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Research Article

Ψ-Stability of Nonlinear Volterra Integro-Differential Systems with Time Delay

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We give some sufficient conditions for Ψ-uniform stability of the trivial solutions of a nonlinear differential system and of nonlinear Volterra integro-differential systems with time delay.

1. Introduction

Akinyele [1] introduced the notion of Ψ-stability of the degree k with respect to a function $\Psi \in C(R_+\text{-}R_+)$, increasing and differentiable on R and such that $\Psi(t) \geq 1$ for $t \geq 0$ and $\lim_{t\to\infty}\Psi(t) = b$, $b \in [1,\infty)$. Constantin [2] introduced the notions of degree of stability and degree of boundedness of solutions of an ordinary differential equation, with respect to a continuous positive and nondecreasing function $\Psi : R_+ \to R_+$; some criteria for these notions are proved there too.

Morchało [3] introduced the notions of Ψ-stability, Ψ-uniform stability, and Ψ-asymptotic stability of trivial solution of the nonlinear system $x' = f(t,x)$. Several new and sufficient conditions for the mentioned types of stability are proved for the linear system $x' = A(t)x$; in this paper Ψ is a scalar continuous function. In [4, 5], Diamandescu gives some sufficient conditions for Ψ-asymptotic stability and Ψ-(uniform) stability of the nonlinear Volterra integro-differential system $x' = A(t)x + \int_0^t F(t,s,x(s))ds$; in these papers Ψ is a matrix function. Furthermore, in [6], sufficient conditions are given for the uniform Lipschitz stability of the system $x' = f(t,x) + g(t,x)$.

In paper [7], for the nonlinear system

$$y' = f(t,y) + g(t,y) \tag{1}$$

and the nonlinear Volterra integro-differential system

$$z' = f(t,z) + \int_0^t F(t,s,z(s))\,ds, \tag{2}$$

by using the knowledge of fundamental matrix and nonlinear variation of constants, we give some sufficient conditions for Ψ-(uniform) stability of trivial solution for the system. The purpose of this paper is to provide sufficient conditions for Ψ-uniform stability of trivial solutions for the nonlinear delayed system

$$x'(t) = f(t,x(t)) + g(t,x(t-\tau(t))) \tag{3}$$

and the nonlinear delayed Volterra integro-differential systems

$$\begin{aligned} x'(t) &= f(t,x(t)) + g(t,x(t-\tau(t))) \\ &\quad + p(t,x(t))\int_0^t q(s,x(s-\tau(s)))\,ds, \end{aligned} \tag{4}$$

$$\begin{aligned} x'(t) &= f(t,x(t)) + g(t,x(t-\tau(t))) \\ &\quad + p(t,x(t-\tau(t)))\int_0^t q(s,x(s))\,ds, \end{aligned} \tag{5}$$

where $f, g, p, q \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, $f(t,0) = g(t,0) = p(t,0) = q(t,0) = 0$ for $t \in \mathbb{R}_+$, and $\tau \in C^1(\mathbb{R}_+,\mathbb{R}_+)$ with

$\tau(t) \le t$ on $\mathbb{R}_+$. The systems studied in [7] do not include time delay, whereas all the systems studied in this paper have time delay.

In this paper, we investigate conditions on the functions f, g, p, q under which the trivial solutions of systems (3), (4), and (5) are Ψ-stability on $\mathbb{R}_+$; the main tool used is the integral inequalities and the integral technique. Here Ψ is a matrix function whose introduction allows us to obtain a mixed behavior for the components of solutions.

Let $\mathbb{R}^n$ denote the Euclidean n-space. For $x = (x_1, x_2, x_3, \dots, x_n)^T \in \mathbb{R}^n$, let $\|x\| = \max\{|x_1|, |x_2|, \dots, |x_n|\}$ be the norm of x. For an $n \times n$ matrix $A = (a_{ij})$, we define the norm $|A| = \sup_{\|x\|\le 1} \|Ax\|$. It is well known that

$$|A| = \max_{1\le i\le n} \sum_{j=1}^{n} \left|a_{ij}\right|. \tag{6}$$

Let $\Psi_i : \mathbb{R}_+ \to (0, \infty)$, $i = 1, 2, \dots, n$, be continuous functions and $\Psi = \operatorname{diag}[\Psi_1, \Psi_2, \dots, \Psi_n]$.

Now we give the definitions of Ψ-(uniform) stability that we will need in the sequel.

Definition 1 (see [4, 8]). The trivial solution of (3) ((4) or (5)) is said to be Ψ-stable on $\mathbb{R}_+$ if for every $\varepsilon > 0$ and any $t_0 \in \mathbb{R}_+$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that any solution $x(t)$ of (3) ((4) or (5)), which satisfies the inequality $\|\Psi(t_0)x(t_0)\| < \delta$, exists and satisfies the inequality $\|\Psi(t)x(t)\| < \varepsilon$ for all $t \ge t_0$.

Definition 2 (see [4, 8]). The trivial solution of (3) ((4) or (5)) is said to be Ψ-uniformly stable on $\mathbb{R}_+$ if it is Ψ-stable on $\mathbb{R}_+$ and the previous δ is independent of t_0.

2. Ψ-Stability of the Systems

To prove our theorems, we need the following lemmas.

Lemma 3. *Let $h, k, p, q \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ with $(t, s) \mapsto \partial_t h(t, s)$, $\partial_t k(t, s)$, $\partial_t p(t, s)$, $\partial_t q(t, s) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$. Assume, in addition, that $b \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing functions and $\alpha(t) \le t$ for $t \ge 0$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$u(t) \le b(t) + \int_0^t h(t,s)\,u(s)\,ds + \int_0^{\alpha(t)} k(t,s)\,u(s)\,ds + \int_0^t p(t,s)\,u(s)\left(\int_0^{\alpha(s)} q(s,v)\,u(v)\,dv\right)ds, \tag{7}$$

for $t \ge 0$, and $b(t)\int_0^t R(s)Q(s)ds < 1$, then

$$u(t) \le \frac{b(t)\,Q(t)}{1 - b(t)\int_0^t R(s)\,Q(s)\,ds}, \quad t \ge 0, \tag{8}$$

where $Q(t) = \exp(\int_0^t h(t,s)ds + \int_0^{\alpha(t)} k(t,s)ds)$, $R(t) = (d/dt)\int_0^t p(t,s)(\int_0^{\alpha(s)} q(s,v)dv)ds$.

Proof. Let $T \ge 0$ be fixed and denote

$$x(t) = \int_0^t h(t,s)\,u(s)\,ds + \int_0^{\alpha(t)} k(t,s)\,u(s)\,ds + \int_0^t p(t,s)\,u(s)\left(\int_0^{\alpha(s)} q(s,v)\,u(v)\,dv\right)ds, \quad t \ge 0, \tag{9}$$

then $u(t) \le b(t) + x(t)$, and x is nondecreasing on $\mathbb{R}_+$. For $t \in [0, T]$, by calculations we get the following:

$$\begin{aligned}
x'(t) &= \left[h(t,t)\,u(t) + \int_0^t \partial_t h(t,s)\,u(s)\,ds\right] \\
&\quad + \left[k(t,\alpha(t))\,u(\alpha(t))\,\alpha'(t) + \int_0^{\alpha(t)} \partial_t k(t,s)\,u(s)\,ds\right] \\
&\quad + \left[p(t,t)\,u(t)\int_0^{\alpha(t)} q(t,v)\,u(v)\,dv \right. \\
&\quad \left. + \int_0^t \partial_t p(t,s)\,u(s)\left(\int_0^{\alpha(s)} q(s,v)\,u(v)\,dv\right)ds\right] \\
&\le [b(T) + x(t)]\left[\frac{d}{dt}\left(\int_0^t h(t,s)\,ds + \int_0^{\alpha(t)} k(t,s)\,ds\right)\right] \\
&\quad + [b(T) + x(t)]^2 \frac{d}{dt}\int_0^t p(t,s)\left(\int_0^{\alpha(s)} q(s,v)\,dv\right)ds.
\end{aligned} \tag{10}$$

Suppose that $b(0) > 0$ (if $b(0) = 0$, carry out the following arguments with $b(t) + \varepsilon$ instead of $b(t)$, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\varepsilon \to 0$ to complete the proof), then we get

$$\begin{aligned}
&\frac{x'(t)}{[b(T) + x(t)]^2} \\
&\quad - \frac{1}{b(T) + x(t)}\frac{d}{dt}\left(\int_0^t h(t,s)\,ds + \int_0^{\alpha(t)} k(t,s)\,ds\right) \\
&\le \frac{d}{dt}\int_0^t p(t,s)\left(\int_0^{\alpha(s)} q(s,v)\,dv\right)ds.
\end{aligned} \tag{11}$$

Let

$$\begin{aligned}
z(t) &= \frac{1}{b(T) + x(t)}, \\
q(t) &= \int_0^t h(t,s)\,ds + \int_0^{\alpha(t)} k(t,s)\,ds, \\
Q(t) &= \exp(q(t)) \\
&= \exp\left(\int_0^t h(t,s)\,ds + \int_0^{\alpha(t)} k(t,s)\,ds\right), \\
R(t) &= \frac{d}{dt}\int_0^t p(t,s)\left(\int_0^{\alpha(s)} q(s,v)\,dv\right)ds,
\end{aligned} \tag{12}$$

then, we have

$$z'(t) + z(t)\left(\frac{d}{dt}q(t)\right) \geq -R(t). \tag{13}$$

Multiplying the above inequality by $e^{q(t)} = Q(t)$, we get

$$\frac{d}{dt}(z(t)Q(t)) \geq -Q(t)R(t). \tag{14}$$

Consider now the integral on the interval $[0,t]$ to obtain

$$z(t)Q(t) \geq z(0) - \int_0^t Q(s)R(s)\,ds, \quad 0 \leq t \leq T, \tag{15}$$

so

$$\begin{aligned} z(t) &= \frac{1}{b(T)+x(t)} \\ &\geq \left[\frac{1}{b(T)} - \int_0^t Q(s)R(s)\,ds\right]\frac{1}{Q(t)} \\ &= \frac{1-b(T)\int_0^t Q(s)R(s)\,ds}{b(T)Q(t)} \end{aligned} \tag{16}$$

for $0 \leq t \leq T$. Let $t = T$, since $b(T)\int_0^T Q(s)R(s)ds < 1$, then we have

$$b(T)+x(T) \leq \frac{b(T)Q(T)}{1-b(T)\int_0^T Q(s)R(s)\,ds}. \tag{17}$$

Since $T \geq 0$ was arbitrarily chosen, considering $u(t) \leq b(t) + x(t)$, we get (8). □

Lemma 4. *Let h, k, p, q, b, α be as in Lemma 3. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$\begin{aligned} u(t) \leq{}& b(t) + \int_0^t h(t,s)u(s)\,ds + \int_0^{\alpha(t)} k(t,s)u(s)\,ds \\ &+ \int_0^{\alpha(t)} p(t,s)u(s)\left(\int_0^s q(s,v)u(v)\,dv\right)ds, \end{aligned} \tag{18}$$

for $t \geq 0$, and $b(t)\int_0^t R(s)Q(s)ds < 1$, then

$$u(t) \leq \frac{b(t)Q(t)}{1-b(t)\int_0^t R(s)Q(s)\,ds}, \quad t \geq 0, \tag{19}$$

where $Q(t) = \exp(\int_0^t h(t,s)ds + \int_0^{\alpha(t)} k(t,s)ds)$, $R(t) = (d/dt)\int_0^{\alpha(t)} p(t,s)(\int_0^s q(s,v)dv)ds$.

The proof is similar to the proof of Lemma 3, we omit the details.

Theorem 5. *If there exist functions $a(t,s), b(t,s) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ with $(t,s) \mapsto \partial_t a(t,s), \partial_t b(t,s) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ such that*

$$\begin{aligned} \left\|\Psi(t) f(s,x)\right\| &\leq a(t,s)\left\|\Psi(s)x\right\|, \\ \left\|\Psi(t) g(s,x)\right\| &\leq b(t,s)\left\|\Psi(s)x\right\|, \end{aligned} \tag{20}$$

for $0 \leq s \leq t$ and for all $x \in \mathbb{R}^n$. Moreover,

$$\begin{gathered} \limsup_{t\to\infty} \int_0^t (a(t,s)+b(t,s))\,ds = L_1, \\ \left|\Psi(t)\Psi^{-1}(s)\right| \leq L_2 \quad \text{for } 0 \leq s \leq t, \end{gathered} \tag{21}$$

and $|\Psi(t)x(\alpha(t))| \leq |\Psi(\alpha(t))x(\alpha(t))|$, where L_1, L_2 are nonnegative constants. If $\alpha(t) = t - \tau(t)$ is an increasing diffeomorphism of $\mathbb{R}_+$. Then, the trivial solution of system (3) is Ψ-uniformly stable on $\mathbb{R}_+$.

Proof. Suppose that $x(t,t_0,x_0) := x(t)$ is the unique solution of system (3) which satisfies $x(t_0) = x_0$, since

$$\begin{aligned} x(t) &= x_0 + \int_{t_0}^t f(s,x(s))\,ds + \int_{t_0}^t g(s,x(\alpha(s)))\,ds \\ &= x_0 + \int_{t_0}^t f(s,x(s))\,ds + \int_{\alpha(t_0)}^{\alpha(t)} \frac{g\left(\alpha^{-1}(r),x(r)\right)}{\alpha'\left(\alpha^{-1}(r)\right)}dr, \end{aligned} \tag{22}$$

after performing the change of variables $r = \alpha(s)$ in the second integral, and α^{-1} is the inverse of the diffeomorphism α then, it follows that

$$\begin{aligned} \|\Psi(t)x(t)\| \leq{}& \left\|\Psi(t)\Psi^{-1}(t_0)\Psi(t_0)x_0\right\| \\ &+ \int_{t_0}^t \left\|\Psi(t) f(s,x(s))\right\| ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} \left\|\Psi(t)\frac{g\left(\alpha^{-1}(r),x(r)\right)}{\alpha'\left(\alpha^{-1}(r)\right)}\right\| ds \\ \leq{}& L_2\left\|\Psi(t_0)x_0\right\| + \int_{t_0}^t a(t,s)\left\|\Psi(s)x(s)\right\| ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} \frac{b\left(t,\alpha^{-1}(r)\right)}{\alpha'\left(\alpha^{-1}(r)\right)}\left\|\Psi(r)x(r)\right\| dr, \end{aligned} \tag{23}$$

this implies by Lemma 3 that

$$\begin{aligned} \|\Psi(t)x(t)\| \leq{}& L_2\left\|\Psi(t_0)x_0\right\| \exp \\ &\times \left(\int_{t_0}^t a(t,s)\,ds + \int_{\alpha(t_0)}^{\alpha(t)} \frac{b\left(t,\alpha^{-1}(r)\right)}{\alpha'\left(\alpha^{-1}(r)\right)}dr\right) \\ ={}& L_2\left\|\Psi(t_0)x_0\right\| \exp\left(\int_{t_0}^t (a(t,s)+b(t,s))\,ds\right) \\ \leq{}& L_2 e^{L_1}\left\|\Psi(t_0)x_0\right\|, \end{aligned} \tag{24}$$

so for every $\varepsilon > 0$, choose $\delta = \varepsilon/(L_2 e^{L_1})$, then

$$\|\Psi(t)x(t)\| \leq L_2 e^{L_1}\left\|\Psi(t_0)x_0\right\| < \varepsilon \tag{25}$$

for $\|\Psi(t_0)x_0\| < \delta$ and for all $0 \leq t_0 \leq t < \infty$. Hence, the conclusion of the theorem follows. □

Theorem 6. *Let all the conditions in Theorem 5 hold. Suppose further that there exist functions* $m(t,s), n(t,s) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ *with* $(t,s) \mapsto \partial_t m(t,s)$, $\partial_t n(t,s) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ *such that*

$$\begin{aligned} \left\|\Psi(t)\, p(s,x)\, \Psi^{-1}(s)\right\| &\le m(t,s)\,\|\Psi(s)\, x\|, \\ \left\|\Psi(t)\, q(s,x)\right\| &\le n(t,s)\,\|\Psi(s)\, x\|, \end{aligned} \tag{26}$$

for $0 \le s \le t$ *and for all* $x \in \mathbb{R}^n$, *moreover,*

$$\limsup_{t\to\infty} \int_0^t m(t,s)\left(\int_0^s n(s,u)\, du\right) ds = L_3, \tag{27}$$

where L_3 *is a nonnegative constant. Then, the trivial solutions of systems (4) and (5) are* Ψ*-uniformly stable on* $\mathbb{R}_+$.

Proof. For that system (4), suppose $x(t,t_0,x_0) := x(t)$ is the unique solution of system (4) which satisfies $x(t_0) = x_0$, since

$$\begin{aligned} x(t) = x_0 &+ \int_{t_0}^t f(s, x(s))\, ds + \int_{t_0}^t g(s, x(\alpha(s)))\, ds \\ &+ \int_{t_0}^t p(s, x(s)) \int_0^s q(u, x(\alpha(u)))\, du\, ds, \quad 0 \le t_0 \le t, \end{aligned} \tag{28}$$

it follows that

$$\begin{aligned} \|\Psi(t)\, x(t)\| \le{} & \left\|\Psi(t)\,\Psi^{-1}(t_0)\,\Psi(t_0)\, x_0\right\| \\ &+ \int_{t_0}^t \|\Psi(t)\, f(s, x(s))\|\, ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} \frac{\left\|\Psi(t)\, g\left(\alpha^{-1}(r), x(r)\right)\right\|}{\alpha'\left(\alpha^{-1}(r)\right)} dr \\ &+ \int_{t_0}^t \left\|\Psi(t)\, p(s, x(s))\,\Psi^{-1}(s)\right\| \\ &\times \left(\int_0^{\alpha(s)} \frac{\left\|\Psi(s)\, q\left(\alpha^{-1}(r), x(r)\right)\right\|}{\alpha'\left(\alpha^{-1}(r)\right)} dr\right) ds \\ \le{} & L_2\left\|\Psi(t_0)\, x_0\right\| + \int_{t_0}^t a(t,s)\,\|\Psi(s)\, x(s)\|\, ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} \frac{b\left(t, \alpha^{-1}(r)\right)}{\alpha'\left(\alpha^{-1}(r)\right)} \|\Psi(r)\, x(r)\|\, dr \\ &+ \int_{t_0}^t m(t,s)\,\|\Psi(s)\, x(s)\| \\ &\times \left(\int_0^{\alpha(s)} \frac{n\left(s, \alpha^{-1}(r)\right)\|\Psi(r)\, x(r)\|}{\alpha'\left(\alpha^{-1}(r)\right)} dr\right) ds \end{aligned} \tag{29}$$

after performing the change of variables $r = \alpha(s)$ (or $r = \alpha(u)$) at some intermediate step, and α^{-1} is the inverse of the diffeomorphism α. Denote

$$\begin{aligned} Q(t) &= \exp\left(\int_{t_0}^t a(t,s)\, ds + \int_{\alpha(t_0)}^{\alpha(t)} \frac{b\left(t, \alpha^{-1}(r)\right)}{\alpha'\left(\alpha^{-1}(r)\right)} dr\right) \\ &= \exp\left(\int_{t_0}^t \left(a(t,s) + b(t,s)\right) ds\right), \\ R(t) &= \frac{d}{dt}\left[\int_{t_0}^t m(t,s)\left(\int_0^{\alpha(s)} \frac{n\left(s, \alpha^{-1}(r)\right)}{\alpha'\left(\alpha^{-1}(r)\right)} dr\right) ds\right] \\ &= \frac{d}{dt}\left[\int_{t_0}^t m(t,s)\left(\int_0^s n(s,u)\, du\right) ds\right]. \end{aligned} \tag{30}$$

This implies by Lemma 3 that

$$\begin{aligned} &\|\Psi(t)\, x(t)\| \\ &\quad\le L_2\left\|\Psi(t_0)\, x_0\right\| \frac{Q(t)}{1 - L_2\left\|\Psi(t_0)\, x_0\right\| \int_0^t Q(v)\, R(v)\, dv} \\ &\quad\le \left\|\Psi(t_0)\, x_0\right\| \frac{L_2 e^{L_1}}{1 - L_2\left\|\Psi(t_0)\, x_0\right\| e^{L_1} \int_0^t R(v)\, dv} \\ &\quad= \left\|\Psi(t_0)\, x_0\right\| \\ &\qquad\times \frac{L_2 e^{L_1}}{1 - L_2\left\|\Psi(t_0)\, x_0\right\| e^{L_1} \int_{t_0}^t m(t,s)\left(\int_0^s n(s,u)\, du\right) ds} \\ &\quad\le \left\|\Psi(t_0)\, x_0\right\| \frac{L_2 e^{L_1}}{1 - L_2 L_3\left\|\Psi(t_0)\, x_0\right\| e^{L_1}} \end{aligned} \tag{31}$$

for $L_2 L_3 \left\|\Psi(t_0) x_0\right\| e^{L_1} < 1$ and $0 \le t_0 \le t$. So, for every $\varepsilon > 0$ and $t_0 \ge 0$, let $0 < q < 1/L_2 L_3 e^{L_1}$ be a constant and choose $\delta = \min\{q, ((1 - q L_2 L_3 e^{L_1})\varepsilon)/L_2 e^{L_1}\}$, then

$$\|\Psi(t)\, x(t)\| < \frac{\left(1 - q L_2 L_3 e^{L_1}\right)\varepsilon}{L_2 e^{L_1}} \times \frac{L_2 e^{L_1}}{1 - q L_2 L_3 e^{L_1}} = \varepsilon \tag{32}$$

for $\left\|\Psi(t_0) x_0\right\| < \delta$ and for all $0 \le t_0 \le t < \infty$. This proves that the trivial solution of system (4) is Ψ-uniformly stable on $\mathbb{R}_+$.

Using Lemma 4, the proof of system (5) is similar to that of system (4) and the details are left to the readers. □

Remark 7. For $\Psi_i = 1$, $i = 1, 2, \ldots, n$, we obtain the theorems of classical stability and uniform stability.

3. Examples

Example 8. Consider the nonlinear differential system

$$\begin{aligned} x_1'(t) &= x_1(t) + x_1\left(\frac{t}{2}\right)\sin t, \\ x_2'(t) &= -x_2(t) + x_2\left(\frac{t}{2}\right)\cos t. \end{aligned} \tag{33}$$

In (33), $f(t,x(t)) = (x_1(t), -x_2(t))^T$, $g(t,x(t/2)) = (x_1(t/2)\sin t, x_2(t/2)\cos t)^T$. Let $\Psi(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix}$, then $a(t,s) = b(t,s) = e^{-(t-s)}$ for $0 \le s \le t \le \infty$, it is easy to verify that $L_1 = 2$, $L_2 = 1$, and all the assumptions in Theorem 5 satisfied, so the trivial solution of system (33) is ψ-uniformly stable on $\mathbb{R}_+$.

Example 9. Consider the nonlinear Volterra integro-differential system as follows:

$$\begin{aligned} x_1'(t) &= x_1(t) + x_1(t)e^{-t}\int_0^t x_1\left(\frac{s}{2}\right)\cos s\,ds, \\ x_2'(t) &= -x_2(t) + x_2(t)e^{-t}\int_0^t x_2\left(\frac{s}{2}\right)\sin s\,ds. \end{aligned} \tag{34}$$

In (34), $f(t,x(t)) = (x_1(t), -x_2(t))^T$, $g \equiv 0$, $p(t,x(t)) = (x_1(t)e^{-t}, x_2(t)e^{-t})^T$, $q(s,x(s/2)) = (x_1(s/2)\cos s, x_2(s/2)\sin s)^T$. Choose the same matrix function $\Psi(t)$, then $a(t,s) = n(t,s) = e^{-(t-s)}$, $b(t,s) \equiv 0$, $m(t,s) = e^{-2(t-s)}$ for $0 \le s \le t \le \infty$, it is easy to verify that $L_1 = L_2 = 1$, $L_3 = 1/2$, and all the assumptions in Theorem 6 are satisfied, so the trivial solution of system (34) is ψ-uniformly stable on $\mathbb{R}_+$.

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