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Research Article

Remark on Existence and Uniqueness of Solutions for a Coupled System of Multiterm Nonlinear Fractional Differential Equations

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The aim of this paper is to extend the work of Sun et al. (2012) to a more general case for a wider range of function classes of f and g. Our results include the case of the previous work, which are essential improvement of the work of Sun et al. (2012), especially.

1. Introduction

Fractional calculus can give a more vivid and accurate description of problems in various fields of sciences than the traditional calculus [1–3]. Recently many complicated dynamic phenomena were modeled by fractional order calculus system and have received more and more attention; see [4–16].

In recent work [12], Sun et al. studied the existence and uniqueness of solutions for a coupled system of multiterm nonlinear fractional differential equations with an initial value condition

$$\begin{aligned} -\mathscr{D}^{\alpha}x(t) &= f\left(t, y(t), \mathscr{D}^{\beta_1}y(t), \ldots, \mathscr{D}^{\beta_N}y(t)\right), \\ &\qquad \mathscr{D}^{\alpha-i}x(0) = 0, \quad i = 1, 2, \ldots, n_1, \\ -\mathscr{D}^{\sigma}y(t) &= g\left(t, x(t), \mathscr{D}^{\rho_1}x(t), \ldots, \mathscr{D}^{\rho_N}x(t)\right), \\ &\qquad \mathscr{D}^{\sigma-j}y(0) = 0, \quad j = 1, 2, \ldots, n_2, \end{aligned} \tag{1}$$

where $t \in (0,1]$, $\alpha > \beta_1 > \beta_2 > \cdots > \beta_N > 0$, $\sigma > \rho_1 > \rho_2 > \cdots > \rho_N > 0$, $n_1 = [\alpha] + 1$, $n_2 = [\sigma] + 1$ for $\alpha, \sigma \notin \mathbb{N}$ and $n_1 = \alpha$, $n_2 = \sigma$ for $\alpha, \sigma \in \mathbb{N}$, $\beta_q, \rho_q < 1$ for any $q \in \{1, 2, \ldots, N\}$, $\mathscr{D}$ is the standard Riemann-Liouville derivative, and $f, g : [0,1] \times \mathbb{R}^{N+1} \to \mathbb{R}$ are given functions. In order to obtain the existence and uniqueness of solutions of (1), the following growth conditions are introduced in [12].

(H1) There exist two nonnegative functions $a(t), b(t) \in L^1[0,1]$ such that

$$\begin{aligned} &\left|f(t, x_0, x_1, \ldots, x_N)\right| \\ &\qquad \le a(t) + c_0|x_0|^{\gamma_0} + c_1|x_1|^{\gamma_1} + \cdots + c_N|x_N|^{\gamma_N}, \\ &\left|G(t, x_0, x_1, \ldots, x_N)\right| \\ &\qquad \le b(t) + d_0|x_0|^{\theta_0} + d_1|x_1|^{\theta_1} + \cdots + d_N|x_N|^{\theta_N}, \end{aligned} \tag{2}$$

where $c_i, d_i \ge 0$, $0 < \gamma_i, \theta_i < 1$ for $i = 0, 1, 2, \ldots, N$.

(H2) The functions f and g satisfy

$$\begin{aligned} \left|f(t, x_0, x_1, \ldots, x_N)\right| &\le c_0|x_0|^{\gamma_0} + c_1|x_1|^{\gamma_1} + \cdots + c_N|x_N|^{\gamma_N}, \\ \left|g(t, x_0, x_1, \ldots, x_N)\right| &\le d_0|x_0|^{\theta_0} + d_1|x_1|^{\theta_1} + \cdots + d_N|x_N|^{\theta_N}, \end{aligned} \tag{3}$$

where $c_i, d_i \ge 0$, $\gamma_i, \theta_i > 1$ for $i = 0, 1, 2, \ldots, N$.

However, there are many functions which cannot satisfy conditions (H1) and (H2); for example,

$$g(t, x_0, x_1) = \frac{t}{6.08} + \frac{1}{25.26}\left[x_0 + e^{x_1}\right]. \tag{4}$$

Hence the results of [12] are limited only to a small subset of functions which satisfy (H1) and (H2). This paper thus aims to

extend the work of Sun et al. [12] to a more general case with more general conditions on f and g. Our major contributions of this paper include three aspects.

(1) We extend the function classes to more general case; that is, the power growth assumptions (H1) and (H2) are replaced by a very general assumption where the functions $\phi(|x_0|, |x_1|, \ldots, |x_N|)$ and $\psi(|x_0|, |x_1|, \ldots, |x_N|)$ are only required to be nondecreasing function classes (see (A1)), which implies that the function classes are extended to more general case and also include the case of [12] as a special case. In mathematics and applied science, this generalization is important and interesting.

(2) In [12], the weight functions considered constants $c_0, c_1, \ldots, c_N$. But in physics, the influence of weight functions for the whole system is important, so in this work, we improve the weight functions to general Lebesgue integral functions $b(t), d(t) \in L^1[0,1]$, which is also an essential improvement.

(3) In this paper, the nonlinearities f and g are allowed to be exponential growth. However, in [12], the nonlinearities f and g are only allowed to be power growth. It is known that in most cases exponential growth is faster than power growth. From this aspect, this is also a major contribution of this paper.

The remaining part of the paper is organized as follows. In Section 2, some preliminary results including definitions, notations, and lemmas are given. Section 3 presents the main results and the proof of the results. In addition, an example is given to illustrate the application of the main results.

2. Preliminaries and Lemmas

Definition 1 (see [1–3]). The fractional integral of order $\alpha > 0$ of a function $x : (a, +\infty) \to R$ is given by

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s)\, ds, \tag{5}$$

provided that the right-hand side is pointwisely defined on $(a, +\infty)$.

Definition 2 (see [1–3]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function x: $(a, +\infty) \to R$ is given by

$$\mathcal{D}^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} x(s)\, ds, \tag{6}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, and $t > a$, provided that the right-hand side is defined on $(a, +\infty)$.

Lemma 3 (see [1]). *Assume that $x \in L^1[0,1]$ with a fractional derivative of order $\alpha > 0$; then*

$$I^\alpha \mathcal{D}^\alpha x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \tag{7}$$

where $c_i \in R$, $i = 1, 2, \ldots, n$, $n = [\alpha] + 1$.

Lemma 4 (see [12]). *Suppose that $h \in L^1[0,1]$. Then the initial value problem*

$$\begin{gathered} \mathcal{D}^\alpha x(t) = h(t), \quad \alpha > 0,\ t \in [a,b], \\ \mathcal{D}^\alpha x(a) = b_k, \quad k = 1, 2, \ldots n, \end{gathered} \tag{8}$$

has a unique solution

$$x(t) = \sum_{j=1}^n \frac{b_j}{\Gamma(\alpha - j + 1)} (t-a)^{\alpha-j} + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{h(s)}{(t-s)^{1-\alpha}} ds, \tag{9}$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$ and $\alpha = n$ for $\alpha \in \mathbb{N}$.

Let $I = [0,1]$ and let $C(I)$ be the space of all continuous functions defined on I. We define the space

$$\begin{aligned} X \times Y = \{ &(x,y) \mid (x,y) \in C(I) \times C(I), \\ &\left(\mathcal{D}^{\rho_j} x(t), \mathcal{D}^{\beta_j} y(t)\right) \in C(I) \\ &\times C(I), j = 1, 2, \ldots, N \} \end{aligned} \tag{10}$$

endowed with the norm $\|(x,y)\|_{X\times Y} = \max\{\|x\|_X, \|y\|_Y\}$, where

$$\begin{aligned} \|x\|_X &= \max_{t\in I} |x(t)| + \sum_{j=1}^N \max_{t\in I} \left|\mathcal{D}^{\rho_j} x(t)\right|, \\ \|y\|_Y &= \max_{t\in I} |y(t)| + \sum_{j=1}^N \max_{t\in I} \left|\mathcal{D}^{\beta_j} y(t)\right|. \end{aligned} \tag{11}$$

Then $X \times Y$ is a Banach space with norm $\|(x,y)\|_{X\times Y}$.

By Lemma 4, system (1) is equivalent to the following coupled system of integral equations:

$$\begin{aligned} &x(t) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s), \mathcal{D}^{\beta_1} y(s), \ldots, \mathcal{D}^{\beta_N} y(s)\right) ds, \\ &y(t) \\ &= \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} g\left(s, x(s), \mathcal{D}^{\rho_1} x(s), \ldots, \mathcal{D}^{\rho_N} x(s)\right) ds. \end{aligned} \tag{12}$$

Define an operator $T : X \times Y \to X \times Y$

$$\begin{aligned} &T(x,y)(t) \\ &= \left(T_1 x(t), T_2 y(t)\right) \\ &= \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\right. \\ &\qquad \times f\left(s, y(s), \mathcal{D}^{\beta_1} y(s), \ldots, \mathcal{D}^{\beta_N} y(s)\right) ds, \end{aligned}$$

$$\frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1} \times g\left(s, x(s), \mathscr{D}^{\rho_1}x(s), \ldots, \mathscr{D}^{\rho_N}x(s)\right) ds\Bigg). \tag{13}$$

It is obvious that a fixed point of operator T is the solution of coupled system (1).

3. Main Result

Theorem 5. *Let $f, g : [0,1] \times \mathbb{R}^{N+1} \to \mathbb{R}$ be continuous. Assume that*

(*A1*) *there exist nonnegative functions $a, b, c, d \in L^1[0,1]$ and nonnegative nondecreasing functions ϕ, ψ with respect to each variable x_i, $i = 0, 1, 2, \ldots, N$, such that*

$$\begin{aligned} \left|f(t, x_0, x_1, \ldots, x_N)\right| &\le a(t) + b(t)\phi\left(|x_0|, |x_1|, \ldots, |x_N|\right), \\ \left|g(t, x_0, x_1, \ldots, x_N)\right| &\le c(t) + d(t)\psi\left(|x_0|, |x_1|, \ldots, |x_N|\right); \end{aligned} \tag{14}$$

(*A2*) *there exists a constant $R_0 > \max\{k_1, l_1\}$ such that*

$$\begin{aligned} \phi(R_0, R_0, \ldots, R_0) &\le \frac{R_0 - k_1}{k_2}, \\ \psi(R_0, R_0, \ldots, R_0) &\le \frac{R_0 - l_1}{l_2}, \end{aligned} \tag{15}$$

where

$$\begin{aligned} k_1 &= \max_{t\in I}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|a(s)|\,ds + \sum_{j=1}^N \frac{1}{\Gamma(\alpha-\rho_j)}\int_0^t (t-s)^{\alpha-\rho_j-1}|a(s)|\,ds\right), \\ k_2 &= \max_{t\in I}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|b(s)|\,ds + \sum_{j=1}^N \frac{1}{\Gamma(\alpha-\rho_j)}\int_0^t (t-s)^{\alpha-\rho_j-1}|b(s)|\,ds\right), \\ l_1 &= \max_{t\in I}\left(\frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}|c(s)|\,ds + \sum_{j=1}^N \frac{1}{\Gamma(\sigma-\beta_j)}\int_0^t (t-s)^{\sigma-\beta_j-1}|c(s)|\,ds\right), \\ l_2 &= \max_{t\in I}\left(\frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}|d(s)|\,ds + \sum_{j=1}^N \frac{1}{\Gamma(\sigma-\beta_j)}\int_0^t (t-s)^{\sigma-\beta_j-1}|d(s)|\,ds\right). \end{aligned} \tag{16}$$

Then the coupled system (1) *has a solution.*

Proof. Define a closed ball of Banach space $X \times Y$

$$B = \left\{(x, y) \in X \times Y : \|(x, y)\|_{X\times Y} \le R_0\right\}. \tag{17}$$

We will prove that $T : B \to B$. In fact, for any $(x, y) \in B$, by (A1), we have

$$\begin{aligned} &\left|T_1x(t)\right| \\ &= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, y(s), \mathscr{D}^{\beta_1}y(s), \ldots, \mathscr{D}^{\beta_N}y(s)\right)ds\right| \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|a(s)|\,ds \\ &\quad + \frac{\phi(R_0, R_0, \ldots, R_0)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|b(s)|\,ds, \\ &\left|\mathscr{D}^{\rho_j}T_1x(t)\right| \\ &= \left|\mathscr{D}^{\rho_j}I^{\alpha} f\left(t, y(t), \mathscr{D}^{\beta_1}y(t), \ldots, \mathscr{D}^{\beta_N}y(t)\right)\right| \\ &= \left|I^{\alpha-\rho_j} f\left(t, y(t), \mathscr{D}^{\beta_1}y(t), \ldots, \mathscr{D}^{\beta_N}y(t)\right)\right| \\ &= \frac{1}{\Gamma(\alpha-\rho_j)} \times \int_0^t (t-s)^{\alpha-\rho_j-1} f\left(s, y(s), \mathscr{D}^{\beta_1}y(s), \ldots, \mathscr{D}^{\beta_N}y(s)\right)ds \\ &\le \frac{1}{\Gamma(\alpha-\rho_j)} \times \int_0^t (t-s)^{\alpha-\rho_j-1}|a(s)|\,ds \\ &\quad + \frac{\phi(R_0, R_0, \ldots, R_0)}{\Gamma(\alpha-\rho_j)}\int_0^t (t-s)^{\alpha-\rho_j-1}|b(s)|\,ds. \end{aligned} \tag{18}$$

Thus it follows from (18) and (A2) that

$$\begin{aligned} \|T_1x\|_X &= \max_{t\in I}\left|T_1x(t)\right| + \sum_{j=1}^N \max_{t\in I}\left|\mathscr{D}^{\rho_j}T_1x(t)\right| \\ &\le k_1 + k_2\phi(R_0, R_0, \ldots, R_0) \le R_0. \end{aligned} \tag{19}$$

In the same way, we also have

$$\begin{aligned} \|T_2y\|_Y &= \max_{t\in I}\left|T_2y(t)\right| + \sum_{j=1}^N \max_{t\in I}\left|\mathscr{D}^{\rho_j}T_2y(t)\right| \\ &\le l_1 + l_2\psi(R_0, R_0, \ldots, R_0) \le R_0. \end{aligned} \tag{20}$$

Consequently, $\|T_1x\|_X \le R_0$ and $\|T_2y\|_Y \le R_0$, and then $\|T\|_{X\times Y} \le R_0$ for any $(x, y) \in B$; that is, $T : B \to B$.

By [12], we know that the operator T is completely continuous. Therefore, the Schauder fixed point theorem implies that coupled system (1) has a solution in B. The proof is completed. □

From Theorem 5, we easily obtain the following corollaries.

Corollary 6. *Let $f, g : [0,1] \times \mathbb{R}^{N+1} \to \mathbb{R}$ be continuous. Assume that*

(A1) there exist nonnegative functions $c, d \in L^1[0,1]$ and nonnegative nondecreasing functions ϕ, ψ with respect to each variable x_i, $i = 0, 1, 2, \ldots, N$, such that

$$\begin{aligned} &\left|f\left(t, x_0, x_1, \ldots, x_N\right)\right| \le b(t)\,\phi\left(\left|x_0\right|, \left|x_1\right|, \ldots, \left|x_N\right|\right), \\ &\left|g\left(t, x_0, x_1, \ldots, x_N\right)\right| \le d(t)\,\psi\left(\left|x_0\right|, \left|x_1\right|, \ldots, \left|x_N\right|\right); \end{aligned} \tag{21}$$

(A2) there exists a positive constant R_0 such that

$$\begin{aligned} &\phi\left(R_0, R_0, \ldots, R_0\right) \le \frac{R_0}{k_2}, \\ &\psi\left(R_0, R_0, \ldots, R_0\right) \le \frac{R_0}{l_2}, \end{aligned} \tag{22}$$

where

$$\begin{aligned} k_2 &= \max_{t\in I}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|b(s)\right| ds \right. \\ &\qquad \left. + \sum_{j=1}^{N}\frac{1}{\Gamma\left(\alpha-\rho_j\right)}\int_0^t (t-s)^{\alpha-\rho_j-1}\left|b(s)\right| ds\right), \\ l_2 &= \max_{t\in I}\left(\frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}\left|d(s)\right| ds \right. \\ &\qquad \left. + \sum_{j=1}^{N}\frac{1}{\Gamma\left(\sigma-\beta_j\right)}\int_0^t (t-s)^{\sigma-\beta_j-1}\left|d(s)\right| ds\right). \end{aligned} \tag{23}$$

Then the coupled system (1) *has a solution.*

Corollary 7. *Let $f, g : [0,1] \times \mathbb{R}^{N+1} \to \mathbb{R}$ be continuous. Assume that*

(A^*1) there exist nonnegative functions $a, c \in L^1[0,1]$ such that

$$\begin{aligned} &\left|f\left(t, x_0, x_1, \ldots, x_N\right)\right| \le a(t), \\ &\left|g\left(t, x_0, x_1, \ldots, x_N\right)\right| \le c(t). \end{aligned} \tag{24}$$

Then the coupled system (1) *has a solution.*

Proof. In fact, let us choose $R_0 = \max\{k_1, l_1\}$, where

$$\begin{aligned} k_1 &= \max_{t\in I}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|a(s)\right| ds \right. \\ &\qquad \left. + \sum_{j=1}^{N}\frac{1}{\Gamma\left(\alpha-\rho_j\right)}\int_0^t (t-s)^{\alpha-\rho_j-1}\left|a(s)\right| ds\right), \\ l_1 &= \max_{t\in I}\left(\frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}\left|c(s)\right| ds \right. \\ &\qquad \left. + \sum_{j=1}^{N}\frac{1}{\Gamma\left(\sigma-\beta_j\right)}\int_0^t (t-s)^{\sigma-\beta_j-1}\left|c(s)\right| ds\right), \end{aligned} \tag{25}$$

and construct a closed ball of Banach space $X \times Y$

$$B = \left\{(x, y) \in X \times Y : \left\|(x, y)\right\|_{X\times Y} \le R_0\right\}. \tag{26}$$

The rest of proof is similar to Theorem 5. □

Remark 8. The condition (A1) is weaker than (H1) and (H2). Clearly, $\phi(|x_0|, |x_1|, \ldots, |x_N|)$ and $\psi(|x_0|, |x_1|, \ldots, |x_N|)$ include $c_0|x_0|^{\gamma_0} + c_1|x_1|^{\gamma_1} + \cdots + c_N|x_N|^{\gamma_N}$ and $d_0|x_0|^{\theta_0} + d_1|x_1|^{\theta_1} + \cdots + d_N|x_N|^{\theta_N}$, $\theta_i, \gamma_i \ne 1$ as special cases. Moreover (A1) also includes the case $\theta_i = 1$ or/and $\gamma_i = 1$, but (H1) and (H2) do not be allowed.

Remark 9. In Corollary 7, for the special case $a, c \in C[0,1]$, clearly $f, g : [0,1]\times\mathbb{R}^{N+1} \to \mathbb{R}$ are continuous and bounded. This leads to the Corollary 3.1 of [12]. Therefore, Corollary 3.1 of [12] is only a special case of Corollary 7.

In the following, we focus on the uniqueness of the solution of the system (1).

Theorem 10. *Let $f, g : [0,1] \times \mathbb{R}^{N+1} \to \mathbb{R}$ be continuous. Assume that*

(B1) there exist nonnegative functions $a, c \in L^1[0,1]$ and nonnegative nondecreasing functions ϕ, ψ with respect to each variable x_i, $i = 0, 1, 2, \ldots, N$, such that

$$\begin{aligned} &\left|f\left(t, u_0, u_1, \ldots, u_N\right) - f\left(t, v_0, v_1, \ldots, v_N\right)\right| \\ &\qquad \le a(t)\,\phi\left(\left|u_0 - v_0\right|, \left|u_1 - v_1\right|, \ldots, \left|u_N - v_N\right|\right), \\ &\left|g\left(t, u_0, u_1, \ldots, u_N\right) - g\left(t, v_0, v_1, \ldots, v_N\right)\right| \\ &\qquad \le b(t)\,\psi\left(\left|u_0 - v_0\right|, \left|u_1 - v_1\right|, \ldots, \left|u_N - v_N\right|\right); \end{aligned} \tag{27}$$

(B2) for any $s > 0$,

$$\phi(s, s, \ldots, s) \le s, \qquad \psi(s, s, \ldots, s) \le s, \tag{28}$$

and $\max\{k_1^2, l_1^2\} < 1$, *where*

$$k_1 = \max_{t\in I}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}a(s)\,ds + \sum_{j=1}^{N}\frac{1}{\Gamma(\alpha-\rho_j)}\int_0^t (t-s)^{\alpha-\rho_j-1}a(s)\,ds\right),$$
$$l_1 = \max_{t\in I}\left(\frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}c(s)\,ds + \sum_{j=1}^{N}\frac{1}{\Gamma(\sigma-\beta_j)}\int_0^t (t-s)^{\sigma-\beta_j-1}c(s)\,ds\right). \tag{29}$$

Then coupled system (1) *has a unique solution.*

Proof. We prove that the operator $T : X\times Y \to X\times Y$ is contraction. To do this, let (x_1, y_1), $(x_2, y_2) \in X\times Y$; we have

$$\begin{aligned}
&\left|T_1x_2(t) - T_1x_1(t)\right| \\
&= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \times f\left(s, y_2(s), \mathcal{D}^{\beta_1}y_2(s), \ldots, \mathcal{D}^{\beta_N}y_2(s)\right)ds \right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \times f\left(s, y_1(s), \mathcal{D}^{\beta_1}y_1(s), \ldots, \mathcal{D}^{\beta_N}y_1(s)\right)ds\right| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}a(s) \times \phi\left(\left|y_2(s)-y_1(s)\right|, \left|\mathcal{D}^{\beta_1}y_2(s) - \mathcal{D}^{\beta_1}y_1(s)\right|, \ldots, \left|\mathcal{D}^{\beta_N}y_2(s) - \mathcal{D}^{\beta_N}y_1(s)\right|\right)ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}a(s) \times \phi\left(\|y_2-y_1\|, \left\|\mathcal{D}^{\beta_1}y_2 - \mathcal{D}^{\beta_1}y_1\right\|, \ldots, \left\|\mathcal{D}^{\beta_N}y_2 - \mathcal{D}^{\beta_N}y_1\right\|\right)ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}a(s) \times \phi\left(\|y_2-y_1\|_Y, \|y_2-y_1\|_Y, \ldots, \|y_2-y_1\|_Y\right)ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}a(s)\,ds\|y_2-y_1\|_Y,
\end{aligned}$$

$$\begin{aligned}
&\left|\mathcal{D}^{\rho_j}T_1x_2(t) - \mathcal{D}^{\rho_j}T_1x_1(t)\right| \\
&= \left|I^{\alpha-\rho_j}\left(f\left(t, y_2(t), \mathcal{D}^{\beta_1}y_2(t), \ldots, \mathcal{D}^{\beta_N}y_2(t)\right) - f\left(t, y_1(t), \mathcal{D}^{\beta_1}y_1(t), \ldots, \mathcal{D}^{\beta_N}y_1(t)\right)\right)\right| \\
&\le \frac{1}{\Gamma(\alpha-\rho_j)} \times \int_0^t (t-s)^{\alpha-\rho_j-1} \times \left|f\left(t, y_2(t), \mathcal{D}^{\beta_1}y_2(t), \ldots, \mathcal{D}^{\beta_N}y_2(t)\right) - f\left(t, y_1(t), \mathcal{D}^{\beta_1}y_1(t), \ldots, \mathcal{D}^{\beta_N}y_1(t)\right)\right|ds \\
&\le \frac{1}{\Gamma(\alpha-\rho_j)} \times \int_0^t (t-s)^{\alpha-\rho_j-1}a(s) \times \phi\left(\|y_2-y_1\|, \left\|\mathcal{D}^{\beta_1}y_2 - \mathcal{D}^{\beta_1}y_1\right\|, \ldots, \left\|\mathcal{D}^{\beta_N}y_2 - \mathcal{D}^{\beta_N}y_1\right\|\right)ds \\
&\le \frac{1}{\Gamma(\alpha-\rho_j)} \times \int_0^t (t-s)^{\alpha-\rho_j-1}a(s) \times \phi\left(\|y_2-y_1\|_Y, \|y_2-y_1\|_Y, \ldots, \|y_2-y_1\|_Y\right)ds \\
&\le \frac{1}{\Gamma(\alpha-\rho_j)}\int_0^t (t-s)^{\alpha-\rho_j-1}a(s)\,ds\|y_2-y_1\|_Y.
\end{aligned} \tag{30}$$

Thus it follows from (30) and (B2) that

$$\begin{aligned}
&\|T_1x_2 - T_1x_1\|_X \\
&= \max_{t\in I}\left|T_1x_2(t) - T_1x_1(t)\right| + \sum_{j=1}^{N}\max_{t\in I}\left|\mathcal{D}^{\rho_j}\left(T_1x_2(t) - T_1x_1(t)\right)\right| \\
&\le \left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}a(s)\,ds + \sum_{j=1}^{N}\frac{1}{\Gamma(\alpha-\rho_j)}\int_0^t (t-s)^{\alpha-\rho_j-1}a(s)\,ds\right)\|y_2-y_1\|_Y \\
&\le k_1\|y_2-y_1\|_Y.
\end{aligned} \tag{31}$$

Similarly, we can get

$$\begin{aligned}
&\left|T_1 y_2(t) - T_1 y_1(t)\right| \\
&\quad \le \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} c(s)\, ds \|x_2 - x_1\|_X, \\
&\left|\mathscr{D}^{\beta_j} T_1 y_2(t) - \mathscr{D}^{\beta_j} T_1 y_1(t)\right| \\
&\quad \le \frac{1}{\Gamma(\sigma-\beta_j)} \int_0^t (t-s)^{\sigma-\beta_j-1} c(s)\, ds \|x_2 - x_1\|_X, \\
&\|T_1 y_2 - T_1 y_1\|_Y \\
&\quad = \max_{t\in I} \left|T_1 y_2(t) - T_1 y_1(t)\right| \\
&\qquad + \sum_{j=1}^{N} \max_{t\in I} \left|\mathscr{D}^{\rho_j}\left(T_1 y_2(t) - T_1 y_1(t)\right)\right| \\
&\quad \le \left(\frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} c(s)\, ds \right. \\
&\qquad \left. + \sum_{j=1}^{N} \frac{1}{\Gamma(\sigma-\beta_j)} \int_0^t (t-s)^{\sigma-\beta_j-1} c(s)\, ds \right) \|x_2 - x_1\|_X \\
&\quad \le l_1 \|x_2 - x_1\|_X.
\end{aligned} \tag{32}$$

Hence, for the Euclidean distance d on $\mathbb{R}^2$, we get

$$\begin{aligned}
&d\left(T(x_2, y_2), T(x_1, y_1)\right) \\
&\quad = \sqrt{\|T_1 x_2 - T_1 x_1\|_X^2 + \|T_1 y_2 - T_1 y_1\|_Y^2} \\
&\quad \le \sqrt{k_1^2 \|x_2 - x_1\|_X^2 + l_1^2 \|y_2 - T_1\|_Y^2} \\
&\quad \le \sqrt{\max\{k_1^2, l_1^2\}} \sqrt{\|x_2 - x_1\|_X^2 + \|y_2 - T_1\|_Y^2} \\
&\quad = \sqrt{\max\{k_1^2, l_1^2\}}\, d\left((x_2, y_2), (x_1, y_1)\right).
\end{aligned} \tag{33}$$

Thus T is a contraction since $\sqrt{\max\{k_1^2, l_1^2\}} < 1$.

By Banach contraction principle, T has a unique fixed point, which is a solution of the coupled system (1). The proof is completed. □

An Example. Consider the existence of solutions for the following coupled system of multiterm nonlinear fractional differential equations:

$$\begin{aligned}
-\mathscr{D}^{3.5} x(t) &= \frac{t}{6.08} + \frac{1}{25.26}\left[y(t) + e^{(\mathscr{D}^{0.8} y(t))}\right], \\
&\mathscr{D}^{3.5} x(0) = 0, \quad i = 1, 2, \ldots, 4, \\
-\mathscr{D}^{4.2} y(t) &= \frac{10000}{5501}\left[t^{-1/2} x^{0.2}(t) + t^2 \left(\mathscr{D}^{0.5} x(t)\right)^{0.5}\right], \\
&\mathscr{D}^{4.2-j} y(0) = 0, \quad j = 1, 2, \ldots, 5,
\end{aligned} \tag{34}$$

where $t \in (0, 1]$.

Let

$$\begin{aligned}
f(t, x_0, x_1) &= \frac{t}{6.08} + \frac{1}{25.26}\left[x_0 + e^{x_1}\right], \\
g(t, x_0, x_1) &= t^{-1/2} x_0^{0.2} + t^2 x_1^{0.5},
\end{aligned} \tag{35}$$

and choose

$$\begin{aligned}
&a(t) = \frac{t}{6.08}, \qquad b(t) = \frac{1}{25.26}, \\
&\phi(x_0, x_1) = x_0 + e^{x_1}, \qquad c(t) = 0, \\
&d(t) = \frac{10000}{5501}\left[t^{-1/2} + t^2\right], \\
&\psi(x_0, x_1) = x_0^{0.2} + x_1^{0.5}.
\end{aligned} \tag{36}$$

Then

$$\begin{aligned}
f(t, x_0, x_1) &\le a(t) + b(t)\phi(x_0, x_1), \\
g(t, x_0, x_1) &\le c(t) + d(t)\psi(x_0, x_1);
\end{aligned} \tag{37}$$

consequently, (A1) holds.

In the following, we check the condition (A1). Since

$$\begin{aligned}
k_1 &= \max\left(\frac{1}{\Gamma(3.5)} \int_0^t \frac{(t-s)^{2.5} s}{6.08} ds + \frac{1}{\Gamma(3)} \int_0^t \frac{(t-s)^2 s}{6.08} ds\right) \\
&= 0.01, \\
k_2 &= \max\left(\frac{1}{\Gamma(3.5)} \int_0^t \frac{(t-s)^{2.5}}{25.26} ds + \frac{1}{\Gamma(3)} \int_0^t \frac{(t-s)^2}{25.26} ds\right) \\
&= 0.01, \\
l_1 &= 0, \\
l_2 &= \frac{10000}{5501} \\
&\quad \times \max\left(\frac{1}{\Gamma(4.2)} \int_0^t (t-s)^{3.2}\left(s^{-1/2} + s^2\right) ds \right. \\
&\qquad \left. + \frac{1}{\Gamma(3.4)} \int_0^t (t-s)^{2.4}\left(s^{-1/2} + s^2\right) ds\right) = 1,
\end{aligned} \tag{38}$$

take $R_0 = 5$; we have

$$\begin{aligned}
\phi(R_0, R_0) &= R_0 + e^{R_0} = 5 + e^5 \\
&= 153.44 < \frac{R_0 - k_1}{k_2} = \frac{5 - 0.01}{0.01} = 499, \\
\psi(R_0, R_0) &= R_0^{0.2} + R_0^{0.5} = 5^{0.2} + 5^{0.5} \\
&= 3.6158 < \frac{R_0 - l_1}{l_2} = 5,
\end{aligned} \tag{39}$$

which implies that (A2) is satisfied. Hence, by Theorem 5, the coupled system of fractional differential equation (34) has a solution.

Remark 11. In the coupled system of fractional differential equation (34), the nonlinear function f involves exponential growth, but the results of [12] are only allowed to be power growth; that is, (34) cannot be solved by using the results of [12]. So the results obtained in this paper give a significant improvement of the previous work in [12].

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