Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 216035, 4 pages
http://dx.doi.org/10.1155/2013/216035

*Research Article*

# Existence and Uniqueness of the Positive Definite Solution for the Matrix Equation $X = Q + A^*(\widehat{X} - C)^{-1}A$

**Dongjie Gao**

*Department of Mathematics, Heze University, Heze, Shandong 274015, China*

Correspondence should be addressed to Dongjie Gao; aizai_2004@126.com

Received 15 May 2013; Revised 23 June 2013; Accepted 4 July 2013

Academic Editor: Vejdi I. Hasanov



We consider the nonlinear matrix equation $X = Q + A^*(\widehat{X} - C)^{-1}A$, where $Q$ is positive definite, $C$ is positive semidefinite, and $\widehat{X}$ is the block diagonal matrix defined by $\widehat{X} = \operatorname{diag}(X, X, \ldots, X)$. We prove that the equation has a unique positive definite solution via variable replacement and fixed point theorem. The basic fixed point iteration for the equation is given.

## 1. Introduction

We consider the matrix equation

$$X = Q + A^*\left(\widehat{X} - C\right)^{-1}A, \tag{1}$$

where $Q$ is an $n \times n$ positive definite matrix, $C$ is an $mn \times mn$ positive semidefinite matrix, $A$ is arbitrary $mn \times n$ matrix, and $\widehat{X}$ is the block diagonal matrix defined by $\widehat{X} = \operatorname{diag}(X, X, \ldots, X)$ in which $X$ is an $n \times n$ matrix. This matrix equation is connected with certain interpolation problem (see [1]).

When $C = 0$, we write $A$ as the $m \times 1$ block matrix

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_4 \end{pmatrix}, \tag{2}$$

where $A_1, \ldots, A_m$ are $n \times n$ matrices. Then (1) is equal to $X = Q + \sum_{i=1}^{m} A_i^* X^{-1} A_i$, which has been investigated by many authors (see [2–9]). However, are few little theoretical results when $C \neq 0$. Ran and Reurings [1] have proved that (1) has a unique positive definite solution. Based on this, Sun [10] presented some perturbation results for the unique solution.

Fixed point theorem is often used to discuss the existence and uniqueness of the solution. However, for (1), it is difficult to prove the uniqueness of the solution directly by using fixed point theorems. In this paper, we turn (1) into its equivalent equation via variable replacement. Then we consider its equivalent equation by using fixed point theorem. This provides a new proof for the existence and uniqueness of the positive definite solution. And this method is shown to be much easier than the way of [1]. In addition, the basic fixed point iteration for the equation is given.

In this paper we use $\mathscr{C}(n)$ to denote $n \times n$ complex matrices, $\mathscr{P}(n)$ to denote $n \times n$ positive definite matrices, and $\overline{\mathscr{P}}(n)$ to denote $n \times n$ positive semidefinite matrices. For $X, Y \in \mathscr{P}(n)$, we write $X \geq Y$ $(X > Y)$ if $X - Y$ is positive semidefinite (definite). $A^*$ denotes the conjugate transpose of a matrix $A$. Let $\overline{\lambda} = \lambda_{\max}(A^*A)$, $\underline{\lambda} = \lambda_{\min}(A^*A)$, $\overline{C_1} = \lambda_{\max}(C_1)$, and $\underline{C_1} = \lambda_{\min}(C_1)$.

## 2. The Case of $\widehat{Q} > C$

In this section, we discuss (1) with $\widehat{Q} > C$. Let $S_1(n) \subset P(n)$ be the set defined by

$$S_1(n) = \left\{X \in P(n) \mid \widehat{X} > C\right\}. \tag{3}$$

We are interested in positive definite solutions of (1) in this set. A matrix $X$ is a solution of (1) if and only if it is a fixed point of the map $G_1$ defined by

$$G_1(X) = Q + A^*\left(\widehat{X} - C\right)^{-1}A. \tag{4}$$

Let $C_1 = \widehat{Q} - C$. For $\widehat{Q} > C$, we know that $C_1$ is a positive definite matrix. Then (1) turns into

$$X - Q = A^*\left(\widehat{X} - \widehat{Q} + C_1\right)^{-1} A. \tag{5}$$

Let $Y = X - Q$. Then (1) eventually becomes

$$Y = A^*\left(\widehat{Y} + C_1\right)^{-1} A. \tag{6}$$

In the following, we consider the existence and uniqueness of the positive definite solution of (6). And then we can easily get corresponding conclusions about (1).

**Theorem 1.** *Equation* (6) *has a positive semidefinite solution for any* $A \in \mathscr{C}(n)$.

*Proof.* A matrix $Y$ is a solution of (6) if and only if it is a fixed point of the map $F$ defined by $F(Y) = A^*(\widehat{Y} + C_1)^{-1} A$. Note that $F$ maps $[0, F(0)]$ into itself, because $F$ is order reversing. Hence it has a fixed point in the $[0, F(0)]$. That is to says (6) has a positive semidefinite solution. □

**Theorem 2.** *If $Y$ is positive semidefinite solution of* (6), *then* $Y \in [0, F(0)]$.

*Proof.* By $Y \geq 0$, and $C_1 > 0$, we know that $(\widehat{Y} + C_1)^{-1} > 0$ and $A^*(\widehat{Y} + C_1)^{-1} A \geq 0$; then $Y = A^*(\widehat{Y} + C_1)^{-1} A \geq 0$. By $Y \geq 0$, we know that $Y = A^*(\widehat{Y} + C_1)^{-1} A \leq A^* C_1^{-1} A = F(0)$. That is, $Y \in [0, F(0)]$. □

For proving the uniqueness, we first verify the following lemma.

**Lemma 3.** *Let* $\eta(t) = (((1-t)\underline{C_1})/(\overline{C_1} + \underline{C_1}^{-1}\overline{\lambda})) > 0$. *For any* $Y \in [0, F(0)]$ *and* $0 < t < 1$, *one has*

$$F^2(tY) \geq t\left(1 + \eta(t)\right) F^2(Y). \tag{7}$$

*Proof.* By Theorem 2, we know $F(Y) \in [0, F(0)]$ for any $Y \in [0, F(0)]$. Then $F^2(Y) \in [0, F(0)]$ for any $Y \in [0, F(0)]$. Let $F(tY) = A^*(\widehat{tY} + C_1)^{-1} A$, and $F_1(tY) = A^*(\widehat{Y} + t^{-1}C_1)^{-1} A$. For any $Y \in [0, F(0)]$ and $0 < t < 1$, we have

$$\begin{aligned}
&F^2(tY) - t\left(1 + \eta(t)\right) F^2(Y) \\
&\quad = A^*\left(\widehat{F(tY)} + C_1\right)^{-1} A - tA^*\left(\widehat{F(Y)} + C_1\right)^{-1} A \\
&\qquad - t\eta(t) F^2(Y) \\
&\quad = tA^*\left(\widehat{F_1(tY)} + tC_1\right)^{-1} A - tA^*\left(\widehat{F(Y)} + C_1\right)^{-1} A \\
&\qquad - t\eta(t) F^2(Y) \\
&\quad \geq tA^*\left(\widehat{F(Y)} + tC_1\right)^{-1} A - tA^*\left(\widehat{F(Y)} + C_1\right)^{-1} A \\
&\qquad - t\eta(t) F^2(Y) \\
&\quad = tA^*\left[\left(\widehat{F(Y)} + tC_1\right)^{-1} - \left(\widehat{F(Y)} + C_1\right)^{-1}\right] A \\
&\qquad - t\eta(t) F^2(Y) \\
&\quad = tA^*\left[\left(\widehat{F(Y)} + tC_1\right)^{-1} (1-t) C_1 \left(\widehat{F(Y)} + C_1\right)^{-1}\right] A \\
&\qquad - t\eta(t) F^2(Y) \\
&\quad \geq tA^*\left[\left(\widehat{F(Y)} + C_1\right)^{-1} (1-t) C_1 \left(\widehat{F(Y)} + C_1\right)^{-1}\right] A \\
&\qquad - t\eta(t) F^2(Y) \\
&\quad \geq t\frac{(1-t)\underline{C_1}}{\overline{C_1} + \underline{C_1}^{-1}\overline{\lambda}} A^*\left(\widehat{F(Y)} + C_1\right)^{-1} A - t\eta(t) F^2(Y) \\
&\quad = t\eta(t) F^2(Y) - t\eta(t) F^2(Y) \\
&\quad = 0.
\end{aligned} \tag{8}$$

□

**Theorem 4.** *Equation* (6) *has a unique positive definite solution* $Y \in [0, F(0)]$, *and for any* $Y_0 \in [0, F(0)]$, *the iteration*

$$Y_{n+1} = A^*\left(\widehat{Y}_n + C_1\right)^{-1} A, \quad n = 0, 1, 2, \ldots, \tag{9}$$

*converges to $Y$; that is,* $\lim_{n\to\infty} Y_n = Y$.

*Proof.* Consider the matrix sequence (9). Let $F(Y_n) = A^*(\widehat{Y_n} + C_1)^{-1} A$. We first suppose that $Y_0 = 0$. Then we get

$$\begin{aligned}
0 \leq Y_1 &= A^*\left(\widehat{Y_0} + C_1\right)^{-1} A = F(0), \\
0 \leq Y_2 &= A^*\left(\widehat{Y_1} + C_1\right)^{-1} A = F^2(0) \leq F(0), \\
F^2(0) \leq Y_3 &= A^*\left(\widehat{Y_2} + C_1\right)^{-1} A = F^3(0) \leq F(0), \\
F^2(0) \leq Y_4 &= A^*\left(\widehat{Y_3} + C_1\right)^{-1} A = F^4(0) \leq F^3(0).
\end{aligned} \tag{10}$$

Analogously one can prove that

$$0 \leq F^{2k}(0) \leq F^{2k+2}(0) \leq F^{2k+1}(0) \leq F^{2k-1}(0) \leq F(0). \tag{11}$$

Hence, the sequence $\{F^{2k}(0), k \geq 0\}$ is monotone increasing and is bounded from above by $F(0)$. Thus, the sequence $\{F^{2k}(0), k \geq 0\}$ has a finite positive definite limit. Moreover, the sequence $\{F^{2k+1}(0), k \geq 0\}$ is monotone decreasing and is bounded from below by 0. Thus, the sequence $\{F^{2k+1}(0), k \geq 0\}$ has a finite positive definite limit. Let

$$\lim_{k\to\infty} F^{2k}(0) = Y^{(1)}, \qquad \lim_{k\to\infty} F^{2k+1}(0) = Y^{(2)}; \tag{12}$$

then $0 \le Y^{(i)} \le F(0)$ for $i = 1, 2$. Clearly, both $Y^{(1)}$ and $Y^{(2)}$ are the positive fixed points of $F^2(Y)$. Then

$$\begin{aligned} Y^{(1)} &= F^2\left(Y^{(1)}\right) \\ &= A^*\left[\widehat{F(Y^{(1)})} + C_1\right]^{-1} A \\ &\ge A^*\left(\overline{\lambda} C_1^{-2} C_1 + C_1\right)^{-1} A \\ &\ge A^*\left(\overline{\lambda} \underline{C_1}^{-2} + 1\right)^{-1} C_1^{-1} A \\ &= \frac{1}{\overline{\lambda} \underline{C_1}^{-2} + 1} A^* C_1^{-1} A \\ &\ge \frac{1}{\overline{\lambda} \underline{C_1}^{-2} + 1} F^2\left(Y^{(2)}\right) \\ &= \frac{1}{\overline{\lambda} \underline{C_1}^{-2} + 1} Y^{(2)}. \end{aligned} \tag{13}$$

Let $t_0 = \sup\{t \mid Y^{(1)} \ge tY^{(2)}\}$; then $0 < t_0 < +\infty$. Now we prove that $t_0 \ge 1$. Assume, on the contrary, that $0 < t_0 < 1$; then $Y^{(1)} \ge t_0 Y^{(2)}$. By Lemma 3 and monotonicity of $F^2(Y)$, we have

$$\begin{aligned} Y^{(1)} &= F^2\left(Y^{(1)}\right) \ge F^2\left(t_0 Y^{(2)}\right) \ge \left(1 + \eta\left(t_0\right)\right) t_0 F^2\left(Y^{(2)}\right) \\ &= \left(1 + \eta\left(t_0\right)\right) t_0 Y_2. \end{aligned} \tag{14}$$

Since $(1 + \eta(t_0))t_0 > t_0$, then it is a contradiction to the definition of $t_0$. Hence we have $t_0 \ge 1$; therefore $Y^{(1)} \ge Y^{(2)}$. Similarly, we can get $Y^{(1)} \le Y^{(2)}$. Therefore, $Y^{(1)} = Y^{(2)}$. Let $\Omega = [0, F(0)]$. By Theorem 1, we know $F(\Omega) \subseteq \Omega$. Hence $F^2(\Omega) \subseteq \Omega$. That is, $F^2$ has only one positive fixed point in $\Omega$. In other words, the equation $Y = F^2(Y)$ has only one positive definite solution. Hence $Y^{(1)} = Y^{(2)}$. It follows that $\lim_{n\to\infty} F^n(0)$ is a fixed point of $F^2$. Since the positive definite solution of $F(Y) = Y$ solves $Y = F^2(Y)$, then $F^2(Y) = Y$ has only one positive definite solution. Hence $\lim_{n\to\infty} F^{(n)}(0)$ is the unique fixed point of $F$.

For any $Y_0 > 0$, we get

$$\begin{gathered} Y_1 \ge 0, \\ 0 \le Y_2 = A^*\left(\widehat{Y_1} + C_1\right)^{-1} A \le A^* C_1^{-1} A = F(0). \end{gathered} \tag{15}$$

By induction, we have for any $k = 1, 2, \ldots$

$$\begin{gathered} F^{2k}(0) \le Y_{2k+1} \le F^{2k-1}(0), \\ F^{2k-2}(0) \le Y_{2k} \le F^{2k-1}(0). \end{gathered} \tag{16}$$

Now letting $k \to \infty$ on both sides of the above inequalities, we can get

$$\lim_{k\to\infty} Y_k = \lim_{n\to\infty} F^{(n)}(0) = Y. \tag{17}$$

Since (1) is equal to (6) when $X = Y + Q$, then we know that $X$ is a positive definite solution of (1) if and only if $Y$ is a positive semidefinite solution of (6). And furthermore, $Y \in [0, F(0)]$ if and only if $X \in [Q, G_1(Q)]$. Thus, we can get the following conclusions about (1). □

**Theorem 5.** *Equation* (1) *with* $\widehat{Q} > C$ *has a positive definite solution for any* $A \in \mathscr{C}(n)$.

**Theorem 6.** *If* $X$ *is positive definite solution of* (1) *with* $\widehat{Q} > C$, *then* $X \in [Q, G_1(Q)] \subset S_1(n)$.

**Theorem 7.** *Let* $Q \in \mathscr{P}(n)$ *and* $C \in \overline{\mathscr{P}}(mn)$ *such that* $\widehat{Q} > C$. *Then* (1) *has a unique solution* $X$ *in* $S_1(n)$ *and for any* $X_0 \in [Q, G_1(Q)]$, *the iteration*

$$X_{n+1} = Q + A^*\left(\widehat{X}_n - C\right)^{-1} A, \quad n = 0, 1, 2, \ldots, \tag{18}$$

*converges to* $X$; *that is,* $\lim_{n\to\infty} X_n = X$.

## 3. The Case of $\widehat{Q} < C$

In this section, we discuss (1) with $\widehat{Q} < C$ and $A^*(C - \widehat{Q})^{-1} A < Q$. In fact, $C \in \overline{\mathscr{P}}(n)$ must be a positive definite matrix in this case, because $\widehat{Q} < C$ and because $Q$ is a positive definite matrix. We consider the matrix equation

$$X = Q - A^*\left(C - \widehat{X}\right)^{-1} A, \tag{19}$$

which is equal to (1). Let $S_2(n) \subset P(n)$ be the set defined by

$$S_2(n) = \left\{X \in P(n) \mid \widehat{X} < C\right\}. \tag{20}$$

We are interested in positive definite solutions of (1) in this set. A matrix $X$ is a solution of (1) if and only if it is a fixed point of the map $G_2$ defined by

$$G_2(X) = Q - A^*\left(C - \widehat{X}\right)^{-1} A. \tag{21}$$

Let $C_2 = C - \widehat{Q}$; For $\widehat{Q} < C$, we have $C_2$ being a positive definite matrix. Then (1) turns into

$$Q - X = A^*\left(\widehat{Q} - \widehat{X} + C_2\right)^{-1} A. \tag{22}$$

Let $Y = Q - X$. Then (1) eventually becomes

$$Y = A^*\left(\widehat{Y} + C_2\right)^{-1} A. \tag{23}$$

We note that (23) should have the same results as (6). Then we directly give the following conclusions without proof.

**Theorem 8.** *Equation* (1) *with* $\widehat{Q} < C$ *and* $A^*(C - \widehat{Q})^{-1} A < Q$ *has a positive definite solution for any* $A \in \mathscr{C}(n)$.

**Theorem 9.** *If* $X$ *is a positive definite solution of* (1) *with* $\widehat{Q} < C$ *and* $A^*(C - \widehat{Q})^{-1} A < Q$, *then* $X \in [G_2(Q), Q] \subset S_2(n)$.

**Theorem 10.** *Let $Q \in \mathscr{P}(n)$ and $C \in \overline{\mathscr{P}}(mn)$ such that $\widehat{Q} < C$. Then* (1) *has a unique solution $X$ in $S_2(n)$, and for any $X_0 \in [G_2(Q), Q]$, the iteration*

$$X_{n+1} = Q - A^*\left(C - \widehat{X}_n\right)^{-1} A, \quad n = 0, 1, 2, \ldots, \tag{24}$$

*converges to $X$; that is,* $\lim_{n\to\infty} X_n = X$.

## 4. Numerical Examples

We now use numerical examples to illustrate our results. All computations were performed using MATLAB, version7.01. We denote $\varepsilon(X) = \|X - A^*(\widehat{X} - C)^{-1}A - Q\|_\infty$ and use the stopping criterion $\varepsilon(X) < 1.0 \times 10^{-10}$.

*Example 1.* Consider (1) with $n = 2$, $m = 3$, and

$$A = \begin{pmatrix} 2 & -2 \\ 1 & 0 \\ -5 & 3 \\ 6 & 2 \\ -3 & 1 \\ 6 & 5 \end{pmatrix}, \qquad Q = \begin{pmatrix} 20 & -1 \\ -1 & 13 \end{pmatrix},$$
$$C = \begin{pmatrix} 6 & 1 & -2 & 0 & 5 & 2 \\ 1 & 9 & -1 & -1 & 3 & -2 \\ -2 & -1 & 7 & 1 & -1 & 2 \\ 0 & -1 & 1 & 6 & 0 & -2 \\ 5 & 3 & -1 & 0 & 9 & 0 \\ 2 & -2 & 2 & -2 & 0 & 4 \end{pmatrix}, \tag{25}$$

where the matrices $Q$ and $C$ satisfy $\widehat{Q} > C$. Considering the iterative method (18) with $X_0 = Q$, after 23 iterations we get the following result:

$$X \approx X_{23} = \begin{pmatrix} 27.28666873488235 & 1.12493304573834 \\ 1.12493304573834 & 16.21667821374470 \end{pmatrix} \tag{26}$$

and $\varepsilon(X_{23}) = \|X_{23} - A^*(\widehat{X_{23}} - C)^{-1}A - Q\|_\infty = 8.622080827080936 \times 10^{-11}$. It is not difficult to verify that $X \in [Q, G_1(Q)] \subset S_1(n)$.

*Example 2.* Consider (1) with $n = 2$, $m = 3$, and

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ 1 & 2 \\ 2 & 1 \\ -1 & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 4 & -1 \\ -1 & 5 \end{pmatrix},$$
$$C = \begin{pmatrix} 15 & 1 & -1 & 2 & 0 & 1 \\ 1 & 19 & -2 & -1 & 3 & -1 \\ -1 & -2 & 10 & 2 & -2 & 2 \\ 2 & -1 & 2 & 19 & 0 & -1 \\ 0 & 3 & -2 & 0 & 23 & 0 \\ 1 & -1 & 2 & -1 & 0 & 30 \end{pmatrix}, \tag{27}$$

where the matrices $Q$ and $C$ satisfy $\widehat{Q} < C$ and $A^*(C - \widehat{Q})^{-1}A < Q$. Considering the iterative method (24) with $X_0 = Q$, after 7 iterations one gets an approximation to the positive definite solution $X$. It is

$$X \approx X_7 = \begin{pmatrix} 3.48707713254602 & -1.33049056598352 \\ -1.33049056598352 & 4.39117111733130 \end{pmatrix} \tag{28}$$

and $\varepsilon(X_7) = \|X_{23} - A^*(\widehat{X_7} - C)^{-1}A - Q\|_\infty = 6.447897771266753 \times 10^{-11}$. It is not difficult to verify that $X \in [G_2(Q), Q] \subset S_2(n)$.

## Acknowledgments

The work was supported by the National Natural Science Foundation of China (11071141), the Natural Science Foundation of Shandong Province of China (ZR2011AL018), the Specialized Research Foundation for the Doctoral Program of Higher Education of China (20123705110001), the Program for Scientific Research Innovation Team in Colleges and Universities of Shandong Province, and the Project of Shandong Province Higher Educational Science and Technology Program (J11LA06 and J13LI02).

## References

[1] A. C. M. Ran and M. C. B. Reurings, "A nonlinear matrix equation connected to interpolation theory," *Linear Algebra and Its Applications*, vol. 379, pp. 289–302, 2004.

[2] A. Ferrante and B. C. Levy, "Hermitian solutions of the equation $X = Q + NX^{-1}N^*$," *Linear Algebra and Its Applications*, vol. 247, pp. 359–373, 1996.

[3] V. I. Hasanov, "Notes on two perturbation estimates of the extreme solutions to the equations $X \pm A^*X^{-1}A = Q$," *Applied Mathematics and Computation*, vol. 216, no. 5, pp. 1355–1362, 2010.

[4] D. J. Gao and Y. H. Zhang, "Hermitian positive definite solutions of the matrix equation $X - A^*X^qA = Q(q > 0)$," *Mathematica Numerica Sinica*, vol. 29, no. 1, pp. 73–80, 2007 (Chinese).

[5] X. Y. Yin and S. Y. Liu, "Positive definite solutions of the matrix equations $X \pm A^*X^{-1}A = Q(q > 0)$," *Computers & Mathematics with Applications*, vol. 59, no. 12, pp. 3727–3739, 2010.

[6] J. Cai and G. L. Chen, "On the Hermitian positive definite solutions of nonlinear matrix equation $X^s + A^*X^{-t}A = Q$," *Applied Mathematics and Computation*, vol. 217, no. 1, pp. 117–123, 2010.

[7] A. M. Sarhan, N. M. El-Shazly, and E. M. Shehata, "On the existence of extremal positive definite solutions of the nonlinear matrix equation $Xr + \sum_{i=1}^m A_i^* X^{\delta_i} A_i = I$," *Mathematical and Computer Modelling*, vol. 51, no. 9-10, pp. 1107–1117, 2010.

[8] Y. Lim, "Solving the nonlinear matrix equation $X = Q + \sum_{i=1}^m M_i X^{\delta_i} M_i^*$ via a contraction principle," *Linear Algebra and its Applications*, vol. 430, no. 4, pp. 1380–1383, 2009.

[9] X. F. Duan and A. P. Liao, "On Hermitian positive definite solution of the matrix equation $X - \sum_{i=1}^m A_i^* X^r A_i = Q$," *Journal of Computational and Applied Mathematics*, vol. 229, no. 1, pp. 27–36, 2009.

[10] J.-G. Sun, "Perturbation analysis of the matrix equation $X = Q + A^H(\widehat{X} - C)^{-1}A$," *Linear Algebra and Its Applications*, vol. 372, pp. 33–51, 2003.