and (17).

Proof. Applying (18) of Theorem 5 for k such that $1 \le k \le n$, we have

$$\begin{aligned}
B^{\delta} &= A^{r_0/2} I_0(p_0, r_0) A^{r_0/2} \\
&\ge I_1(p_1, r_1) \\
&= A_1^{-r_1/2}\left(A_1^{r_1/2} B^{p_1} A_1^{r_1/2}\right)^{(\delta+r_1)/(p_1+r_1)} A_1^{-r_1/2} \\
&\ge A_1^{-r_1/2} I_2(p_2, r_2) A_1^{-r_1/2} \\
&= A_1^{-r_1/2} A_2^{-r_2/2}\left[A_2^{r_2/2}\left(A_1^{r_1/2} B^{p_1} A_1^{r_1/2}\right)^{p_2} \times A_2^{r_2/2}\right]^{(\delta+r_1+r_2)/((p_1+r_1)p_2+r_2)} \\
&\quad \times A_2^{-r_2/2} A_1^{-r_1/2} \\
&\ \ \vdots \\
&\ge A_1^{-r_1/2} A_2^{-r_2/2} A_3^{-r_3/3} \cdots A_{n-1}^{-r_{n-1}/2} I_n(p_n, r_n) \\
&\quad \times A_{n-1}^{-r_{n-1}/2} \cdots A_3^{-r_3/3} A_2^{-r_2/2} A_1^{-r_1/2} \\
&= A_1^{-r_1/2} A_2^{-r_2/2} A_3^{-r_3/3} \cdots A_{n-1}^{-r_{n-1}/2} A_n^{-r_n/2} \\
&\quad \times C_{A_i,B}[n]^{(\delta+r_1+r_2+\cdots+r_n)/q[n]} \\
&\quad \times A_n^{-r_n/2} A_{n-1}^{-r_{n-1}/2} \cdots A_3^{-r_3/3} A_2^{-r_2/2} A_1^{-r_1/2}.
\end{aligned} \tag{27}$$

□

3. Monotonicity Property on Operator Functions

We would like to emphasize that the condition of Theorem 7 is stronger than Theorem 5, and moreover when we discuss monotonicity property on operator functions, we can only apply Theorem 7.

Theorem 7. *If $A_n \gg A_{n-1} \gg \cdots \gg A_2 \gg A_1 \gg B$ and $r_1, r_2, \ldots, r_n \ge 0$, $p_1, p_2, \ldots, p_n \ge 0$ for a natural number n. Then the following inequality holds:*

$$A_n^{r_n} \ge C_{A_i,B}[n]^{r_n/q[n]}, \tag{28}$$

where $C_{A_i,B}[n]$ and $q[n]$ are defined in (2) *and* (4).

Proof. We will show (28) by mathematical induction. In the case $n = 1$.

Since $A_1 \gg B$ implies

$$A_1 \ge \left(A_1^{r_1/2} B^{p_1} A_1^{r_1/2}\right)^{r_1/(p_1+r_1)} \tag{29}$$

holds for any, $p_1 \ge 0$ and $r_1 \ge 0$ by (i) of Theorem A, whence (28) for $n = 1$.

Assume that (28) holds for a natural number k $(1 \le k < n)$. We will show (28) for $r_1, r_2, \ldots, r_{k+1} \ge 0$ and $p_1, p_2, \ldots, p_k$, $p_{k+1} \ge 0$ for $k + 1$.

We can obtain the following inequality from the hypothesis (28) for the case $n = k$:

$$A_k^{r_k} \ge C_{A_i,B}[k]^{r_k/q[k]}, \tag{30}$$

hence we have $A_{k+1} \gg A_k \gg C_{A_i,B}[k]^{1/q[k]}$, and (i) of Theorem A ensures

$$A_{k+1}^{r} \ge \left(A_{k+1}^{r/2} C_{A_i,B}[k]^{p/q[k]} A_{k+1}^{r/2}\right)^{r/(p+r)} \quad \text{for } p, r \ge 0. \tag{31}$$

Putting $r = r_{k+1}$ and $p = q[k]p_{k+1}$, then we have the following inequality:

$$\begin{aligned}
A_{k+1}^{r_{k+1}} &\ge \left(A_{k+1}^{r_{k+1}/2} C_{A_i,B}[k]^{p_{k+1}} A_{k+1}^{r_{k+1}/2}\right)^{r_{k+1}/(q[k]p_{k+1}+r_{k+1})} \\
&= C_{A_i,B}[k+1]^{r_{k+1}/q[k+1]},
\end{aligned} \tag{32}$$

so that (32) shows (28) for $k + 1$. □

Theorem 8. *If $A_n \gg A_{n-1} \gg \cdots \gg A_2 \gg A_1 \gg B$ and $r_1, r_2, \ldots, r_n \ge 0$ for a natural number n. For any fixed $\delta \ge 0$, let $p_1, p_2, \ldots, p_n$ be satisfied by* (16).

Then

$$I_n(p_n, r_n) = A_n^{-r_n/2} C_{A_i,B}[n]^{(\delta+r_1+r_2+\cdots+r_n)/q[n]} A_n^{-r_n/2} \tag{33}$$

is a decreasing function of both $r_n \ge 0$ and p_n which satisfies

$$p_n \ge \frac{\delta + r_1 + r_2 + \cdots + r_{n-1}}{q[n-1]}, \tag{34}$$

where $C_{A_i,B}[n]$ and $q[n]$ are defined in (2) *and* (4).

Proof. Since the condition (16) with $\delta \ge 0$ suffices (28) in Theorem 7, we have the following inequality by Theorem 7; see (28).

We state the following important inequality (35) for the forthcoming discussion which is the inequality in (16):

$$q[n] = q[n-1]p_n + r_n \ge \delta + r_1 + r_2 + \cdots + r_{n-1} + r_n \tag{35}$$

because the inequality in (35) follows by (ii) of Lemma 2, and the inequality follows by

$$q[n-1]p_n \ge \delta + r_1 + r_2 + \cdots + r_{n-1} \tag{36}$$

obtained by (34).

(a) Proof of the result that $I_n(p_n, r_n)$ is a decreasing function of p_n.

Without loss of generality, we can assume that $p_n > 0$. We can obtain the following inequality by (28) and by (i) of Lemma 2:

$$\begin{aligned}
A_n^{r_n} &\ge C_{A_i,B}[n]^{r_n/q[n]} = \left(A_n^{r_n/2} C_{A_i,B}[n-1]^{p_n} A_n^{r_n/2}\right)^{r_n/q[n]} \\
&= A_n^{r_n/2} C_{A_i,B}[n-1]^{p_n/2} \\
&\quad \times \left(C_{A_i,B}[n-1]^{p_n/2} A_n^{r_n} C_{A_i,B}[n-1]^{p_n/2}\right)^{(r_n-q[n])/q[n]} \\
&\quad \times C_{A_i,B}[n-1]^{p_n/2} A_n^{r_n/2},
\end{aligned} \tag{37}$$

and (37) implies

$$\begin{aligned}&\left(C_{A_i,B}[n-1]^{p_n/2}A_n^{r_n}C_{A_i,B}[n-1]^{p_n/2}\right)^{(q[n]-r_n)/q[n]}\\&\quad\geq C_{A_i,B}[n-1]^{p_n}.\end{aligned}\tag{38}$$

Put $\alpha=\omega/p_n\in[0,1]$ for $p_n\geq\omega\geq 0$, then we raise each side of (38) to the power $\alpha=\omega/p_n\in[0,1]$, then

$$\begin{aligned}&\left(C_{A_i,B}[n-1]^{p_n/2}A_n^{r_n}C_{A_i,B}[n-1]^{p_n/2}\right)^{((q[n]-r_n)\omega)/(q[n]p_n)}\\&\quad\geq C_{A_i,B}[n-1]^{\omega}.\end{aligned}\tag{39}$$

Whence we have

$$\begin{aligned}
&I_n(p_n,r_n)\\
&=A_n^{-r_n/2}\left(A_n^{r_n/2}C_{A_i,B}[n-1]^{p_n}A_n^{r_n/2}\right)^{(\delta+r_1+r_2+\cdots+r_n)/q[n]}A_n^{-r_n/2}\\
&=A_n^{-r_n/2}\\
&\quad\times\left\{\left(A_n^{r_n/2}C_{A_i,B}[n-1]^{p_n}\right.\right.\\
&\qquad\left.\left.\times A_n^{r_n/2}\right)^{(q[n]+q[n-1]\omega)/q[n]}\right\}^{(\delta+r_1+r_2+\cdots+r_n)/(q[n]+q[n-1]\omega)}\\
&\quad\times A_n^{-r_n/2}\\
&=A_n^{-r_n/2}\left\{A_n^{r_n/2}C_{A_i,B}[n-1]^{p_n/2}\right.\\
&\qquad\times\left(C_{A_i,B}[n-1]^{p_n/2}A_n^{r_n}\right.\\
&\qquad\quad\left.\times C_{A_i,B}[n-1]^{p_n/2}\right)^{(q[n-1]\omega)/q[n]}\\
&\qquad\left.\times C_{A_i,B}[n-1]^{p_n/2}A_n^{r_n/2}\right\}^{(\delta+r_1+r_2+\cdots+r_n)/(q[n]+q[n-1]\omega)}\\
&\quad\times A_n^{-r_n/2}\ \text{by Lemma B}\\
&=A_n^{-r_n/2}\left\{A_n^{r_n/2}C_{A_i,B}[n-1]^{p_n/2}\right.\\
&\qquad\times\left(C_{A_i,B}[n-1]^{p_n/2}A_n^{r_n}\right.\\
&\qquad\quad\left.\times C_{A_i,B}[n-1]^{p_n/2}\right)^{((q[n]-r_n)\omega)/(q[n]p_n)}\\
&\qquad\times C_{A_i,B}[n-1]^{p_n/2}\\
&\qquad\left.\times A_n^{r_n/2}\right\}^{(\delta+r_1+r_2+\cdots+r_n)/(q[n]+q[n-1]\omega)}A_n^{-r_n/2}\\
&\geq A_n^{-r_n/2}\left(A_n^{r_n/2}C_{A_i,B}[n-1]^{p_n/2}C_{A_i,B}[n-1]^{\omega}\right.\\
&\qquad\left.\times C_{A_i,B}[n-1]^{p_n/2}A_n^{r_n/2}\right)^{(\delta+r_1+r_2+\cdots+r_n)/(q[n-1](p_n+\omega)+r_n)}\\
&\quad\times A_n^{-r_n/2}\\
&=I_n(p_n+\omega,r_n),
\end{aligned}\tag{40}$$

and the last inequality holds by LH because (39) and $(\delta+r_1+r_2+\cdots+r_n)/(q[n-1](p_n+\omega)+r_n)\in[0,1]$ which is ensured by (35) and $q[n]+q[n-1]\omega=q[n-1](p_n+\omega)+r_n\geq q[n]$ by (4), so that $I_n(p_n,r_n)$ is a decreasing function of p_n.

(b) Proof of the result that $I_n(p_n,r_n)$ is a decreasing function of r_n.

Without loss of generality, we can assume that $r_n>0$. Raise each side of (28) to the power $\mu/r_n\in[0,1]$ for $r_n\geq\mu\geq 0$ by LH, then

$$A_n^{\mu}\geq\left(A_n^{r_n/2}C_{A_i,B}[n-1]^{p_n}A_n^{r_n/2}\right)^{\mu/q[n]}.\tag{41}$$

We state the following inequality by (ii) of Lemma 3 and (35):

$$\begin{aligned}&q[n]-(\delta+r_1+r_2+\cdots+r_n)\\&\quad=q[n-1]p_n+r_n-(\delta+r_1+r_2+\cdots+r_n)\\&\quad=q[n-1]p_n-(\delta+r_1+r_2+\cdots+r_{n-1})\geq 0.\end{aligned}\tag{42}$$

Then we have

$$\begin{aligned}
&I_n(p_n,r_n)\\
&=A_n^{-r_n/2}C_{A_i,B}[n]^{(\delta+r_1+r_2+\cdots+r_n)/q[n]}A_n^{-r_n/2}\\
&=A_n^{-r_n/2}\left(A_n^{r_n/2}C_{A_i,B}[n-1]^{p_n}A^{r_n/2}\right)^{(\delta+r_1+r_2+\cdots+r_n)/q[n]}A_n^{-r_n/2}\\
&=C_{A_i,B}[n-1]^{p_n/2}\\
&\quad\times\left(C_{A_i,B}[n-1]^{p_n/2}A_n^{r_n}\right.\\
&\qquad\left.\times C_{A_i,B}[n-1]^{p_n/2}\right)^{(\delta+r_1+r_2+\cdots+r_n-q[n])/q[n]}C_{A_i,B}[n-1]^{p_n/2}\\
&=C_{A_i,B}[n-1]^{p_n/2}\\
&\quad\times\left\{\left(C_{A_i,B}[n-1]^{p_n/2}A_n^{r_n}\right.\right.\\
&\qquad\left.\left.\times C_{A_i,B}[n-1]^{p_n/2}\right)^{(q[n]+\mu)/q[n]}\right\}^{(\delta+r_1+r_2+\cdots+r_n-q[n])/(q[n]+\mu)}\\
&\quad\times C_{A_i,B}[n-1]^{p_n/2}\\
&=C_{A_i,B}[n-1]^{p_n/2}\\
&\quad\times\left\{C_{A_i,B}[n-1]^{p_n/2}A_n^{r_n/2}\right.\\
&\qquad\times\left(A_n^{r_n/2}C_{A_i,B}[n-1]^{p_n}A_n^{r_n/2}\right)^{\mu/q[n]}A_n^{r_n/2}\\
&\qquad\left.\times C_{A_i,B}[n-1]^{p_n/2}\right\}^{(\delta+r_1+r_2+\cdots+r_n-q[n])/(q[n]+\mu)}\\
&\quad\times C_{A_i,B}[n-1]^{p_n/2}
\end{aligned}$$

$$\begin{aligned}
&\geq C_{A_i,B}[n-1]^{p_n/2} \\
&\quad \times \left\{ C_{A_i,B}[n-1]^{p_n/2} A_n^{r_n+\mu} \right. \\
&\qquad \left. \times\, C_{A_i,B}[n-1]^{p_n/2} \right\}^{(\delta+r_1+r_2+\cdots+r_n-q[n])/(q[n]+\mu)} \\
&\quad \times C_{A_i,B}[n-1]^{p_n/2} \\
&= I_n\left(p_n, r_n+\mu\right),
\end{aligned} \tag{43}$$

and the last inequality holds by LH because (41) and

$$\begin{aligned}
&\frac{\delta+r_1+r_2+\cdots+r_n-q\,[n]}{q\,[n]+\mu} \\
&\qquad = -\frac{q\,[n]-\left(\delta+r_1+r_2+\cdots+r_n\right)}{q\,[n]+\mu} \in [-1,0],
\end{aligned} \tag{44}$$

so that $I_k(p_k, r_k)$ is a decreasing function of r_n. □

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References

[1] E. Heinz, "Beiträge zur Störungstheorie der Spektralzerlegung," *Mathematische Annalen*, vol. 123, pp. 415–438, 1951.

[2] K. Löwner, "Über monotone Matrixfunktionen," *Mathematische Zeitschrift*, vol. 38, no. 1, pp. 177–216, 1934.

[3] G. K. Pedersen, "Some operator monotone functions," *Proceedings of the American Mathematical Society*, vol. 36, pp. 309–310, 1972.

[4] T. Furuta, "$A \geq B$ assures $(B^r A^p B^r)^{1/q} \geq B^{p+r/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$," *Proceedings of the American Mathematical Society*, vol. 101, no. 1, pp. 85–88, 1987.

[5] T. Furuta, "An elementary proof of an order preserving inequality," *Proceedings of the Japan Academy*, vol. 65, no. 5, p. 126, 1989.

[6] M. Fujii, "Furuta's inequality and its mean theoretic approach," *Journal of Operator Theory*, vol. 23, no. 1, pp. 67–72, 1990.

[7] E. Kamei, "A satellite to Furuta's inequality," *Mathematica Japonica*, vol. 33, no. 6, pp. 883–886, 1988.

[8] K. Tanahashi, "Best possibility of the Furuta inequality," *Proceedings of the American Mathematical Society*, vol. 124, no. 1, pp. 141–146, 1996.

[9] T. Furuta, "Applications of order preserving operator inequality," *Operator Theory*, vol. 59, pp. 180–190, 1992.

[10] M. Uchiyama, "Some exponential operator inequalities," *Mathematical Inequalities and Applications*, vol. 2, no. 3, pp. 469–471, 1999.

[11] T. Furuta, "Extension of the Furuta inequality and Ando-Hiai log-majorization," *Linear Algebra and Its Applications*, vol. 219, pp. 139–155, 1996.

[12] M. Fujii, T. Furuta, and E. Kamei, "Furuta's inequality and its application to Ando's theorem," *Linear Algebra and Its Applications*, vol. 179, pp. 161–169, 1993.

[13] T. Furuta, "Operator functions on chaotic order involving order preserving operator inequalities," *Journal of Mathematical Inequalities*, vol. 6, no. 1, pp. 15–31, 2012.