Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 157140, 4 pages
http://dx.doi.org/10.1155/2013/157140

Research Article

Affine-Periodic Solutions for Dissipative Systems

Yu Zhang,[1] Xue Yang,[1,2] and Yong Li[1]

[1] *College of Mathematics, Jilin University, Changchun 130012, China*
[2] *School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China*

Correspondence should be addressed to Xue Yang; yangxuemath@163.com

Received 4 November 2013; Accepted 10 December 2013

Academic Editor: Bingwen Liu

As generalizations of Yoshizawa's theorem, it is proved that a dissipative affine-periodic system admits affine-periodic solutions. This result reveals some oscillation mechanism in nonlinear systems.

1. Introduction

Consider the system

$$x' = f(t,x), \quad ' = \frac{\mathrm{d}}{\mathrm{d}t}, \tag{1}$$

where $f : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and ensures the uniqueness of solutions with respect to initial values. Fix $T > 0$. The system (1) is said to be T-periodic if $f(t+T,x) = f(t,x)$ for all $(t,x) \in \mathbb{R}^1 \times \mathbb{R}^n$. For this T-periodic system, a major problem is to seek the existence of T-periodic solutions. Actually, some physical systems also admit the certain affine-periodic invariance. For example, let $Q \in \mathrm{GL}(n)$, and

$$f(t+T,x) = Qf\left(t, Q^{-1}x\right), \quad \forall (t,x). \tag{2}$$

This affine-periodic invariance exhibits two characters: periodicity in time and symmetry in space. Obviously, when $Q = id$, the invariance is just the usual periodicity; when $Q = -id$, the invariance implies the usual antisymmetry in space. When $Q \in \mathrm{SO}(n)$, the invariance shows the rotating symmetry in space. Hence, the invariance also reflects some properties of solutions in geometry. Now, (2) is said to possess the affine-periodic structure. For this affine-periodic system, we are concerned with the existence of affine-periodic solutions $x(t)$ with

$$x(t+T) = Qx(t), \quad \forall t. \tag{3}$$

In the qualitative theory, it is a basic result that the dissipative periodic systems admit the existence of periodic solutions. The related topics had ever captured the main field in periodic solutions theory from the 1960s to the 1990s. For some litratures, see, for example, [1–12].

In the present paper, we will see whether (1) admits affine-periodic solutions or not if (1) is affine-dissipative. Here, (1) is said to be affine-dissipative if $Q^{-m}x(t+mT)$ are ultimately bounded. Our main result is the following.

Theorem 1. *Let $Q \in GL(n)$. If the system* (1) *is Q-affine-periodic, that is,*

$$f(t+T,x) = Qf\left(t, Q^{-1}x\right), \tag{4}$$

and affine-dissipative, then it admits a Q-affine-periodic solution $x_*(t)$; that is,

$$x_*(t+T) = Qx_*(t), \quad \forall t. \tag{5}$$

The paper is organized as follows. In Section 2, we use the asymptotic fixed-point theorem, for example, Horn's fixed-point theorem to prove Theorem 1. Section 3 deals with the case of functional differential equations, where an anagolous version is given and the proof is sketched. Finally, in Section 4, we illustrate some applications.

2. Proof of Theorem 1

In order to prove Theorem 1, we first recall some preliminaries.

Lemma 2 (Horn's fixed-point theorem [13]). *Let X be a Banach space, and let $S_0 \subset S_1 \subset S_2 \subset X$ be convex sets, where*

S_0 is compact, S_1 relatively open with respect to S_2, and S_2 closed. Assume that $P : S_0 \to X$ is continuous and satisfies

$$\begin{aligned} P^j\left(\overline{S}_1\right) &\subset S_2, \quad j = 0, 1, \ldots, N-1, \\ P^j\left(\overline{S}_1\right) &\subset S_0, \quad j = N, \ldots, 2N-1. \end{aligned} \tag{6}$$

Then, P has a fixed point in S_0.

The following is a usual definition.

Definition 3. The system (1) is said to be dissipative or ultimately bounded, if there is $B_0 > 0$ and for any $B > 0$, there are $M = M(B) > 0$ and $L = L(B) > 0$ such that for $|x_0| \le B$,

$$\begin{aligned} \left|x\left(t, x_0\right)\right| &\le M, \quad \forall t \in [0, L], \\ \left|x\left(t, x_0\right)\right| &\le B_0, \quad \forall t \in [L, \infty), \end{aligned} \tag{7}$$

where $x(t, x_0)$ denotes the solution of (1) with the initial value $x(0) = x_0$.

For the affine-periodic system (1), we have the following.

Definition 4. The system (1) is said to be Q-affine-dissipative, if there is $B_0 > 0$ and for any $B > 0$, there are $M = M(B) > 0$ and $L = L(B) > 0$ such that

$$\begin{aligned} \left|x\left(t, x_0\right)\right| &\le M, \quad \forall t \in [0, L], \\ \left|Q^{-m} x\left(t + mT, x_0\right)\right| &\le B_0, \quad \forall t \in [L, \infty), \ m \in \mathbb{Z}_+^1, \end{aligned} \tag{8}$$

whenever $|x_0| \le B$.

Proof of Theorem 1. Define the map $P : \mathbb{R}^n \to \mathbb{R}^n$ by

$$P\left(x_0\right) = Q^{-1} x\left(T, x_0\right), \quad \forall x_0 \in \mathbb{R}^n, \tag{9}$$

and set

$$\begin{aligned} S_0 &= \left\{y \in \mathbb{R}^n : |y| \le B_0\right\}, \\ S_1 &= \left\{y \in \mathbb{R}^n : |y| < B_1\right\}, \\ S_2 &= \left\{y \in \mathbb{R}^n : |y| \le B_2\right\}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} B_1 &= B_0 + 1, \\ B_2 &= \sup\left\{\left|Q^{-m} x\left(mT, x_0\right)\right| : m \in \{0, \ldots, N\}, \right. \\ &\qquad \left. \left|x_0\right| \le B_0 + 1\right\} + B_0 + 2, \\ N &= \left[L\left(B_1\right)\right] + 1. \end{aligned} \tag{11}$$

By uniqueness and the affine periodicity of $f(t, x)$, $Q^{-m} x(t + mT, x_0)$ is still the solution of (1) for each $m \in \mathbb{Z}_+^1$. Therefore,

$$P^i\left(x_0\right) = Q^{-i} x\left(iT, x_0\right), \quad i = 0, 1, \ldots. \tag{12}$$

It follows from (8) that

$$\begin{aligned} P^j\left(\overline{S}_1\right) &\subset S_2, \quad j = 0, \ldots, N-1, \\ P^j\left(\overline{S}_1\right) &\subset S_0, \quad j = N, \ldots, 2N-1. \end{aligned} \tag{13}$$

Thus, Horn's fixed-point theorem implies that P has a fixed point $\overline{x}_0$ in S_0; that is, $Q^{-1} x(T, \overline{x}_0) = \overline{x}_0$. Also, uniqueness yields

$$\begin{aligned} &Q^{-1} x\left(t + T, \overline{x}_0\right) = x\left(t, \overline{x}_0\right), \\ &\Longleftrightarrow x\left(t + T, \overline{x}_0\right) = Q x\left(t, \overline{x}_0\right), \quad \forall t. \end{aligned} \tag{14}$$

This completes the proof of Theorem 1. □

3. A Version to Functional Differential Equations

Consider the functional differential equation (FDE)

$$x' = F\left(t, x_t\right), \tag{15}$$

where $F : \mathbb{R}^1 \times \mathbb{C} \to \mathbb{R}^n$ is continuous, takes any bounded set in $\mathbb{C}$ to a bounded set in $\mathbb{R}^n$, and ensures the uniqueness of solutions with respect to initial values, where $\mathbb{C} = \mathbb{C}([-r, 0], \mathbb{R}^n)$, $x_t(s) = x(t + s)$, and $s \in [-r, 0]$. Moreover, F is Q-affine-periodic; that is,

$$F\left(t + T, \varphi\right) = Q F\left(t, Q^{-1} \varphi\right), \quad \forall (t, \varphi) \in \mathbb{R}^1 \times \mathbb{C}. \tag{16}$$

Definition 5. The system (15) is said to be Q-affine-dissipative; if there is $B_0 > 0$ and for any $B > 0$, there are $M = M(B) > 0$ and $L = L(B) > 0$ such that

$$\begin{aligned} \left|x\left(t, \varphi\right)\right| &\le M, \quad \forall t \in [0, L], \\ \left|Q^{-m} x\left(t + mT, \varphi\right)\right| &\le B_0, \quad \forall t \in [L, \infty), \end{aligned} \tag{17}$$

whenever $\|\varphi\| = \max_{[-r,0]} |\varphi(s)| \le B$; here, $x(t, \varphi)$ denotes the solution of (15) at initial value $x_0 = \varphi$.

We are in position to state another main result.

Theorem 6. *If the system* (15) *is Q-affine-periodic-dissipative, then it admits a Q-affine-periodic solution $x(t)$; that is,*

$$x(t + T) = Q x(t), \quad \forall t. \tag{18}$$

Proof. Define the map $P : \mathbb{C} \to \mathbb{C}$ by

$$P(\varphi) = Q^{-1} x_T(\cdot, \varphi), \quad \forall \varphi \in \mathbb{C}, \tag{19}$$

and set

$$\begin{aligned} S_0 &= \{\varphi \in \mathbb{C} : \|\varphi\| \le B_0, \\ &\qquad \left|\varphi\left(s_1\right) - \varphi\left(s_2\right)\right| \le h\left|s_1 - s_2\right|, \forall s_1, s_2 \in [-r, 0]\}, \\ S_1 &= \{\varphi \in \mathbb{C} : \|\varphi\| < B_1, \\ &\qquad \left|\varphi\left(s_1\right) - \varphi\left(s_2\right)\right| < h_1\left|s_1 - s_2\right|, \forall s_1, s_2 \in [-r, 0]\}, \\ S_2 &= \{\varphi \in \mathbb{C} : \|\varphi\| \le B_2, \\ &\qquad \left|\varphi\left(s_1\right) - \varphi\left(s_2\right)\right| \le h_2\left|s_1 - s_2\right|, \forall s_1, s_2 \in [-r, 0]\}, \end{aligned} \tag{20}$$

where

$$h = \sup\left\{\left|F(t,\varphi)\right| : t \in \mathbb{R}^1, \|\varphi\| \le B_0\right\},$$
$$h_1 = \sup\left\{\left|F(t,\varphi)\right| : t \in \mathbb{R}^1, \|\varphi\| \le B_0 + 1\right\}, \tag{21}$$
$$B_1 = B_0 + 1,$$

$$B_2 = \sup\{\|Q^{-m}x_{mT}(\cdot,\varphi)\| : m \in \{0,1,\ldots,N\},$$
$$\|\varphi\| \le B_0 + 1\} + B_0 + 2, \tag{22}$$
$$h_2 = \sup\left\{\left|F(t,\varphi)\right| : t \in \mathbb{R}^1, \|\varphi\| \le B_2\right\},$$

where $N = [L(B_1) + r] + 2$. Then, (17) and the constructions imply that

$$P^j\left(\overline{S}_1\right) \subset S_2, \quad j = 0,\ldots,N-1,$$
$$P^j\left(\overline{S}_1\right) \subset S_0, \quad j = N,\ldots,2N-1. \tag{23}$$

Hence, P has a fixed point $\varphi_* \in S_0$ via Horn's theorem. The uniqueness implies that $x(t,\varphi_*)$ is the desired affine-periodic solution of (15). The proof is complete. □

4. Some Applications

First, we observe a simple example to show the meanings of affine-periodic solutions.

Example 7. Consider the equation

$$x' + 2x = e^{-t}. \tag{24}$$

Put $f(t,x) = -2x + e^{-t}$. The general solution of (24) is

$$x(t) = e^{-2t}c + e^{-t} \quad (c \text{ is any constant}). \tag{25}$$

Obviously, for given $\tau > 0$,

$$f(t+\tau,x) = e^{-\tau}f\left(t,e^{\tau}x\right), \tag{26}$$

and any solution $x(t)$ satisfies

$$\begin{aligned}\left|\left(e^{-\tau}\right)^{-m}x(t+m\tau)\right| &= \left|e^{m\tau}e^{-2(t+m\tau)}c + e^{m\tau}e^{-(t+m\tau)}\right| \\ &\le e^{-(2t+m\tau)}|c| + 1 \\ &\le e^{-2t}|c| + 1,\end{aligned} \tag{27}$$

which implies that (24) is $e^{-\tau}$-periodic-dissipative. By Theorem 1, (24) has an $e^{-\tau}$-affine-periodic solution. This solution is just $x(t) = e^{-t}$ and different from the usual periodic solutions!

As usual, Lyapunov's method is flexible in studying the existence of affine-periodic solutions. The following results illustrate applications in this aspect.

Theorem 8. *Assume that there exists a Lyapunov's function $V : \mathbb{R}^1_+ \times \mathbb{R}^n \to \mathbb{R}^1_+$ such that*

(i) *$V(t,x)$ is of $\mathbb{C}^1$;*

(ii) *$V'(t,x) \le -W(t,x)$, $|x| \ge M > 0$, where $W(t,x)$ is continuous in $\mathbb{R}^1_+ \times \{|x| \le M\}$, and $W(t,x) \ge \alpha > 0$, $|x| \ge M$;*

(iii) *Uniformly in t,*

$$\liminf_{|x|\to\infty} V(t,x) > \sup\left\{V(t,x) : t \in \mathbb{R}^1_+, |x| \le M\right\}. \tag{28}$$

Then, the system (1) *has a Q-affine-periodic solution.*

Proof. Let $x(t,x_0)$ denote the solution of (1) with the initial value $x(0) = x_0$. Put

$$K = \sup\left\{V(t,x) : t \in \mathbb{R}^1_+, |x| \le M\right\},$$
$$G = \left\{x \in \mathbb{R}^n : V(t,x) \le K\right\}. \tag{29}$$

By assumption (iii), G is bounded and closed. In the following, we will prove that for each $B > 0$, there are $M = M(B) > 0$ and $N = N(B) > 0$ such that

$$\left|x(t,x_0)\right| \le M, \quad \forall t \in [0,N],$$
$$x(t,x_0) \in G, \quad \forall t \ge N, \tag{30}$$

whenever $|x_0| \le B$.

In fact, given that $x_0 \in \mathbb{R}^n$, $|x_0| > M$ implies on the maximal interval $[0,L)$ that $|x(t,x_0)| > M$; we have

$$\begin{aligned}0 \le V(t,x(t,x_0)) &\le V(0,x_0) - \int_0^t W(s,x(s,x_0))\,ds \\ &\le V(0,x_0) - \alpha t.\end{aligned} \tag{31}$$

This shows that there is $t_1 \in (0,\infty)$ such that

$$\left|x(t,x_0)\right| > M, \quad \forall t \in [0,t_1),$$
$$\left|x(t_1,x_0)\right| = M. \tag{32}$$

Note that

$$V(t,x(t,x_0)) \le V(t_1,x(t_1,x_0)), \quad \text{if } \left|x(t_1,x_0)\right| \ge M, \tag{33}$$

which together with the construction of G yields

$$x(t,x_0) \in G, \quad t \in [t_1,\infty). \tag{34}$$

If $|x_0| < M$, and there is a $\bar{t} \in (0,\infty)$ such that

$$\left|x(t,x_0)\right| < M, \quad t \in (0,\bar{t}), \quad \left|x(\bar{t},x_0)\right| = M, \tag{35}$$

then we also have

$$x(t,x_0) \in G, \quad \forall t \in [\bar{t},\infty). \tag{36}$$

Of course, in case of $|x_0| = M$, we have

$$x(t,x_0) \in G, \quad \forall t \in [0,\infty). \tag{37}$$

Taking these cases into account, we choose

$$N = t_1. \tag{38}$$

Now, the existence of affine-periodic solutions is an immediate consequence. The proof is complete. □

Theorem 9. *Assume that*

$$\langle x, f(t,x)\rangle \le -a(t)|x|^2, \tag{39}$$

where $a \in \mathrm{Loc}(\mathbb{R}_+^1)$ *satisfies*

$$\int_0^\infty a(s)\,\mathrm{d}s = \infty, \qquad \int_0^\infty a^-(s)\,\mathrm{d}s < \infty. \tag{40}$$

Then, (1) *has an affine-periodic solution.*

Proof. Let

$$V(t,x) = \frac{1}{2}|x|^2. \tag{41}$$

Then,

$$\begin{aligned} &V'(t,x) = \langle x, f(t,x)\rangle \le -2a(t)V(t,x) \\ &\Longrightarrow V(t,x(t,x_0)) \le e^{-\int_0^t 2a(s)\mathrm{d}s} V(0,x_0), \quad \forall t \ge 0. \end{aligned} \tag{42}$$

By assumption, $\int_0^\infty a(s)\mathrm{d}s = \infty$, there is $t_1 \in (0,\infty)$ such that

$$e^{-\int_0^{t_1} 2a(s)\mathrm{d}s}\frac{1}{2}|x|^2 \le 1. \tag{43}$$

Thus,

$$\begin{aligned} &V(t,x(t,x_0)) \le e^{-\int_{t_1}^\infty 2a^-(s)\mathrm{d}s}, \quad \forall t \ge t_1, \\ &\Longrightarrow |x(t,x_0)| \le \sqrt{2}e^{-\int_{t_1}^\infty a^-(s)\mathrm{d}s}, \quad \forall t \ge t_1. \end{aligned} \tag{44}$$

By Theorem 1, (1) has an affine-periodic solution. This finishes the proof. □

Example 10. Consider the system

$$x' = \pm|x|^{2\beta}x + \left(e^{\sqrt{-1}2\pi\Theta t}\right) \equiv f(t,x), \qquad (*)_\pm$$

where $\beta \ge 0$; $x \in \mathbb{C}^n$; $\Theta = (\theta_1, \theta_2, \ldots, \theta_n)^T$, $\theta_i > 0$, $i = 1, 2, \ldots, n$. Let

$$Q = e^{\sqrt{-1}2\pi\Theta T}, \quad T > 0. \tag{45}$$

Then

$$f(t+T,x) = Qf\left(t, Q^{-1}x\right). \tag{46}$$

In the following, we only consider the case $(*)_-$. Otherwise, set $t \to -t$ for $(*)_+$. Take $V(t,x) = (1/2)|x|^2$. Notice that for $|x| \ge \sqrt{2} = M$, $\alpha = \sqrt{2}$,

$$\begin{aligned} V'(t,x) &= \langle x, f(t,x)\rangle = x^T\overline{f}(t,x) \\ &= -|x|^{2\beta+2} + x^T e^{-\sqrt{-1}2\pi\Theta t} \\ &\le -|x|^{2\beta+2} + |x| \\ &\le -|x| = -W(t,x) \le -\alpha. \end{aligned} \tag{47}$$

Hence, by Theorem 8, $(*)_-$ has a Q-affine T-periodic solution. Now, if leting p/q be a reduced fraction and $\theta_i T = p/q$, $i = 1, 2, \ldots, n$, then the Q-affine T-periodic solutions are just q-subharmonic ones; if $\Theta T \in \mathbb{Q}^n$ (the set of rational vectors), then there is a K such that these affine T-periodic solutions are K-periodic ones; if $\Theta T \in \mathbb{R}^n \setminus \mathbb{Q}^n$, then these solutions are quasiperiodic ones with frequency ΘT.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work was supported by the National Natural Science Foundation of China (Grants nos. 11171132 and 11201173).

References

[1] O. A. Arino, T. A. Burton, and J. R. Haddock, "Periodic solutions to functional differential equations," *Proceedings of the Royal Society of Edinburgh A*, vol. 101, pp. 253–271, 1985.

[2] T. A. Burton and S. Zhang, "Unified boundedness, periodicity, and stability in ordinary and functional differential equations," *Annali di Matematica Pura ed Applicata*, vol. 145, no. 1, pp. 129–158, 1986.

[3] J. Chu, J. Sun, and P. J. Y. Wong, "Existence for singular periodic problems: a survey of recent reaults," *Abstract and Applied Analysis*, Article ID 420835, 17 pages, 2013.

[4] H. Ergoren and C. Tunc, "Anti-periodic solutions for a class of fourth-order nonlinear differential equations with multiple deviating arguments," *Azerbaijan Journal of Mathematics*, vol. 3, no. 1, pp. 111–126, 2013.

[5] H. Gao, Y. Li, and B. Zhang, "A fixed point theorem of Krasnoselskii-Schaefer type and its applications in control and periodicity of integral equations," *Fixed Point Theory*, vol. 12, no. 1, pp. 91–112, 2011.

[6] J. K. Hale and O. Lopes, "Fixed point theorems and dissipative processes," *Journal of Differential Equations*, vol. 13, no. 2, pp. 391–402, 1973.

[7] T. Küpper, Y. Li, and B. Zhang, "Periodic solutions for dissipative-repulsive systems," *Tohoku Mathematical Journal*, vol. 52, no. 3, pp. 321–329, 2000.

[8] Y. Li, Q. D. Zhou, and X. R. Lu, "Periodic solutions and equilibrium states for functional differential inclusions with nonconvex right-hand side," *Quarterly of Applied Mathematics*, vol. 55, no. 1, pp. 57–68, 1997.

[9] G. Makay, "Periodic solutions of dissipative functional differential equations," *Tohoku Mathematical Journal*, vol. 46, pp. 417–426, 1994.

[10] T. I. Seidman and O. Klein, "Periodic solutions of isotone hybrid systems, Discrete Contin," *Dynamic Systems Service B*, vol. 18, no. 2, pp. 483–493, 2013.

[11] T. Yoshizawa, *Stability Theory by Liapunov's Second Method*, The Mathematical Society of Japan, Tokyo, Japan, 1966.

[12] Y. Yuan and Z. Guo, "On the existence and stability of periodic solutions for a nonlinear neutral functional differential equation," *Abstract and Applied Analysis*, vol. 2013, Article ID 175479, 8 pages, 2013.

[13] W. A. Horn, "Some fixed point theorems for compact maps and flows in Banach spaces," *Transactions of the American Mathematical Society*, vol. 149, pp. 391–404, 1970.