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Letter to the Editor

Comment on "Continuous g-Frame in Hilbert C^*-Modules"

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The continuous g-frames in Hilbert C^*-modules were introduced and investigated by Kouchi and Nazari (2011). They also studied the continuous g-Riesz basis and a characterization for it was presented by using the synthesis operator. However, we found that there is an error in the proof. The purpose of this paper is to improve their result by introducing the so-called modular continuous g-Riesz basis.

Kouchi and Nazari in [1] introduced the continuous g-frames in Hilbert C^*-modules and investigated some of their properties. The following lemma is a useful tool in their study.

Lemma 1 (see [2]). *Let A be a C^*-algebra, $\mathscr{U}$ and $\mathscr{V}$ two Hilbert A-modules, and $T \in \mathrm{End}_A^*(\mathscr{U},\mathscr{V})$. The following statements are equivalent:*

(1) *T is surjective;*

(2) *T^* is bounded below with respect to norm, that is, there is $m > 0$ such that $\|T^* f\| \geq m\|f\|$ for all $f \in \mathscr{V}$;*

(3) *T^* is bounded below with respect to inner product, that is, there is $m' > 0$ such that $\langle T^* f, T^* f\rangle \geq m'\langle f, f\rangle$ for all $f \in \mathscr{V}$.*

The authors also defined the continuous g-Riesz basis in Hilbert C^*-modules as follows.

Definition 2. A continuous g-frame $\{\Lambda_m \in \mathrm{End}_A^*(\mathscr{U},V_m) : m \in \mathscr{M}\}$ for Hilbert C^*-module $\mathscr{U}$ with respect to $\{V_m : m \in \mathscr{M}\}$ is said to be a continuous g-Riesz basis if it satisfies the following:

(1) $\Lambda_m \neq 0$ for any $m \in \mathscr{M}$;

(2) if $\int_{m\in\mathscr{K}} \Lambda_m^* g_m d\mu(m) = 0$, then $\Lambda_m^* g_m$ is equal to zero for each $m \in \mathscr{M}$, where $\{g_m\}_{m\in\mathscr{K}} \in \oplus_{m\in\mathscr{M}} V_m$ and $\mathscr{K}$ is a measurable subset of $\mathscr{M}$.

By using the synthesis operator T_Λ for a sequence $\{\Lambda_m \in \mathrm{End}_A^*(\mathscr{U},V_m) : m \in \mathscr{M}\}$ defined by

$$T_\Lambda(g) = \int_{m\in\mathscr{M}} \Lambda_m^* g_m d\mu(m), \quad \forall g = \{g_m\} \in \bigoplus_{m\in\mathscr{M}} V_m, \tag{1}$$

they gave a characterization of continuous g-Riesz basis [1, Theorem 4.6].

Theorem 3. *A family $\{\Lambda_m \in \mathrm{End}_A^*(\mathscr{U},V_m) : m \in \mathscr{M}\}$ is a continuous g-Riesz basis for $\mathscr{U}$ with respect to $\{V_m : m \in \mathscr{M}\}$ if and only if the synthesis operator T_Λ is a homeomorphism.*

We note, however, that in the proof of the above theorem, they said that "$\Lambda_m^* f_m = 0$ for any $m \in \mathscr{M}$, and $\Lambda_m \neq 0$, so $f_m = 0$", which is not true, because if Λ_m has a dense range, then Λ_m^* is one-to-one. We can improve their result by introducing the following modular continuous g-Riesz basis.

Definition 4. One calls a family $\{\Lambda_m \in \mathrm{End}_A^*(\mathscr{U},V_m) : m \in \mathscr{M}\}$ in Hilbert C^*-module $\mathscr{U}$ a modular continuous g-Riesz basis if

(1) $\{f \in \mathscr{U} : \Lambda_m f = 0, m \in \mathscr{M}\} = \{0\}$;

(2) there exist constants $A, B > 0$ such that for any $g = \{g_m\} \in \oplus_{m\in\mathscr{M}} V_m$,

$$A\|g\|^2 \leq \left\|\int_{m\in\mathscr{M}} \Lambda_m^* g_m d\mu(m)\right\|^2 \leq B\|g\|^2. \tag{$\star$}$$

Theorem 5. *A sequence $\{\Lambda_m \in \mathrm{End}_A^*(\mathscr{U},V_m) : m \in \mathscr{M}\}$ is a modular continuous g-Riesz basis for $\mathscr{U}$ with respect to*

$\{V_m : m \in \mathscr{M}\}$ if and only if the synthesis operator T_Λ is a homeomorphism.

Proof. Suppose first that $\{\Lambda_m \in \mathrm{End}_A^*(\mathscr{U}, V_m) : m \in \mathscr{M}\}$ is a modular continuous g-Riesz basis for $\mathscr{U}$ with synthesis operator T_Λ. Then $(\star)$ turns to be

$$A\|g\|^2 \leq \|T_\Lambda(g)\|^2 \leq B\|g\|^2, \quad \forall g = \{g_m\} \in \bigoplus_{m\in\mathscr{M}} V_m, \tag{2}$$

showing that T_Λ is bounded below with respect to norm. Hence, by Lemma 1, its adjoint operator T_Λ^* is surjective. Since the condition (1) in Definition 4 implies that T_Λ^* is injective, it follows that T_Λ^* is invertible and so T_Λ is invertible.

Conversely, let T_Λ be a homeomorphism. Then T_Λ is surjective, and again by Lemma 1, T_Λ^* is injective. So the condition (1) in Definition 4 holds. Now for any $g = \{g_m\} \in \oplus_{m\in\mathscr{M}} V_m$,

$$\begin{aligned} \|T_\Lambda^{-1}\|^{-2}\|g\|^2 &\leq \left\| \int_{m\in\mathscr{M}} \Lambda_m^* g_m d\mu(m) \right\|^2 \\ &= \|T_\Lambda(g)\|^2 \leq \|T_\Lambda\|^2 \|g\|^2. \end{aligned} \tag{3}$$

Therefore, $\{\Lambda_m \in \mathrm{End}_A^*(\mathscr{U}, V_m) : m \in \mathscr{M}\}$ is a modular continuous g-Riesz basis for $\mathscr{U}$ with respect to $\{V_m : m \in \mathscr{M}\}$. □

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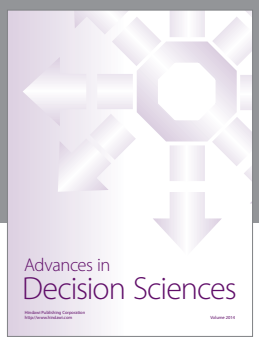

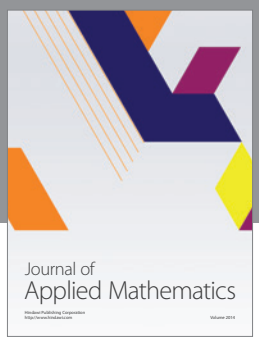

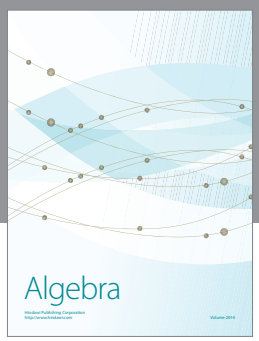

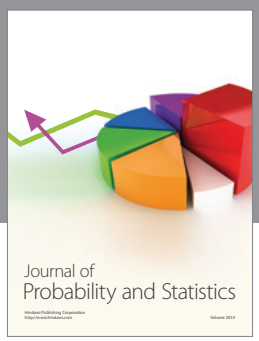

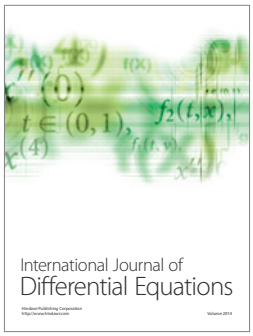

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