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Research Article

Regular Functions with Values in Ternary Number System on the Complex Clifford Analysis

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We define a new modified basis $\hat{i}$ which is an association of two bases, e_1 and e_2. We give an expression of the form $z = x_0 + \hat{i}\overline{z_0}$, where x_0 is a real number and $\overline{z_0}$ is a complex number on three-dimensional real skew field. And we research the properties of regular functions with values in ternary field and reduced quaternions by Clifford analysis.

1. Introduction

The noncommutative three-dimensional real field $\mathbb{R}^3$ of the hypercomplex numbers is called a ternary number system $\mathbb{T}$. The quaternions are represented by the form $z = \sum_{j=0}^{3} e_j x_j$, where x_j $(j = 0, \ldots, 3)$ are real numbers on four dimensional real field $\mathbb{R}^4$. Fueter [1] has given a definition of quaternionic functions in $\mathbb{R}^4$ and Deavours [2] and Sudbery [3] have developed theories of quaternionic analysis. Naser [4] investigated some properties of hyperholomorphic functions and Koriyama et al. [5] researched properties of hyperholomorphic functions and holomorphic functions in quaternionic analysis. Nôno [6] obtained several results for regular functions which have a complex number form in quaternion analysis. Cho [7] researched some properties of Euler's formula and De moivre's formula for quaternions. Sangwine and Bihan [8] obtained some results for the quaternionic polar representation with a complex modulus and complex argument inspired by the cayley-dickson form. Fueter [9] obtained some properties of the three variables which are called the Fueter variables and researched the fact that structures lead to the set of all Fueter-regular functions in the general cases of Clifford analysis. By Brackx et al. [10], the theory of Fueter-regularity has been developed and generalized as quaternionic variables for theories of Clifford-valued regular functions.

Lim and Shon [11–13] researched the existence of hyperconjugate harmonic functions of octonion variables, properties of dual quaternion functions, and regularity of functions with values in a noncommutative subalgebra of complex matrix algebras.

We consider that ternary numbers are generated by a new basis $\hat{i}$ and give some properties of regular functions with values in $\mathbb{T}$. Also, we represent the corresponding Euler's formula for the form $z = x_0 + \hat{i}\overline{z_0}$ and investigate calculating rules for regular functions in Clifford analysis. We research new representations of Fueter variables in reduced quaternions with $\hat{i}$ and some characteristics of regularity of functions on the Fueter variable system.

2. Preliminaries

The ternary number system $\mathbb{T}$ is a three dimensional noncommutative and associative real field by three bases e_0, e_1, and e_2 with the following rules:

$$\begin{gathered} e_1^2 = e_2^2 = -1, \qquad e_1 e_2 = -e_2 e_1, \\ \overline{e_0} = e_0, \qquad \overline{e_j} = -e_j \quad (j = 1, 2). \end{gathered} \tag{1}$$

The element e_0 is the identity of $\mathbb{T}$ and e_1 identifies the imaginary unit $\sqrt{-1}$ in the complex field. We consider an association of two bases e_1 and e_2 as follows:

$$\hat{i} := \frac{a e_1 + b e_2}{\sqrt{a^2 + b^2}} = \alpha e_1 + \beta e_2 \quad \text{with } \hat{i}^2 = -1, \tag{2}$$

where $\alpha := a/\sqrt{a^2 + b^2}$, $\beta := b/\sqrt{a^2 + b^2}$, and a, b are real numbers except both zeros.

The number of the skew field $\mathbb{T}$ is

$$z = x_0 + e_1x_1 + e_2x_2 \\ = x_0 + \widehat{i}\overline{z_0}, \tag{3}$$

where x_j $(j = 0, 1, 2)$ are real variables, $\overline{z_0} = \gamma(x_1 - x_2e_1e_2)$, and $\gamma := \alpha + \beta e_1e_2$.

We define the ternary number system

$$\mathbb{T} := \left\{z \mid z = x_0 + \widehat{i}\overline{z_0}\right\}. \tag{4}$$

The conjugate number z^* of z in $\mathbb{T}$ is given by the form:

$$z^* = x_0 - \widehat{i}\overline{z_0}. \tag{5}$$

And the norm $|z|$ of z and the inverse z^{-1} of z are given by the following forms:

$$|z| = \sqrt{zz^*} = \sqrt{x_0^2 + \overline{z_0}z_0} = \sqrt{\sum_{j=0}^{2} x_j^2}, \\ z^{-1} = \frac{z^*}{|z|^2} \quad (z \neq 0), \tag{6}$$

where $z_0 = \overline{\gamma}(x_1 + x_2e_1e_2)$ and $\overline{\gamma} = \alpha - \beta e_1e_2$.

We define the addition and multiplication of two ternary numbers $z = x_0 + \widehat{i}\overline{z_0}$ and $w = y_0 + \widehat{i}\overline{w_0}$ as follows:

$$z + w = (x_0 + y_0) + \widehat{i}\left(\overline{z_0} + \overline{w_0}\right), \\ zw = \left(x_0y_0 - z_0\overline{w_0}\right) + \widehat{i}\left(x_0\overline{w_0} + \overline{z_0}y_0\right). \tag{7}$$

Theorem 1. *Let z be an arbitrary number in $\mathbb{T}$. Then the corresponding Euler formula for z is*

$$e^z = e^{x_0}\left(\cos|z_0| + \frac{z_0}{|z_0|}\widehat{i}\sin|z_0|\right). \tag{8}$$

Moreover, taking logarithms of both sides, one obtains the equation as follows:

$$\ln z = \ln|z| + \frac{z_0}{|z_0|}\widehat{i}\cos^{-1}\left(\frac{x_0}{|z|}\right). \tag{9}$$

Proof. For the number $z = x_0 + \widehat{i}\overline{z_0}$ in $\mathbb{T}$, we get $|\widehat{i}\overline{z_0}| = |\overline{z_0}| = |z_0|$ and $((z_0/|z_0|)\widehat{i})^2 = -1$. Then,

$$e^z = e^{x_0 + \widehat{i}\overline{z_0}} = e^{x_0}e^{(\widehat{i}\overline{z_0}/|\widehat{i}\overline{z_0}|)|\widehat{i}\overline{z_0}|} \\ = e^{x_0}\left(\cos|z_0| + \frac{z_0}{|z_0|}\widehat{i}\sin|z_0|\right). \tag{10}$$

From

$$z = |z|\left(\frac{x_0}{|z|} + \frac{z_0}{|z_0|}\widehat{i}\frac{|z_0|}{|z|}\right) \\ = |z|\left\{\cos\left(\cos^{-1}\left(\frac{x_0}{|z|}\right)\right)\right. \\ \left. + \frac{z_0}{|z_0|}\widehat{i}\sin\left(\cos^{-1}\left(\frac{x_0}{|z|}\right)\right)\right\}, \tag{11}$$

we have

$$\ln z = \ln|z| + \frac{z_0}{|z_0|}\widehat{i}\cos^{-1}\left(\frac{x_0}{|z|}\right). \tag{12}$$

We consider the following differential operators:

$$D := \frac{1}{2}\sum_{j=0}^{2}\overline{e_j}\frac{\partial}{\partial x_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} - \widehat{i}\frac{\partial}{\partial z_0}\right), \\ D^* = \frac{1}{2}\sum_{j=0}^{2}e_j\frac{\partial}{\partial x_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + \widehat{i}\frac{\partial}{\partial z_0}\right), \tag{13}$$

where $\partial/\partial z_0 = \gamma(\partial/\partial x_1 - e_1e_2(\partial/\partial x_2))$ and $\partial/\partial\overline{z_0} = \overline{\gamma}(\partial/\partial x_1 + e_1e_2(\partial/\partial x_2))$. Then the Laplacian operator is

$$4\Delta := DD^* = D^*D = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial z_0\partial\overline{z_0}} = \sum_{j=0}^{2}\frac{\partial^2}{\partial x_j^2}. \tag{14}$$

Let Ω be an open set in $\mathbb{R}^3$. The function $f(z)$ that is defined by the following form in Ω with values in $\mathbb{T}$:

$$f : \Omega \longrightarrow \mathbb{T} \tag{15}$$

satisfies

$$z = \left(x_0, \overline{z_0}\right) \in \Omega \longmapsto f(z) = u_0\left(x_0, \overline{z_0}\right) + \widehat{i}\overline{f_0}\left(x_0, \overline{z_0}\right) \in \mathbb{T}, \tag{16}$$

where u_j $(j = 0, 1, 2)$ are real-valued functions and

$$f_0 = \overline{\gamma}\left(u_1 + u_2e_1e_2\right), \qquad \overline{f_0} = \gamma\left(u_1 - u_2e_1e_2\right) \tag{17}$$

are complex-valued functions with values in $\mathbb{T}$. □

Remark 2. The operators D and D^* act for the function $f(z)$ on $\mathbb{T}$ as follows:

$$Df = \frac{1}{2}\left\{\left(\frac{\partial u_0}{\partial x_0} + \frac{\partial\overline{f_0}}{\partial\overline{z_0}}\right) + \widehat{i}\left(\frac{\partial\overline{f_0}}{\partial x_0} - \frac{\partial u_0}{\partial z_0}\right)\right\}, \\ D^*f = \frac{1}{2}\left\{\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial\overline{f_0}}{\partial\overline{z_0}}\right) + \widehat{i}\left(\frac{\partial\overline{f_0}}{\partial x_0} + \frac{\partial u_0}{\partial z_0}\right)\right\}, \\ fD = \frac{1}{2}\left\{\left(\frac{\partial u_0}{\partial x_0} + \frac{\partial f_0}{\partial z_0}\right) + \widehat{i}\left(\frac{\partial\overline{f_0}}{\partial x_0} - \frac{\partial u_0}{\partial z_0}\right)\right\}, \\ fD^* = \frac{1}{2}\left\{\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial f_0}{\partial z_0}\right) + \widehat{i}\left(\frac{\partial\overline{f_0}}{\partial x_0} + \frac{\partial u_0}{\partial z_0}\right)\right\}. \tag{18}$$

3. Properties of Regular Functions with Values in $\mathbb{T}$

Definition 3. Let Ω be an open set in $\mathbb{R}^3$. A function $f(z) = u_0(x_0, \overline{z_0}) + \widehat{i}\overline{f_0}(x_0, \overline{z_0})$ is said to be L(R)-regular in Ω, if the following two conditions are satisfied:

(i) u_0 and f_0 are continuously differential functions on Ω;

(ii) $D^*f(z) = 0$ ($f(z)D^* = 0$) on Ω.

Remark 4. The left equation (ii) of Definition 3 is equivalent to the following:

$$\frac{\partial u_0}{\partial x_0} = \frac{\partial \overline{f_0}}{\partial \overline{z_0}}, \qquad \frac{\partial \overline{f_0}}{\partial x_0} = -\frac{\partial u_0}{\partial z_0}. \tag{19}$$

The equations in (19) are called the corresponding Cauchy-Riemann system for $f(z)$ in $\mathbb{T}$. The right equation (ii) of Definition 3 is equivalent to (19). When the function $f(z) = u_0(x_0,\overline{z_0}) + \widehat{i}\,\overline{f_0}(x_0,\overline{z_0})$ is a L-regular function on $\Omega \subset \mathbb{R}^3$, simply we say that $f(z)$ is a regular function on $\Omega \subset \mathbb{R}^3$. In this case, we often say that $f(z)$ is a biregular function on $\Omega \subset \mathbb{R}^3$.

Remark 5. Let Ω be an open set in $\mathbb{R}^3$ and let $f(z)$ be a regular function on Ω. Then we can replace the corresponding Cauchy-Riemann system in $\mathbb{R}^3$ as follows:

$$\begin{aligned} &\frac{\partial u_0}{\partial x_0} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \qquad \frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1}, \\ &\frac{\partial u_0}{\partial x_1} = -\frac{\partial u_1}{\partial x_0}, \qquad \frac{\partial u_0}{\partial x_2} = -\frac{\partial u_2}{\partial x_0}, \end{aligned} \tag{20}$$

where u_j ($j = 0, 1, 2$) are real-valued functions.

Theorem 6. *Let Ω be an open set in $\mathbb{R}^3$ and let f be a regular function on Ω. Then the derivative f' of f defined by Df is*

$$f' = \frac{\partial f}{\partial x_0} = -\widehat{i}\frac{\partial f}{\partial z_0} \tag{21}$$

on Ω.

Proof. By the definition of regular function with values in $\mathbb{T}$, we have

$$\begin{aligned} Df &= \frac{1}{2}\left\{\left(\frac{\partial u_0}{\partial x_0} + \frac{\partial \overline{f_0}}{\partial \overline{z_0}}\right) + \widehat{i}\left(\frac{\partial \overline{f_0}}{\partial x_0} - \frac{\partial u_0}{\partial z_0}\right)\right\} \\ &= \frac{\partial u_0}{\partial x_0} + \widehat{i}\frac{\partial \overline{f_0}}{\partial x_0} = \frac{\partial f}{\partial x_0} \end{aligned} \tag{22}$$

on Ω. And

$$Df = \frac{\partial \overline{f_0}}{\partial \overline{z_0}} - \widehat{i}\frac{\partial u_0}{\partial z_0} = -\widehat{i}\left(\frac{\partial}{\partial z_0}\widehat{i}\,\overline{f_0} + \frac{\partial u_0}{\partial z_0}\right) = -\widehat{i}\frac{\partial f}{\partial z_0} \tag{23}$$

on Ω. □

Theorem 7. *Let Ω be an open set in $\mathbb{R}^3$ and let $f = u_0 + \widehat{i}\,\overline{f_0}$ be a function with values in $\mathbb{T}$. Suppose that $\partial f/\partial x_0$ and $\partial f/\partial z_0$ exist and are continuous on Ω. If*

$$\frac{\partial f}{\partial x_0} = -\widehat{i}\frac{\partial f}{\partial z_0} \tag{24}$$

on Ω, then f is regular on Ω.

Proof. Since $\partial f/\partial x_0 = -\widehat{i}(\partial f/\partial z_0)$, we have

$$\frac{\partial f}{\partial x_0} = \frac{\partial u_0}{\partial x_0} + \widehat{i}\frac{\partial \overline{f_0}}{\partial x_0}. \tag{25}$$

Hence, we have $D^* f = 0$ and then f is regular on Ω. □

Definition 8. Let Ω be an open set in $\mathbb{R}^3$. A function $f = u_0 + \widehat{i}\,\overline{f_0}$ is said to be harmonic on Ω if all its components u_0 and $\overline{f_0}$ of f are harmonic on Ω.

Proposition 9. *Let Ω be an open set in $\mathbb{R}^3$. If the function f is regular on Ω, then f is harmonic on Ω.*

Proof. Since f is regular function on Ω, we have

$$\begin{aligned} DD^*\overline{f_0} = \frac{1}{4}&\left\{\left(\frac{\partial}{\partial x_0}\frac{\partial \overline{f_0}}{\partial x_0} + \frac{\partial}{\partial \overline{z_0}}\frac{\partial \overline{f_0}}{\partial z_0}\right)\right. \\ &\left. + \widehat{i}\left(\frac{\partial}{\partial x_0}\frac{\partial \overline{f_0}}{\partial z_0} - \frac{\partial}{\partial z_0}\frac{\partial \overline{f_0}}{\partial x_0}\right)\right\} = 0. \end{aligned} \tag{26}$$

Similarly, we can prove that $DD^* u_0 = 0$. So, we obtain the result. □

Proposition 10. *Let Ω be an open set in $\mathbb{R}^3$ and let $f = u_0 + \widehat{i}\,\overline{f_0}$ and $g = v_0 + \widehat{i}\,\overline{g_0}$ be regular functions on Ω. Then the following properties hold:*

(i) *$f\alpha$ is regular on Ω, if α is any ternary constant;*

(ii) *αf is not regular on Ω, if α is any ternary constant;*

(iii) *$f \pm g$ is regular on Ω;*

(iv) *fg is not regular on Ω. Moreover, if g is a real-valued function, then fg is regular on Ω.*

Proof. It is sufficient to show the second condition of Definition 3.

(i) Let α be a ternary constant with $\alpha = a_0 + \widehat{i}\,\overline{\alpha_0}$, where

$$\alpha_0 = \frac{c_1 a_1 + c_2 a_2}{\sqrt{c_1^2 + c_2^2}} + \frac{c_2 a_1 - c_1 a_2}{\sqrt{c_1^2 + c_2^2}} e_1 e_2 \tag{27}$$

and a_0, a_1, a_2, c_1, and c_2 are real numbers. Then the equation

$$\begin{aligned} D^*(f\alpha) &= \frac{1}{2}\left(\frac{\partial}{\partial x_0} + \widehat{i}\frac{\partial}{\partial z_0}\right) \\ &\quad \times \left\{(u_0 a_0 - f_0\overline{\alpha_0}) + \widehat{i}\left(u_0\overline{\alpha_0} + \overline{f_0}a_0\right)\right\} \\ &= \frac{1}{2}\left(\left(\frac{\partial u_0}{\partial x_0}a_0 - \frac{\partial f_0}{\partial x_0}\overline{\alpha_0} - \frac{\partial u_0}{\partial \overline{z_0}}\overline{\alpha_0} - \frac{\partial \overline{f_0}}{\partial \overline{z_0}}a_0\right)\right. \\ &\quad \left. + \widehat{i}\left(\frac{\partial u_0}{\partial x_0}\overline{\alpha_0} + \frac{\partial \overline{f_0}}{\partial x_0}a_0 + \frac{\partial u_0}{\partial z_0}a_0 - \frac{\partial f_0}{\partial z_0}\overline{\alpha_0}\right)\right) \\ &= 0. \end{aligned} \tag{28}$$

Hence, $f\alpha$ is regular on Ω.

(ii) Since

$$D^*(\alpha f) = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + \hat{i}\frac{\partial}{\partial z_0}\right) \times \left\{\left(a_0 u_0 - \alpha_0 \overline{f_0}\right) + \hat{i}\left(a_0 \overline{f_0} + \overline{\alpha_0} u_0\right)\right\}$$
$$= \frac{1}{2}\left(\left(a_0 \frac{\partial u_0}{\partial x_0} - \alpha_0 \frac{\partial \overline{f_0}}{\partial x_0} - \overline{\alpha_0}\frac{\partial f_0}{\partial x_0} - \frac{\partial u_0}{\partial z_0}\alpha_0\right) + \hat{i}\left(a_0 \frac{\partial \overline{f_0}}{\partial x_0} + \overline{\alpha_0}\frac{\partial u_0}{\partial x_0} + a_0 \frac{\partial u_0}{\partial z_0} - \alpha_0 \frac{\partial \overline{f_0}}{\partial z_0}\right)\right) \quad (29)$$

is not zero, αf is not always regular on Ω.

(iii) Since

$$D^*(f \pm g) = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + \hat{i}\frac{\partial}{\partial z_0}\right)\left\{(u_0 \pm v_0) + \hat{i}\left(\overline{f_0} \pm \overline{g_0}\right)\right\}$$
$$= \frac{1}{2}\left(\left(\frac{\partial u_0}{\partial x_0} \pm \frac{\partial v_0}{\partial x_0} - \frac{\partial \overline{f_0}}{\partial \overline{z_0}} \mp \frac{\partial \overline{g_0}}{\partial \overline{z_0}}\right) + \hat{i}\left(\frac{\partial u_0}{\partial z_0} \pm \frac{\partial v_0}{\partial z_0} + \frac{\partial \overline{f_0}}{\partial x_0}a_0 \pm \frac{\partial \overline{g_0}}{\partial x_0}\right)\right) = 0, \quad (30)$$

$f \pm g$ is regular on Ω.

(iv) Since

$$D^*(fg) = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + \hat{i}\frac{\partial}{\partial z_0}\right) \times \left\{\left(u_0 v_0 - f_0 \overline{g_0}\right) + \hat{i}\left(u_0 \overline{g_0} + \overline{f_0} v_0\right)\right\}$$
$$= \frac{1}{2}\Bigg(\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial \overline{f_0}}{\partial \overline{z_0}}\right)v_0 + u_0\left(\frac{\partial v_0}{\partial x_0} - \frac{\partial \overline{g_0}}{\partial \overline{z_0}}\right) - \left(\frac{\partial f_0}{\partial x_0} + \frac{\partial u_0}{\partial \overline{z_0}}\right)\overline{g_0} - \left(f_0 \frac{\partial \overline{g_0}}{\partial x_0} + \overline{f_0}\frac{\partial v_0}{\partial \overline{z_0}}\right)$$
$$+ \hat{i}\Bigg\{\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial f_0}{\partial z_0}\right)\overline{g_0} + u_0\left(\frac{\partial \overline{g_0}}{\partial x_0} + \frac{\partial v_0}{\partial z_0}\right) + \left(\frac{\partial \overline{f_0}}{\partial x_0} + \frac{\partial u_0}{\partial z_0}\right)v_0 + \left(\overline{f_0}\frac{\partial v_0}{\partial x_0} - f_0 \frac{\partial \overline{g_0}}{\partial z_0}\right)\Bigg\}\Bigg)$$
$$= \frac{1}{2}\left(-\left(f_0 \frac{\partial \overline{g_0}}{\partial x_0} + \overline{f_0}\frac{\partial v_0}{\partial \overline{z_0}}\right) + \hat{i}\left(\overline{f_0}\frac{\partial v_0}{\partial x_0} - f_0 \frac{\partial \overline{g_0}}{\partial z_0}\right)\right) \quad (31)$$

is not zero, fg is not always regular on Ω. □

Theorem 11. *Let Ω be an open set in $\mathbb{R}^3$ and let f and g be regular functions on Ω. Then we have the following equations:*

$$2D^*(fg) = (D^* f)g + f\frac{\partial g}{\partial x_0} + \hat{i}\left(u_0 \frac{\partial g}{\partial z_0} + \widehat{i}\overline{f_0}\frac{\partial g}{\partial \overline{z_0}}\right). \quad (32)$$

$$2D(fg) = (Df)g + f\frac{\partial g}{\partial x_0} - \hat{i}\left(u_0 \frac{\partial g}{\partial z_0} + \widehat{i}\overline{f_0}\frac{\partial g}{\partial \overline{z_0}}\right). \quad (33)$$

Proof. From the proof of Proposition 10, we have the following equations:

$$2D^*(fg) = \left\{\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial \overline{f_0}}{\partial \overline{z_0}}\right) + \hat{i}\left(\frac{\partial \overline{f_0}}{\partial x_0} + \frac{\partial u_0}{\partial z_0}\right)\right\}\left(v_0 + \widehat{i}\overline{g_0}\right) - \left(f_0 \frac{\partial \overline{g_0}}{\partial x_0} + \overline{f_0}\frac{\partial v_0}{\partial \overline{z_0}}\right) + \hat{i}\left(\overline{f_0}\frac{\partial v_0}{\partial x_0} - f_0 \frac{\partial \overline{g_0}}{\partial z_0}\right)$$
$$= \left\{\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial \overline{f_0}}{\partial \overline{z_0}}\right) + \hat{i}\left(\frac{\partial \overline{f_0}}{\partial x_0} + \frac{\partial u_0}{\partial z_0}\right)\right\}\left(v_0 + \widehat{i}\overline{g_0}\right) + u_0\left(\frac{\partial v_0}{\partial x_0} - \frac{\partial \overline{g_0}}{\partial \overline{z_0}}\right) - \left(f_0 \frac{\partial \overline{g_0}}{\partial x_0} + \overline{f_0}\frac{\partial v_0}{\partial \overline{z_0}}\right)$$
$$+ \hat{i}u_0\left(\frac{\partial \overline{g_0}}{\partial x_0} + \frac{\partial v_0}{\partial z_0}\right) + \hat{i}\left(\overline{f_0}\frac{\partial v_0}{\partial x_0} - f_0 \frac{\partial \overline{g_0}}{\partial z_0}\right)$$
$$= (D^* f)g + f\frac{\partial g}{\partial x_0} + \hat{i}\left(u_0 \frac{\partial g}{\partial z_0} + \widehat{i}\overline{f_0}\frac{\partial g}{\partial \overline{z_0}}\right). \quad (34)$$

Similarly, we can prove (33).

We let

$$k = e_1 e_2 \frac{1}{2} dz_0 \wedge d\overline{z_0} + e_2 \alpha dx_0 \wedge d\overline{z_0} - e_1 \beta dx_0 \wedge d\overline{z_0}. \quad (35)$$

□

Theorem 12. *Let Ω be an open set in $\mathbb{R}^3$ and U be any domain in Ω with smooth boundary bU such that $U \subset \Omega$. If $f = u_0 + \widehat{i}\overline{f_0}$ is a regular function on Ω, then*

$$\int_{bU} kf = 0, \quad (36)$$

where kf is the ternary product of the form k on the function $f(z)$.

Proof. Since the function $f = u_0 + e_1 \alpha \overline{f_0} + e_2 \beta \overline{f_0}$ exists, we have

$$kf = \left(e_1 e_2 \frac{1}{2} u_0 - e_2 \frac{1}{2}\alpha \overline{f_0} + e_1 \frac{1}{2}\beta \overline{f_0}\right) dz_0 \wedge d\overline{z_0} + \left(e_2 \alpha u_0 - e_1 \beta u_0\right) dx_0 \wedge d\overline{z_0} + \left(-e_1 e_2 \alpha^2 \overline{f_0} - e_1 e_2 \beta^2 \overline{f_0}\right) dx_0 \wedge dz_0. \quad (37)$$

Then

$$
\begin{aligned}
d(kf) &= e_1 e_2 \left(\frac{\partial u_0}{\partial x_0} - \alpha^2 \frac{\partial \overline{f_0}}{\partial \overline{z_0}} - \beta^2 \frac{\partial \overline{f_0}}{\partial \overline{z_0}} \right) dV \\
&\quad + e_2 \left(-\alpha \frac{\partial \overline{f_0}}{\partial x_0} - \alpha \frac{\partial u_0}{\partial z_0} \right) dV \\
&\quad + e_1 \left(\beta \frac{\partial \overline{f_0}}{\partial x_0} + \beta \frac{\partial u_0}{\partial z_0} \right) dV \\
&\quad + \left(-\alpha\beta \frac{\partial \overline{f_0}}{\partial \overline{z_0}} + \alpha\beta \frac{\partial \overline{f_0}}{\partial \overline{z_0}} \right) dV \\
&= \left\{ e_1 e_2 \left(\frac{\partial u_0}{\partial x_0} - \frac{\partial \overline{f_0}}{\partial \overline{z_0}} \right) - e_2 \alpha \left(\frac{\partial \overline{f_0}}{\partial x_0} + \frac{\partial u_0}{\partial z_0} \right) \right. \\
&\quad \left. + e_1 \beta \left(\frac{\partial \overline{f_0}}{\partial x_0} + \frac{\partial u_0}{\partial z_0} \right) \right\} dV,
\end{aligned}
\tag{38}
$$

where $dV = dx_0 \wedge dz_0 \wedge d\overline{z_0}$ in U, and by the corresponding Cauchy-Riemann system for $f(z)$ in $\mathbb{T}$, $d(kf) = 0$. By Stokes theorem, we obtain the result. □

Remark 13. Since

$$
\left(\hat{i}\overline{z_0}\right)^k = \begin{cases} (-1)^{k/2}\left(|z_0|\right)^k, & k\text{: even} \\ (-1)^{[k/2]}\hat{i}\left(|z_0|\right)^{k-1}\overline{z_0}, & k\text{: odd}, \end{cases} \tag{39}
$$

we have

$$
z^n = \sum_{k=0}^{n} \alpha(k)\, x_0^{n-k} |z_0|^{[k/2]} \overline{z_0}^{\delta_k}, \tag{40}
$$

where

$$
\alpha(k) = \begin{cases} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^{k/2}, & k\text{: even} \\ \begin{pmatrix} n \\ k \end{pmatrix} (-1)^{[k/2]}\hat{i}, & k\text{: odd}, \end{cases}
\qquad
\delta_k = \begin{cases} 0, & k\text{: even} \\ 1, & k\text{: odd}. \end{cases} \tag{41}
$$

And $[k/2]$ is the greatest integer that is less than or equal to $k/2$.

Theorem 14. *Let f be a homogeneous polynomial of degree n with respect to the variables x_0 and $\overline{z_0}$. If f is regular on Ω, then*

$$
f(z) = \frac{1}{n!} \frac{\partial^n f(z)}{\partial x_0^n} z^n, \tag{42}
$$

$$
f(z) = \left(-\hat{i}\right)^n \frac{1}{n!} \frac{\partial^n f(z)}{\partial z_0^{n-r} \partial \overline{z_0}^r} z^n, \tag{43}
$$

where r is a nonnegative integer.

Proof. Since $f(z)$ is a homogeneous polynomial, then

$$
f(z) = \frac{1}{n} \frac{\partial f(z)}{\partial x_0} z. \tag{44}
$$

Also, since $\partial f(z)/\partial x_0$ is a homogeneous polynomial of degree $n-1$, we have

$$
\frac{\partial f(z)}{\partial x_0} = \frac{1}{n-1} \frac{\partial^2 f(z)}{\partial x_0^2} z. \tag{45}
$$

Then we have

$$
f(z) = \frac{1}{n(n-1)} \frac{\partial^2 f(z)}{\partial x_0^2} z^2. \tag{46}
$$

Continuing this process, we can get the result (42). Similarly, we obtain the result (43). □

4. Properties of Regular Functions with Values in $\mathbb{T}(\mathbb{C})$

We define the number system

$$
\mathbb{T}(\mathbb{C}) = \left\{ z \mid z = \hat{i}\gamma\left(z_1 - e_1 e_2 z_2\right) \right\}, \tag{47}
$$

where $z_1 = x_1 - (1/2)e_1 x_0$ and $z_2 = x_2 - (1/2)e_2 x_0$.

The non-commutative multiplication of two numbers $z = \hat{i}\gamma(z_1 - e_1 e_2 z_2)$ and $w = \hat{i}\gamma(w_1 - e_1 e_2 w_2)$ is defined by

$$
\begin{aligned}
zw &= -\left\{ \left(z_1 w_1 + z_2 w_2\right) + e_1 e_2 \left(\overline{z_2} w_1 - \overline{z_1} w_2\right) \right\}, \\
wz &= -\left\{ \left(w_1 z_1 + w_2 z_2\right) + e_1 e_2 \left(\overline{w_2} z_1 - \overline{w_1} z_2\right) \right\}.
\end{aligned} \tag{48}
$$

The conjugate number z^* of z in $\mathbb{T}(\mathbb{C})$ is given by the following:

$$
z^* = -\hat{i}\gamma\left(\overline{z_1} - e_1 e_2 \overline{z_2}\right). \tag{49}
$$

And the norm $|z|$ of z and the inverse z^{-1} of z are given by the following forms:

$$
\begin{aligned}
|z| &= \sqrt{zz^*} = \sqrt{z^* z} \\
&= \sqrt{\left(z_1 \overline{z_1} + z_2 \overline{z_2}\right) + e_1 e_2 \left(\overline{z_2 z_1} - \overline{z_1 z_2}\right)} \\
&= \sqrt{\sum_{j=0}^{2} x_j^2}, \\
z^{-1} &= \frac{z^*}{|z|^2} \quad (z \neq 0).
\end{aligned} \tag{50}
$$

We consider the following differential operators:

$$
D = -\frac{1}{2}\hat{i}\gamma\left(D_{z_1} - e_1 e_2 D_{z_2}\right), \qquad D^* = \frac{1}{2}\hat{i}\gamma\left(D_{\overline{z_1}} - e_1 e_2 D_{\overline{z_2}}\right), \tag{51}
$$

where

$$
D_{z_1} = \frac{1}{2} e_1 \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1}, \qquad D_{z_2} = \frac{1}{2} e_2 \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_2}. \tag{52}
$$

Then the Laplacian operator is

$$4\Delta := DD^* = D^*D = \sum_{j=0}^{2} \frac{\partial^2}{\partial x_j^2}. \tag{53}$$

Let G be an open set in $\mathbb{C}^2$. The function $f(z)$ that is defined by the following form in G with values in $\mathbb{T}(\mathbb{C})$:

$$f : G \to \mathbb{T}(\mathbb{C}) \tag{54}$$

satisfies

$$\begin{aligned} z = (z_1, z_2) \in G \longmapsto f(z) &= f(z_1, z_2) \\ &= \hat{i}\gamma (f_1(z_1, z_2) - e_1e_2 f_2(z_1, z_2)), \end{aligned} \tag{55}$$

where $f_1 = u_1 - (1/2)e_1u_0$ and $f_2 = u_2 - (1/2)e_2u_0$ are complex-valued functions with values in $\mathbb{T}(\mathbb{C})$ and u_j $(j = 0, 1, 2)$ are real-valued functions.

Remark 15. The operators D and D^* act for a function $f(z)$ on $\mathbb{T}(\mathbb{C})$ as follows:

$$\begin{aligned} Df &= -\hat{i}^2 \left\{ \left(D_{z_1} f_1 + D_{z_2} f_2 \right) + e_1e_2 \left(D_{\overline{z_2}} f_1 - D_{\overline{z_1}} f_2 \right) \right\}, \\ D^* f &= \hat{i}^2 \left\{ \left(D_{\overline{z_1}} f_1 + D_{\overline{z_2}} f_2 \right) + e_1e_2 \left(D_{z_2} f_1 - D_{z_1} f_2 \right) \right\}. \end{aligned} \tag{56}$$

We define a commutative multiplication of two numbers $z = \hat{i}\gamma(z_1 - e_1e_2z_2)$ and $w = \hat{i}\gamma(w_1 - e_1e_2w_2)$ by

$$\begin{aligned} z \odot w = w \odot z &= \frac{1}{2} (zw + wz) \\ &= \frac{1}{2} \hat{i}^2 \{ (z_1w_1 + z_2w_2 + w_1z_1 + w_2z_2) \\ &\quad + e_1e_2 (\overline{z_2}w_1 - \overline{z_1}w_2 \\ &\quad + \overline{w_2}z_1 - \overline{w_1}z_2) \}. \end{aligned} \tag{57}$$

Remark 16. The operators D and D^* act for a function $f(z)$ on $\mathbb{T}(\mathbb{C})$ as follows:

$$\begin{aligned} D \odot f &= \frac{1}{2} (Df + fD) \\ &= \left\{ \left(D_{z_1} f_1 + D_{z_2} f_2 \right) \right. \\ &\quad + \frac{1}{2} e_1e_2 \left(D_{\overline{z_2}} f_1 - D_{\overline{z_1}} f_2 \right. \\ &\quad \left. \left. + \overline{f_2} D_{z_1} - \overline{f_1} D_{z_2} \right) \right\} \\ &= \left\{ \left(D_{z_1} f_1 + D_{z_2} f_2 + \frac{1}{2} \frac{\partial u_0}{\partial x_0} \right) \right. \\ &\quad + \frac{1}{2} e_1e_2 \left(D_{\overline{z_2}} f_1 - D_{\overline{z_1}} f_2 \right. \\ &\quad \left. \left. + D_{z_1} \overline{f_2} - D_{z_2} \overline{f_1} \right) \right\}, \end{aligned}$$

$$\begin{aligned} D^* \odot f &= \frac{1}{2} (D^* f + fD^*) \\ &= -\left\{ \left(D_{\overline{z_1}} f_1 + D_{\overline{z_2}} f_2 \right) \right. \\ &\quad + \frac{1}{2} e_1e_2 \left(D_{z_2} f_1 - D_{z_1} f_2 \right. \\ &\quad \left. \left. + \overline{f_2} D_{\overline{z_1}} - \overline{f_1} D_{\overline{z_2}} \right) \right\} \\ &= -\left\{ \left(D_{\overline{z_1}} f_1 + D_{\overline{z_2}} f_2 - \frac{1}{2} \frac{\partial u_0}{\partial x_0} \right) \right. \\ &\quad + \frac{1}{2} e_1e_2 \left(D_{z_2} f_1 - D_{z_1} f_2 \right. \\ &\quad \left. \left. + D_{\overline{z_1}} \overline{f_2} - D_{\overline{z_2}} \overline{f_1} \right) \right\}. \end{aligned} \tag{58}$$

Definition 17. Let G be a domain in $\mathbb{C}^2$. A function $f = \hat{i}\gamma(f_1 - e_1e_2f_2)$ is said to be dot-regular in G if the following two conditions are satisfied:

(i) f_1 and f_2 are differential functions in G,

(ii) $D^* \odot f = 0$ in G.

Remark 18. The above equation (ii) of Definition 17 is equivalent as follows:

$$\begin{aligned} D_{\overline{z_1}} f_1 + D_{\overline{z_2}} f_2 &= \frac{1}{2} \frac{\partial u_0}{\partial x_0}, \\ D_{z_2} f_1 - D_{z_1} f_2 &= D_{\overline{z_2}} \overline{f_1} - D_{\overline{z_1}} \overline{f_2}. \end{aligned} \tag{59}$$

Theorem 19. *Let G be an open set in $\mathbb{C}^2$ and let f be a dot-regular function on G. Then the derivative f' of f defined by $D \odot f$ is*

$$\begin{aligned} f' &= 2\hat{i}\gamma \left(D_{\overline{z_1}} - D_{z_1} \right) f = 2e_1 \left(D_{\overline{z_1}} - D_{z_1} \right) f, \\ f' &= -2\hat{i}\gamma \left(D_{z_2} - D_{\overline{z_2}} \right) f = 2e_2 \left(D_{\overline{z_2}} - D_{z_2} \right) f. \end{aligned} \tag{60}$$

Proof. By the definition of a dot-regular function with values in $\mathbb{T}(\mathbb{C})$, we have

$$\begin{aligned} D \odot f &= \left(D_{\overline{z_1}} f_1 + D_{\overline{z_2}} f_2 + e_1 \frac{\partial u_1}{\partial x_0} + e_2 \frac{\partial u_2}{\partial x_0} + \frac{3}{2} \frac{\partial u_0}{\partial x_0} \right) \\ &\quad + \frac{1}{2} e_1e_2 \left(D_{z_2} f_1 - D_{z_1} f_2 + D_{\overline{z_1}} \overline{f_2} \right. \\ &\quad \left. - D_{\overline{z_2}} \overline{f_1} - 2e_2 \frac{\partial u_1}{\partial x_0} + 2e_1 \frac{\partial u_2}{\partial x_0} \right) \\ &= 2\hat{i}\gamma \left(D_{\overline{z_1}} - D_{z_1} \right) f \end{aligned} \tag{61}$$

on G. And, similarly, we have

$$D \odot f = -2\hat{i}\gamma \left(D_{z_2} - D_{\overline{z_2}} \right) f \tag{62}$$

on G. □

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