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Research Article

Lie Group Analysis and Similarity Solutions for Mixed Convection Boundary Layers in the Stagnation-Point Flow toward a Stretching Vertical Sheet

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An analysis for the mixed convection boundary layers in the stagnation-point flow toward a stretching vertical sheet is carried out via symmetry analysis. By employing Lie group method to the given system of nonlinear partial differential equations, we can obtain information about the invariants and symmetries of these equations. This information can be used to determine the similarity variables that will reduce the number of independent variables in the system. The transformed ordinary differential equations are solved numerically for some values of the parameters involved using fifth-order Improved Runge-Kutta Method (IRK5) coupled with shooting method. The features of the flow and heat transfer characteristics are analyzed and discussed in detail. Both cases of assisting and opposing flows are considered. This paper' results in comparison with known results are excellent.

1. Introduction

Over the course of the past several decades, there has been a considerable amount of effort to investigate the process of flow and heat transfer of a viscous and incompressible fluid over a continuously moving surface through a quiescent fluid. The proliferation of research on this particular phenomenon has been sparked by its vast array of pragmatic applications to a myriad of manufacturing processes. Examples of such processes include the extrusion of polymers, continuous casting, cooling of metallic plates, glass fiber production, hot rolling, paper production, wire drawing, aerodynamic extrusion of plastic sheets, crystal glowing, and others. The significance of studying heat transfer and flow field is that it is essential in determining the degree of quality of the end results of processes such as the ones explicated by Karwe and Jaluria [1]. Sakiadis [2] was the first to explore the flow induced by a semi-infinite horizontally moving wall in an ambient fluid. Crane [3] subsequently examined the flow over a linearly stretching sheet in an ambient fluid and came up with a solution which bore its likeness in closed analytical form for the steady two-dimensional problem. Numerous authors, such as Carragher and Crane [4], Elbashbeshy and Bazid [5], P. S. Gupta and A. S. Gupta [6], Magyari and Keller [7, 8], Magyari et al. [9], Liao and Pop [10], and Nazar et al. [11], looked into this problem by considering its various facts, such as uniform heat flux, permeability of the surface, flow, and heat transfer unsteadiness. Pop [12], Andersson [13], Takhar and Nath [14], and Nazar et al. [15] have directed their attention to other physical characteristics such as magnetic field, fluid viscoelasticity, suction, and three-dimensional flow. In contrast, the stretching vertical

plate has suffered a paucity of research. The problems dealt with by Chen [16, 17], Lin and Chen [18], Ali and Al-Yousef [19, 20], Ali [21, 22], and Abo-Eldahab [23] shall be addressed in this class. Of additional noteworthy interest is the unsteady boundary layer flow and heat transfer over a stretching vertical sheet, a phenomenon that has recently been treated by Ishak et al. [24] in a paper. As of late, Mahapatra and Gupta [25, 26] carried out research on the heat transfer in the steady two-dimensional stagnation point flow of a viscous and incompressible Newtonian and viscoelastic fluids over a horizontal stretching sheet considering the case of constant surface temperature.

2. Mathematical Formulation of the Heat Transfer in Steady Laminar Flow over a Moving Surface

Consider the steady, two-dimensional flow of a viscous and incompressible fluid near the stagnation point on a stretching vertical surface placed in the plane $y = 0$ of a Cartesian system of coordinates O_{xy} $(y = 0)$ with the x-axis along the sheet as shown in Figure 1. The fluid occupies the half plane $(y > 0)$. It is assumed that the velocity $u_w(x)$ and the temperature $T_w(x)$ of the stretching sheet is proportional to the distance x from the stagnation-point, where $T_w(x) > T_\infty$ with T_∞ being the uniform temperature of the ambient fluid. The velocity of the flow external to the boundary layer is $u_e(x)$. Under these assumptions along with the Boussinesq and boundary layer approximations, the system of equations, which model the boundary layer flow are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \pm g\beta\left(T - T_\infty\right), \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}, \tag{3}$$

where u and v are the velocity components along x- and y-axes, respectively, T is the fluid temperature, g is the gravity acceleration, α, ν, and β are the thermal diffusivity, kinematic viscosity, and thermal expansion coefficient, respectively, and the "+" and "−" signs in (2) correspond to assisting buoyant flow and to opposing buoyant flow, respectively. We shall assume that the boundary conditions of (1)–(3) are

$$\begin{gathered} v(x,0) = 0, \qquad u(x,0) = u_w(x) = cx, \\ T(x,0) = T_w(x) = T_\infty + bx, \\ u(x,\infty) = u_e(x) = ax, \qquad T(x,\infty) = T_\infty, \end{gathered} \tag{4}$$

where a, b, and c are positive constants. The continuity equation can be satisfied by introducing a stream function Ψ, such that

$$u = \frac{\partial \Psi}{\partial y}, \qquad v = -\frac{\partial \Psi}{\partial x}. \tag{5}$$

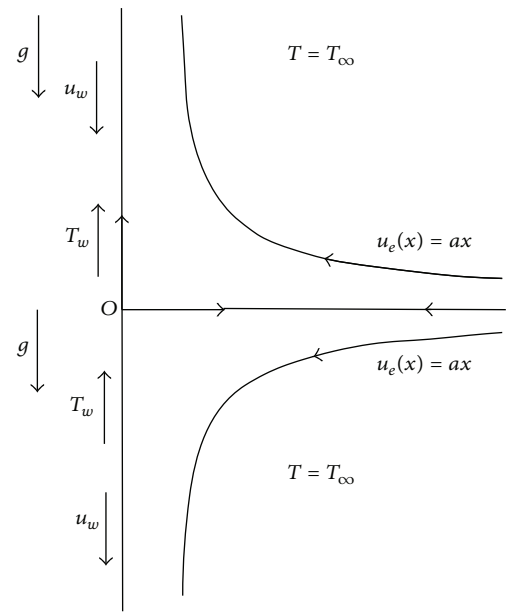

Figure 1: Physical model and coordinate system.

Therefore, from (1)–(3) with (5) we have

$$\begin{gathered} \Psi_y\Psi_{yx} - \Psi_x\Psi_{yy} + \nu\Psi_{yyy} \\ - u_e(x)\frac{d}{dx}u_e(x) \pm \beta\left(T - T_\infty\right) = 0, \\ \Psi_y T_x - \Psi_x T_y - \alpha T_{yy} = 0. \end{gathered} \tag{6}$$

The boundary conditions (4) will be as

$$\begin{gathered} \Psi_x(x,0) = 0, \qquad \Psi_y(x,0) = cx, \\ T(x,0) = T_w(x) = T_\infty + bx, \\ \Psi_y(x,\infty) = ax, \qquad T(x,\infty) = T_\infty. \end{gathered} \tag{7}$$

3. Main Results

3.1. Solution of the Problem by the Lie Point Symmetries. At first, we derive the similarity solutions using Lie group method under which (6) and the boundary conditions (7) are invariant, and then we use these symmetries to determine similarity variables. Now, consider the one-parameter ε, Lie group infinitesimal transformation in $(x, y; \Psi, u_e, T)$ given by

$$\begin{gathered} x^* = x + \varepsilon\xi\left(x, y; \Psi, u_e, T\right) + O\left(\varepsilon^2\right) = e^{\varepsilon X}x, \\ y^* = y + \varepsilon\gamma\left(x, y; \Psi, u_e, T\right) + O\left(\varepsilon^2\right) = e^{\varepsilon X}y, \\ \Psi^* = \Psi + \varepsilon\Phi\left(x, y; \Psi, u_e, T\right) + O\left(\varepsilon^2\right) = e^{\varepsilon X}\Psi, \\ u_e^* = u_e + \varepsilon U\left(x, y; \Psi, u_e, T\right) + O\left(\varepsilon^2\right) = e^{\varepsilon X}u_e, \\ T^* = T + \varepsilon\Upsilon\left(x, y; \Psi, u_e, T\right) + O\left(\varepsilon^2\right) = e^{\varepsilon X}T, \end{gathered} \tag{8}$$

here ε is the group parameter, and X is vector filed. A system of partial differential equations (6) is said to admit a symmetry generated by the vector filed as

$$X \equiv \xi \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial y} + \phi \frac{\partial}{\partial \Psi} + U \frac{\partial}{\partial u_e} + \Upsilon \frac{\partial}{\partial T}. \tag{9}$$

Equivalently, we can obtain $(x^*, y^*; \Psi^*, u_e^*, T^*)$ by solving

$$\begin{aligned} \frac{dx^*}{d\varepsilon} &= \xi(x, y; \Psi, u_e, T), \\ \frac{dy^*}{d\varepsilon} &= \gamma(x, y; \Psi, u_e, T), \\ \frac{d\Psi^*}{d\varepsilon} &= \Phi(x, y; \Psi, u_e, T), \\ \frac{du_e^*}{d\varepsilon} &= U(x, y; \Psi, u_e, T), \\ \frac{dT^*}{d\varepsilon} &= \Upsilon(x, y; \Psi, u_e, T), \end{aligned} \tag{10}$$

subject to initial conditions, $(x^*, y^*; \Psi^*, u_e^*, T^*)|_{\varepsilon=0} \equiv (x, y; \Psi, u_e, T)$. If X is left invariant by the transformation $(x, y; \Psi, u_e, T) \rightarrow (x^*, y^*; \Psi^*, u_e^*, T^*)$, then the solutions $\Psi = \Psi(x, y)$, $u_e = u_e(x)$, and $T = T(x, y)$ are invariant under the symmetry (9) if

$$\begin{aligned} \varphi_\Psi &= X(\Psi - \Psi(x, y))|_{\Psi=\Psi(x,y)} = 0, \\ \varphi_{u_e} &= X(u_e - u_e(x))|_{u_e=u_e(x,y)} = 0, \\ \varphi_T &= X(T - T(x, y))|_{T=T(x,y)} = 0. \end{aligned} \tag{11}$$

Assume

$$\begin{aligned} \Pi_1 &= \Psi_y \Psi_{yx} - \Psi_x \Psi_{yy} + \nu \Psi_{yyy} \\ &\quad - u_e (u_e)_x \pm \beta (T - T_\infty) = 0, \\ \Pi_2 &= \Psi_y T_x - \Psi_x T_y - \alpha T_{yy} = 0. \end{aligned} \tag{12}$$

The vector X as (9) is a Lie point symmetry vector filed for, (6) if

$$X^{[3]} (\Pi_j)|_{\Pi_j=0} = 0, \quad j = 1, 2, \tag{13}$$

where

$$\begin{aligned} X^{[3]} &\equiv \xi \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial y} + \Phi \frac{\partial}{\partial \Psi} + U \frac{\partial}{\partial u_e} + \Upsilon \frac{\partial}{\partial T} \\ &\quad + \Phi^x \frac{\partial}{\partial \Psi_x} + \Phi^y \frac{\partial}{\partial \Psi_y} + \Upsilon^x \frac{\partial}{\partial T_x} + \Upsilon^y \frac{\partial}{\partial T_y} + \Upsilon^{yy} \frac{\partial}{\partial T_{yy}} \\ &\quad + U^x \frac{\partial}{\partial (u_e)_x} + \Phi^{yx} \frac{\partial}{\partial \Psi_{yx}} + \Phi^{yy} \frac{\partial}{\partial \Psi_{yy}} + \Phi^{yyy} \frac{\partial}{\partial \Psi_{yyy}} \end{aligned} \tag{14}$$

is the third prolongation of X. The components Φ^x, Φ^y, Υ^x, Υ^y, U^x, Φ^{yx}, Φ^{yy}, and Φ^{yyy} can be determined from the following expressions

$$\begin{aligned} \Phi^x &= D_x \Phi - \Psi_x D_x \xi - \Psi_y D_x \gamma, \\ \Phi^y &= D_y \Phi - \Psi_x D_y \xi - \Psi_y D_y \gamma, \\ \Upsilon^x &= D_x \Upsilon - T_x D_x \xi - T_y D_x \gamma, \\ \Upsilon^y &= D_y \Upsilon - T_x D_y \xi - T_y D_y \gamma, \\ \Upsilon^{yy} &= D_y \Upsilon^y - T_{yx} D_y \xi - T_{yy} D_y \gamma, \\ U^x &= D_x U - (u_e)_x D_x \xi - (u_e)_y D_x \gamma, \\ \Phi^{xy} &= D_y \Phi^x - \Psi_{xx} D_y \xi - \Psi_{xy} D_y \gamma, \\ \Phi^{xy} &= D_x \Phi^y - \Psi_{yx} D_x \xi - \Psi_{yy} D_x \gamma, \\ \Phi^{yy} &= D_y \Phi^y - \Psi_{yx} D_y \xi - \Psi_{yy} D_y \gamma, \\ \Phi^{yyy} &= D_y \Phi^{yy} - \Psi_{yyx} D_y \xi - \Psi_{yyy} D_y \gamma. \end{aligned} \tag{15}$$

Here, D_x and D_y are introduced as the following total derivatives:

$$\begin{aligned} D_x &\equiv \partial_x + \Psi_x \partial_\Psi + (u_e)_x \partial_{u_e} + T_x \partial_T \\ &\quad + \Psi_{xx} \partial_{\Psi_x} + (u_e)_{xx} \partial_{(u_e)_x} + T_{xx} \partial_{T_x} \\ &\quad + \Psi_{xy} \partial_{\Psi_y} + T_{xy} \partial_{T_y} + \cdots, \\ D_y &\equiv \partial_y + \Psi_y \partial_\Psi + (u_e)_y \partial_{u_e} \\ &\quad + T_y \partial_T + \Psi_{yy} \partial_{\Psi_y} + (u_e)_{yy} \partial_{(u_e)_y} + T_{yy} \partial_{T_y} \\ &\quad + \Psi_{yx} \partial_{\Psi_x} + T_{yx} \partial_{T_x} + \cdots. \end{aligned} \tag{16}$$

Form (13) we have the system of linear differential equations as follows:

$$\begin{aligned} &- U(u_e)_x \pm \beta \Upsilon - \Phi^x \Psi_{yy} + \Phi^y \Psi_{yx} - U^x u_e \\ &\quad + \Phi^{yx} \Psi_y - \Phi^{yy} \Psi_x + \nu \Phi^{yyy} = 0, \\ &- \Phi^x T_y + \Phi^y T_x + \Upsilon^x \Psi_y - \Upsilon^y \Psi_x - \alpha \Upsilon^{yy} = 0. \end{aligned} \tag{17}$$

Replacing the functions ϕ^x, ϕ^y, ϕ^{yx}, ϕ^{yy}, ϕ^{yyy}, U^x, Υ^x, and Υ^y given by the relation (15) and eliminating any dependence between partial differential derivatives of the functions Ψ, u_e, and T, we obtain the new partial differential equations corresponding to (6) (see the Appendix).

Looking at this conditions as a polynomial in the partial derivatives of the functions Ψ, u_e, and T and identifying with

the polynom zero, we obtain the PDE system of ζ, γ, Φ, U, and Υ. The general solution of this PDE system is

$$\begin{gathered}\zeta = K_0 + K_1 x, \qquad \gamma = K_2,\\ \Phi = K_3 + K_4 x + K_5 y + \frac{K_1}{2}\Psi + K_6 u_e,\\ U = K_7 + K_8 x + K_9 y + K_{10}\Psi + K_{11} u_e + \frac{K_1}{2}T,\\ \Upsilon = K_{12} + K_{13} x + K_{14}\Psi + K_{15} u_e + K_{16} T,\end{gathered} \tag{18}$$

where $K_0, \ldots, K_{16} \in \mathbb{R}$, and consequently the infinitesimal generator of the symmetry group G is

$$\begin{aligned}X = {} & K_0\frac{\partial}{\partial x} + K_1\left(x\frac{\partial}{\partial x} + \frac{1}{2}\Psi\frac{\partial}{\partial \Psi} + \frac{1}{2}T\frac{\partial}{\partial u_e}\right) + K_2\frac{\partial}{\partial y},\\ & + K_3\frac{\partial}{\partial \Psi} + K_4 x\frac{\partial}{\partial \Psi} + K_5 y\frac{\partial}{\partial \Psi} + K_6 u_e\frac{\partial}{\partial \Psi} + K_7\frac{\partial}{\partial u_e},\\ & + K_8 x\frac{\partial}{\partial u_e} + K_9 y\frac{\partial}{\partial u_e} + K_{10}\Psi\frac{\partial}{\partial u_e} + K_{11} u_e\frac{\partial}{\partial u_e} + K_{12}\frac{\partial}{\partial T},\\ & + K_{13} x\frac{\partial}{\partial T} + K_{14}\Psi\frac{\partial}{\partial T} + K_{15} u_e\frac{\partial}{\partial T} + K_{16} T\frac{\partial}{\partial T}.\end{aligned} \tag{19}$$

From which, the system of nonlinear (6) has the sixteen-parameter Lie group of point symmetries generated by

$$\begin{gathered}X_1 \equiv \frac{\partial}{\partial x}, \qquad X_2 \equiv x\frac{\partial}{\partial x} + \frac{1}{2}\Psi\frac{\partial}{\partial \Psi} + \frac{1}{2}T\frac{\partial}{\partial u_e},\\ X_3 \equiv \frac{\partial}{\partial y}, \qquad X_4 \equiv \frac{\partial}{\partial \Psi}, \qquad X_5 \equiv x\frac{\partial}{\partial \Psi},\\ X_6 \equiv y\frac{\partial}{\partial \Psi}, \qquad X_7 \equiv u_e\frac{\partial}{\partial \Psi}, \qquad X_8 \equiv \frac{\partial}{\partial u_e},\\ X_9 \equiv x\frac{\partial}{\partial u_e}, \qquad X_{10} \equiv y\frac{\partial}{\partial u_e}, \qquad X_{11} \equiv \Psi\frac{\partial}{\partial u_e},\\ X_{12} \equiv u_e\frac{\partial}{\partial u_e}, \qquad X_{13} \equiv \frac{\partial}{\partial T}, \qquad X_{14} \equiv x\frac{\partial}{\partial T},\\ X_{15} \equiv \Psi\frac{\partial}{\partial T}, \qquad X_{16} \equiv u_e\frac{\partial}{\partial T}, \qquad X_{17} \equiv T\frac{\partial}{\partial T}.\end{gathered} \tag{20}$$

For X_1 up to X_{17}, respectively, the characteristic $\varphi = (\varphi_\Psi, \varphi_{u_e}, \varphi_T)$ has the components as follows:

$$\begin{gathered}\varphi_{X_1} = \left(-\Psi_x, -(u_e)_x, -T_x\right),\\ \varphi_{X_2} = \left(-x\Psi_x + \Psi, -x(u_e)_x + \frac{1}{2}T, -xT_x\right),\\ \varphi_{X_3} = \left(-\Psi_y, 0, -T_y\right), \qquad \varphi_{X_4} = (1,0,0),\\ \varphi_{X_5} = (x,0,0), \qquad \varphi_{X_6} = (y,0,0),\\ \varphi_{X_7} = (u_e,0,0), \qquad \varphi_{X_8} = (0,1,0),\\ \varphi_{X_9} = (0,x,0), \qquad \varphi_{X_{10}} = (0,y,0),\\ \varphi_{X_{11}} = (0,\Psi,0), \qquad \varphi_{X_{12}} = (0,u_e,0),\\ \varphi_{X_{13}} = (0,0,1), \qquad \varphi_{X_{14}} = (0,0,x),\\ \varphi_{X_{15}} = (0,0,\Psi), \qquad \varphi_{X_{16}} = (0,0,u_e), \qquad \varphi_{X_{17}} = (0,0,T).\end{gathered} \tag{21}$$

Equation (21) shows that no solutions are invariant under the groups generated by X_1 and X_3 up to X_{17}. For X_2, the characteristic $\varphi = (\varphi_\Psi, \varphi_{u_e}, \varphi_T)$ has the components

$$\varphi_\Psi = -x\Psi_x + \Psi, \qquad \varphi_{u_e} = -x(u_e)_x + \frac{1}{2}T, \qquad \varphi_T = -xT_x. \tag{22}$$

Therefore, the general solutions of the invariant surface conditions (11) by using the boundary conditions (7) are as follows:

$$\begin{gathered}\Psi(x,y) = cxF(y), \qquad u_e(x) = ax,\\ T(x,y) = (T_w - T_\infty)H(y) + T_\infty.\end{gathered} \tag{23}$$

Substitution from (23) into (6) yields

$$\begin{gathered}cx\left[c\left(F'(y)\right)^2 - cF(y)F''(y) - \nu F'''(y)\right]\\ - a^2 x \pm g\beta(T_w - T_\infty)H(y) = 0,\\ cF(y)H'(y) + \alpha H''(y) = 0.\end{gathered} \tag{24}$$

For simplifying we can use $F(y) = \sqrt{\nu/c}\,f(\eta)$ and $H(y) = \theta(\eta)$ with $\eta = \sqrt{c/\nu}\,y$. Therefore, we have

$$f'''(\eta) + f(\eta)f''(\eta) - \left(f'(\eta)\right)^2 + \left(\frac{a}{c}\right)^2 \pm \lambda\theta(\eta) = 0, \tag{25}$$

$$\frac{1}{\mathrm{Pr}}\theta''(\eta) + f(\eta)\theta'(\eta) = 0, \tag{26}$$

where $\mathrm{Pr} = \nu/c$ is the Prandtl number and $\lambda = G_{r_x}/\mathrm{Re}_x^2$ is the buoyancy parameter with $G_{r_x} = g\beta(T_w - T_\infty)x^3/\nu^2$ is the local Grashof number and $\mathrm{Re}_x^2 = u_w x/\nu$ is the local Reynolds number. Equations (25) and (26) subject to the boundary (7) become

$$\begin{gathered}f(0) = 0, \qquad f'(0) = 1, \qquad \theta(0) = 1,\\ f'(\infty) = \frac{a}{c}, \qquad \theta(\infty) = 0.\end{gathered} \tag{27}$$

When $\lambda = 0$ and $a/c = 1$, the solution of (25) subject to

Table 1: Values of $f''(0)$ for different values of a/c when the buoyancy force term $\lambda\theta$ in (25) is absent.

a/c	Mahapatra and Gupta [25]	Nazar et al. [11]	Ishak et al. [28]	Present work
0				−1.0000
0.1	−0.9694	−0.9694	−0.9694	−0.9694
0.2	−0.9181	−0.9181	−0.9181	−0.9181
0.5	−0.6673	−0.6673	−0.6673	−0.6673
2	2.0175	2.0176	2.0175	2.0175
3	4.7293	4.7296	4.7294	4.7293
4	—	—	—	8.0004
10	—	—	—	36.2574

Table 2: Values of $f''(0)$ and $-\theta'(0)$ for $a/c = 1$, $\lambda = 1$ and various Pr.

Pr	Buoyancy assisting flow		Buoyancy opposing flow	
	$f''(0)$	$-\theta'(0)$	$f''(0)$	$-\theta'(0)$
0	0.604	0.200	−0.663	0.200
0.72	0.419	0.702	−0.448	0.647
6.8	0.219	2.101	−0.224	2.059
20	0.145	3.584	−0.146	3.552
40	0.109	5.058	−0.109	5.034
60	0.091	6.191	−0.091	6.169
80	0.080	7.146	−0.080	7.127
100	0.072	7.987	−0.073	7.970
200	0.053	11.290	−0.053	11.277
400	0.038	15.962	−0.038	15.953
1000	0.024	25.234	−0.024	25.228

boundary condition (27) is given by

$$f(\eta) = \eta. \tag{28}$$

3.2. Numerical Results. Equations (25) and (26) subject to boundary conditions (27) have been solved numerically using the shooting method coupled with fifth-order Improved Runge-Kutta Method (IRK5) [27]. For the validation of the Lie group method used in this study, the case when the buoyancy term $\lambda\theta$ in (25) is absent has been also considered and compared with the results reported by Mahapatra and Gupta [25, 26], Nazar et al. [11], and Ishak et al. [28]. This comparison is shown in Table 1. It is seen that the present values of $f''(0)$ are in very good agreement with those obtained by Mahapatra and Gupta [25, 26], Nazar et al. [11], and Ishak et al. [28]. Therefore, it can be concluded that the present Lie group method can be used with great confidence to study the problem discussed in this paper.

The values of the skin friction coefficient and $-\theta'(0)$ for various Pr when $a/c = 1$ and $\lambda = 1$ are tabulated in Table 2, for both cases of assisting and opposing flows. The values of

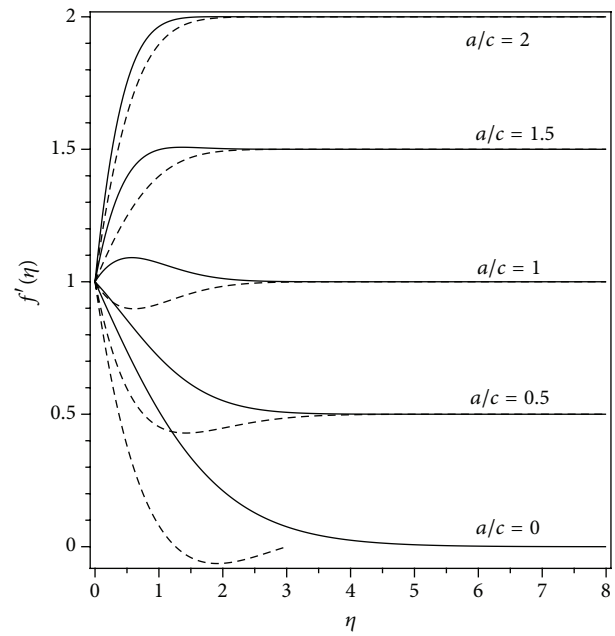

Figure 2: Velocity profiles for some values of a/c when Pr = 1 and $\lambda = 1$.

$-\theta'(0)$ are positive in all cases discussed in this study. Also, the effects of λ on the skin friction coefficient are found to be more significant for fluids having smaller Pr, since the viscosity is less than the fluids with larger Pr.

The resulting profiles of dimensionless velocity $f'(\eta)$ and dimensionless temperature $\theta(\eta)$ are shown in Figures 2 and 3 for various values of a/c, λ, and Pr. From Figure 2, it is seen that for assisting flow, the velocity increases at the beginning until it achieves a certain value then decreases until the value becomes constant, that is unity at the outside of the boundary layer. From Figure 2, it can be seen that when $a/c > 1$, the flow has a boundary layer structure, and the thickness of the boundary layer decreases with increase in a/c. According to Mahapatra and Gupta [25, 26], it can be explained as follows: for a fixed value of c corresponding to the stretching of the surface, an increase in a in relation to c implies an increase in straining motion near the stagnation region resulting in increased acceleration of the external stream, and this leads to thinning of the boundary layer with increase in a/c.

The opposite trend occurs for opposing flow. From Figure 3, it is observed that the temperature of the fluid decreases, as the distance from the surface increases, for both cases of assisting and opposing flow, for all values of a/c, λ, and Pr until it achieves a constant value, namely, zero. This is not surprising, since the fluid receives the heat from the surface, and then the heat energy is changed into other energy forms such as kinetic energy.

The skin friction coefficient and $-\theta'(0)$ are shown in Figures 4, 5, 6, and 7. Figures 4 and 6 suggest that an assisting buoyancy flow produces an increase in the skin friction coefficient, while an opposing buoyant flow gives rise to a decrease in the skin friction coefficient. This is because, the fluid velocity increases when the buoyancy force increases

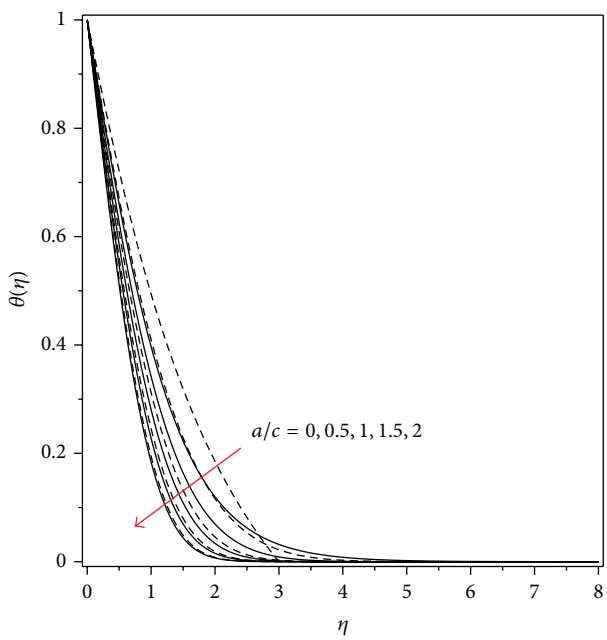

Figure 3: Temperature profiles for some values of a/c when Pr = 1 and $\lambda = 1$; solid line: assisting flow and dash line: opposing flow.

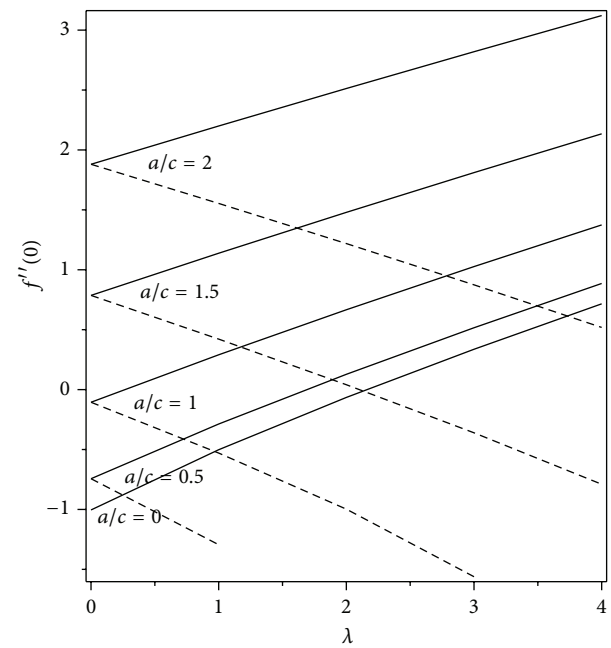

Figure 4: Variation with λ of the skin friction coefficient for some values of a/c when Pr = 1; solid line: assisting flow and dash line: opposing flow.

and hence increases the wall shear stress, which increases the skin friction coefficient. Figure 4 shows that all curves intersect at a point where $\lambda = 0$; that is, when the buoyancy force is zero. This is because (25) and (26) are uncoupled when $\lambda = 0$; in other words, the solutions to the flow field are not affected by the thermal field in which the buoyancy force is lacking. Also in this case, the value of $f''(0) = 0$ remains constant, namely, zero. This value agreed with the exact solution (25), which implies $f''(\eta) = 0$, for all η. Moreover, for assisting flow, it can be seen that $f''(0)$ decreases when Pr increases for a fixed value of λ. This is because when Pr increases, the viscosity increases and slows down the flow hence reduces the surface shear stress and thus reduces the skin friction coefficient $f''(0)$. The opposite trends can be observed for opposing flow. In addition, from Figure 7 the effects of Pr can be examined; that is, increasing Pr enhances the rate of heat transfer, since increasing of Pr will cause the increasing of viscosity then reduces the thermal conductivity, and thus $-\theta'(0)$ increases.

The resulting profiles of dimensionless velocity $f'(\eta)$ and dimensionless temperature $\theta(\eta)$ are shown in Figures 8 and 9 for various values of λ. Figure 8 shows that the velocity profiles increases and decreases for assisting flow and opposing flow, respectively, when λ increases. In Figure 9, it is observed that, for a particular value of Pr, the temperature profiles is slightly increased, as the buoyancy parameter λ is increased, for the case of assisting flow. The opposite trend occurs for opposing flow. This is clear from the fact that assisting buoyant flow produces a favorable pressure gradient that enhances the momentum transport, which in turn increases the surface heat transfer rate.

The values of $f''(0)$ and $-\theta'(0)$ are shown in Table 3 for $a/c = 1$, Pr = 1, and various λ. Table 3 shows

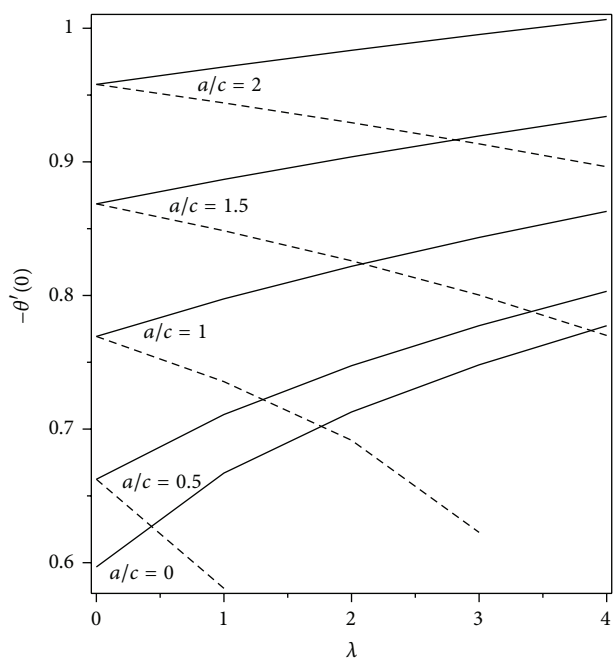

Figure 5: Variation with λ of $-\theta'(0)$ for some values of a/c when Pr = 1; solid line: assisting flow and dash line: opposing flow.

that the functions $f''(0)$ and $-\theta'(0)$ increases and decreases for assisting flow and opposing flow, respectively, when λ increases. The values of $f''(0)$ and $-\theta'(0)$ are shown in Table 4 for $\lambda = 1$, Pr = 1, and various a/c. Table 4 shows that the functions $f''(0)$ and $-\theta'(0)$ increase for both assisting flow and opposing flow when a/c increases.

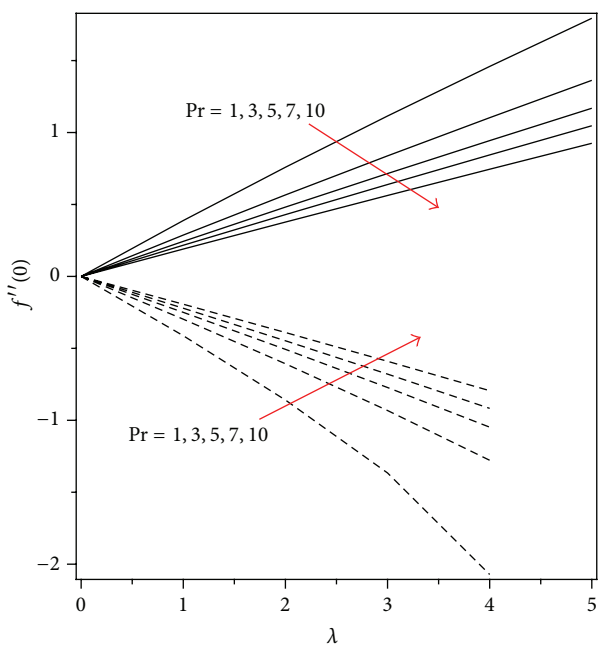

Figure 6: Variation with λ of the skin friction coefficient for some values of Pr when $a/c = 1$; solid line: assisting flow and dash line: opposing flow.

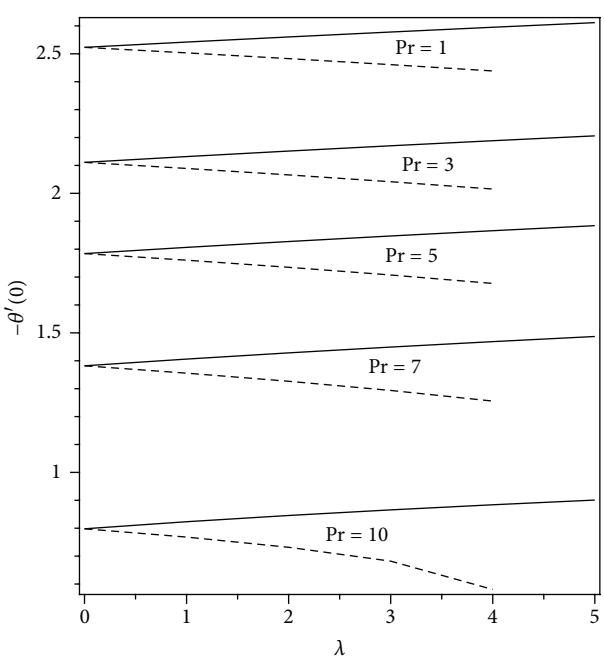

Figure 7: Variation with λ of $-\theta'(0)$ for some values of Pr when $a/c = 1$; solid line: assisting flow and dash line: opposing flow.

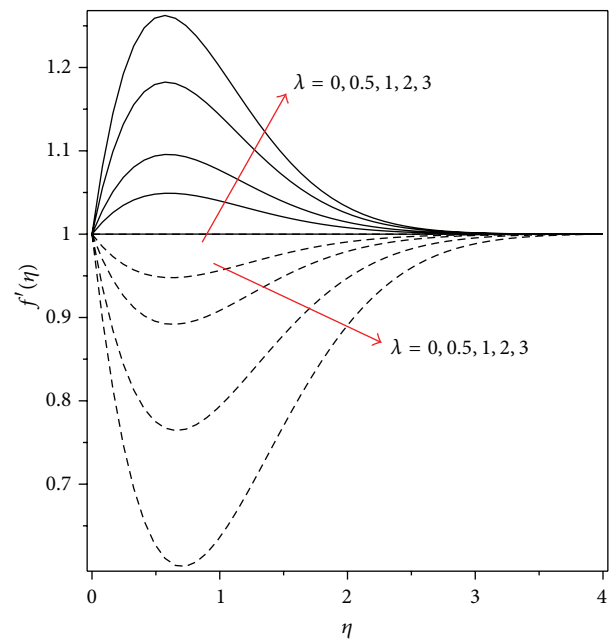

Figure 8: Velocity profiles for some values of λ when Pr $= 1$ and $a/c = 1$.

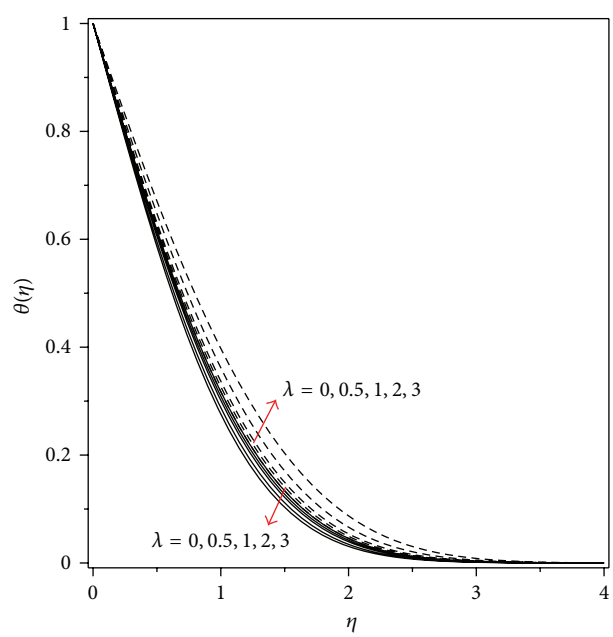

Figure 9: Temperature profiles for some values of λ when Pr $= 1$ and $a/c = 1$.

4. Conclusions

Lie group method is applicable to both linear and nonlinear partial differential equations, which leads to similarity variables that used to reduce the number of independent variables in partial differential equations. By determining the transformation group under which the given partial differential equations are invariant, we can obtain information about the invariants and symmetries of these equations. This information can be used to determine the similarity variables that will reduce the number of independent variables in the system. In this work, we have used Lie group method to obtain similarity reductions of nonlinear boundary layer equations (1)–(3), for the two-dimensional boundary layer equations of the liquid flow for the mixed convection boundary layers in

Table 3: Values of $f''(0)$ and $-\theta'(0)$ for $a/c = 1$, Pr = 1, and various λ.

λ	Buoyancy assisting flow		Buoyancy opposing flow	
	$f''(0)$	$-\theta'(0)$	$f''(0)$	$-\theta'(0)$
0	10.6673	1.2346	2.6696	2.1381
2	16.7304	1.3282	0.65818	2.1298
4	22.1328	1.3987	−1.4727	2.1207
6	27.1134	1.4562	−3.7690	2.1104
8	31.7902	1.5051	−6.3244	2.0982
10	36.2321	1.5478	−9.4295	2.0816

Table 4: Values of $f''(0)$ and $-\theta'(0)$ for $\lambda = 1$, Pr = 1, and various a/c.

a/c	Buoyancy assisting flow		Buoyancy opposing flow	
	$f''(0)$	$-\theta'(0)$	$f''(0)$	$-\theta'(0)$
0.0	6.8627	1.2032	1.0149	0.9356
0.5	8.3387	1.2788	3.3862	1.1005
1.0	11.379	1.3952	7.5631	1.2931
1.5	15.567	1.5207	12.692	1.4596
2.0	20.578	1.6420	18.479	1.6032

the stagnation-point flow toward a stretching vertical sheet. By determining the transformation group under which the given partial differential equations are invariant, we obtained the invariants and the symmetries of these equations. In turn, we used these invariants and symmetries to determine the similarity variables that reduced the number of independent variables. Therefore, the governing partial differential equations (1)–(3) are reduced to a set of two nonlinear ordinary differential equations (25) and (26). The resulting system of nonlinear ordinary differential equations (25) and (26) subjected to the boundary conditions (27) is solved numerically using the shooting method coupled with fifth-order Improved Runge-Kutta Method (IRK5). Effects of the parameters λ, Pr, and a/c of the fluid on the flow and heat transfer characteristics have been examined and discussed in detail. Our results are in complete agreement with those reported by Ishak et al. [28]. Therefore, it can be concluded that the Lie group method can be used with great confidence to study the problem discussed in this paper.

Appendix

$$
\begin{aligned}
&-U(u_e)_x \pm \beta\Upsilon \\
&\quad - \Psi_{yy}\big[\Phi_x + \Psi_x\Phi_\Psi + (u_e)_x\Phi_{u_e} + T_x\Phi_T - \zeta_x\Psi_x - \zeta_\Psi\Psi_x\Psi_x \\
&\qquad - \zeta_{u_e}(u_e)_x\Psi_x - \zeta_T T_x\Psi_x - \gamma_x\Psi_y - \gamma_\Psi\Psi_x\Psi_y \\
&\qquad -\gamma_{u_e}(u_e)_x\Psi_x - \gamma_T T_x\Psi_y\big] \\
&+ \Psi_{yx}\big[\Phi_y + \Psi_y\Phi_\Psi + (u_e)_y\Phi_{u_e} + T_y\Phi_T - \zeta_y\Psi_x - \zeta_\Psi\Psi_x\Psi_y \\
&\qquad - \zeta_{u_e}(u_e)_y\Psi_x - \zeta_T T_y\Psi_x - \gamma_y\Psi_y \\
&\qquad -\gamma_\Psi\Psi_y\Psi_y - \gamma_{u_e}(u_e)_y\Psi_y - \gamma_T T_y\Psi_y\big] \\
&- u_e\big[U_x + \Psi_x U_\Psi + (u_e)_x U_{u_e} + T_x U_T \\
&\qquad - \zeta_x(u_e)_x - \zeta_\Psi\Psi_x(u_e)_x - \zeta_{u_e}(u_e)_x(u_e)_x - \zeta_T T_x(u_e)_x \\
&\qquad -\gamma_x(u_e)_y - \gamma_\Psi(u_e)_y\Psi_x - \gamma_{u_e}(u_e)_x(u_e)_y - \gamma_T T_x(u_e)_y\big] \\
&+ \Psi_y\big[\Phi_{yx} + \Phi_\Psi\Psi_{yx} + \Phi_{\Psi x}\Psi_y + \Phi_{u_e}(u_e)_{yx} \\
&\qquad + \Phi_{u_e x}(u_e)_y + \Phi_T T_{yx} + \Phi_{Tx}T_y - \zeta_{yx}\Psi_x - \zeta_y\Psi_{xx} \\
&\qquad - \zeta_{\Psi x}\Psi_x\Psi_y - \zeta_{\Psi x}\Psi_x\Psi_y - \zeta_\Psi\Psi_{xx}\Psi_y - \zeta_\Psi\Psi_x\Psi_{yx} \\
&\qquad - \zeta_{u_e x}(u_e)_y\Psi_x - \zeta_{u_e}(u_e)_{yx}\Psi_x - \zeta_{u_e}(u_e)_y\Psi_{xx} \\
&\qquad - \zeta_{Tx}T_y\Psi_x - \zeta_T T_{yx}\Psi_x - \zeta_T T_y\Psi_{xx} - \gamma_{yx}\Psi_y - \gamma_y\Psi_{yx} \\
&\qquad - \gamma_{\Psi x}\Psi_y\Psi_y - \gamma_\Psi\Psi_{yx}\Psi_y - \gamma_\Psi\Psi_y\Psi_{yx} - \gamma_{u_e x}(u_e)_y\Psi_y \\
&\qquad - \gamma_{u_e}(u_e)_{yx}\Psi_y - \gamma_{u_e}(u_e)_y\Psi_{yx} - \gamma_{yx}T_y\Psi_y - \gamma_y T_{yx}\Psi_y \\
&\qquad - \gamma_y T_y\Psi_{yx} - \zeta_x\Psi_{yx} - \zeta_\Psi\Psi_x\Psi_y - \zeta_{u_e}(u_e)_x + \zeta_x T_x \\
&\qquad -\gamma_x\Psi_{yy} - \gamma_\Psi\Psi_x\Psi_{yy} - \gamma_{u_e}(u_e)_x\Psi_{yy} - \gamma_T T_x\Psi_{yy}\big] \\
&- \Psi_x\big[\Phi_{yx} + \Phi_{y\Psi}\Psi_x + \Phi_{yu_e}(u_e)_x + \Phi_{yT}T_x + \Phi_T T_{yx} \\
&\qquad + \Phi_{Tx}T_y + \Phi_{T\Psi}T_y\Psi_x + \Phi_{Tu_e}(u_e)_x T_y - \zeta_{yx}\Psi_x \\
&\qquad - \zeta_y\Psi_{xx} - \zeta_{y\Psi}\Psi_x\Psi_x - \zeta_{yu_e}(u_e)_x\Psi_x - \zeta_{yT}T_x\Psi_x \\
&\qquad - \zeta_{\Psi x}\Psi_x\Psi_y - \zeta_\Psi\Psi_{xx}\Psi_y - \zeta_\Psi\Psi_x\Psi_{yx} - \zeta_{\Psi\Psi}\Psi_x\Psi_x\Psi_y \\
&\qquad - \zeta_{\Psi u_e}(u_e)_x\Psi_x\Psi_y - \zeta_{\Psi T}T_x\Psi_x\Psi_y - \gamma_{yx}\Psi_y - \gamma_{yx}\Psi_y \\
&\qquad - \gamma_y\Psi_{yx} - \gamma_{y\Psi}\Psi_x\Psi_y - \gamma_{yu_e}(u_e)_x\Psi_y - \gamma_{yT}T_x\Psi_y \\
&\qquad - \gamma_{\Psi x}\Psi_y\Psi_y - \gamma_\Psi\Psi_{yx}\Psi_y - \gamma_\Psi\Psi_y\Psi_{yx} - \gamma_{\Psi\Psi}\Psi_x\Psi_y\Psi_y \\
&\qquad - \gamma_{\Psi u_e}(u_e)_x\Psi_y\Psi_y - \gamma_{\Psi T}T_x\Psi_y\Psi_y - \gamma_{Tx}T_y\Psi_y \\
&\qquad - \gamma_T T_{yx}\Psi_y - \gamma_T T_y\Psi_{yx} - \gamma_{T\Psi}\Psi_x T_y\Psi_y \\
&\qquad -\gamma_{Tu_e}(u_e)_x T_y\Psi_y - \gamma_{TT}T_x T_y T_y\big] \\
&+ \nu\big[\Phi_{yxy} + \Phi_{y\Psi}\Psi_{xy} + \Phi_{y\Psi y}\Psi_x + \Phi_{yu_e}(u_e)_{xy} \\
&\qquad + \Phi_{yu_e y}(u_e)_x + \Phi_{yT}T_{xy} + \Phi_{yTy}T_x + \Phi_T T_{yxy} \\
&\qquad + \Phi_{Ty}T_{yx} + \Phi_{Tx}T_{yy} + \Phi_{Txy}T_y + \Phi_{T\Psi y}T_y\Psi_x \\
&\qquad + \Phi_{T\Psi}T_{yy}\Psi_x + \Phi_{T\Psi}T_y\Psi_{xy} + \Phi_{Tu_e y}(u_e)_x T_y \\
&\qquad + \Phi_{Tu_e}(u_e)_{xy}T_y + \Phi_{Tu_e}(u_e)_x T_{yy} - \zeta_{yxy}\Psi_x \\
&\qquad - \zeta_{yx}\Psi_{xy} - \zeta_{yy}\Psi_{xx} - \zeta_y\Psi_{xxy} - \zeta_{y\Psi y}\Psi_x\Psi_x \\
&\qquad - \zeta_{y\Psi}\Psi_{xy}\Psi_x - \zeta_{y\Psi}\Psi_x\Psi_{xy} - \zeta_{yu_e y}(u_e)_x\Psi_x \\
&\qquad - \zeta_{yu_e}(u_e)_{xy}\Psi_x - \zeta_{yu_e}(u_e)_x\Psi_{xy} - \zeta_{yTy}T_x\Psi_x \\
&\qquad - \zeta_{yT}T_{xy}\Psi_x - \zeta_{yT}T_x\Psi_{xy} - \zeta_{\Psi xy}\Psi_x\Psi_y \\
&\qquad - \zeta_{\Psi x}\Psi_{xy}\Psi_y - \zeta_{\Psi x}\Psi_x\Psi_{yy} - \zeta_{\Psi xy}\Psi_{xx}\Psi_y
\end{aligned}
$$

$$
\begin{aligned}
&- \zeta_{\Psi}\Psi_{xxy}\Psi_{y} - \zeta_{\Psi}\Psi_{xx}\Psi_{yy} - \zeta_{\Psi y}\Psi_{x}\Psi_{yx} \\
&- \zeta_{\Psi}\Psi_{xy}\Psi_{yx} - \zeta_{\Psi}\Psi_{x}\Psi_{yxy} - \zeta_{\Psi\Psi y}\Psi_{x}\Psi_{x}\Psi_{y} \\
&- \zeta_{\Psi\Psi}\Psi_{xy}\Psi_{x}\Psi_{y} - \zeta_{\Psi\Psi}\Psi_{x}\Psi_{xy}\Psi_{y} - \zeta_{\Psi\Psi}\Psi_{x}\Psi_{x}\Psi_{yy} \\
&- \zeta_{\Psi u_e y}(u_e)_{x}\Psi_{x}\Psi_{y} - \zeta_{\Psi u_e}(u_e)_{xy}\Psi_{x}\Psi_{y} \\
&- \zeta_{\Psi u_e}(u_e)_{x}\Psi_{xy}\Psi_{y} - \zeta_{\Psi u_e}(u_e)_{x}\Psi_{x}\Psi_{yy} \\
&- \zeta_{\Psi T y}T_{x}\Psi_{x}\Psi_{y} - \zeta_{\Psi T}T_{xy}\Psi_{x}\Psi_{y} - \zeta_{\Psi T}T_{x}\Psi_{xy}\Psi_{y} \\
&- \zeta_{\Psi T}T_{x}\Psi_{x}\Psi_{yy} - \zeta_{Txy}T_{y}\Psi_{x} - \zeta_{Tx}T_{yy}\Psi_{x} \\
&- \zeta_{Tx}T_{y}\Psi_{xy} - \zeta_{Ty}T_{yx}\Psi_{x} - \zeta_{T}T_{yxy}\Psi_{x} - \zeta_{T}T_{yx}\Psi_{xy} \\
&- \zeta_{Ty}T_{y}\Psi_{xx} - \zeta_{T}T_{yy}\Psi_{xx} - \zeta_{T}T_{y}\Psi_{xxy} \\
&- \zeta_{T\Psi y}T_{y}\Psi_{xx} - \zeta_{T\Psi}T_{yy}\Psi_{xx} - \zeta_{T\Psi}T_{y}\Psi_{xxy} \\
&- \zeta_{Tu_e y}(u_e)_{x}T_{y}\Psi_{x} - \zeta_{Tu_e}(u_e)_{xy}T_{y}\Psi_{x} \\
&- \zeta_{Tu_e}(u_e)_{x}T_{yy}\Psi_{x} - \zeta_{Tu_e}(u_e)_{x}T_{y}\Psi_{xy} \\
&- \zeta_{Tu_e y}T_{x}T_{y}\Psi_{x} - \zeta_{Tu_e}T_{xy}T_{y}\Psi_{x} - \zeta_{Tu_e y}T_{x}T_{yy}\Psi_{x} \\
&- \zeta_{Tu_e y}T_{x}T_{y}\Psi_{xy} - \gamma_{yxy}\Psi_{y} - \gamma_{yx}\Psi_{yy} - \gamma_{yy}\Psi_{yx} \\
&- \gamma_{y}\Psi_{yxy} - \gamma_{y\Psi y}\Psi_{x}\Psi_{y} - \gamma_{y\Psi}\Psi_{xy}\Psi_{y} - \gamma_{y\Psi}\Psi_{x}\Psi_{yy} \\
&- \gamma_{\Psi xy}\Psi_{y}\Psi_{y} - \gamma_{\Psi x}\Psi_{yy}\Psi_{y} - \gamma_{\Psi x}\Psi_{y}\Psi_{yy} - \gamma_{\Psi y}\Psi_{yx}\Psi_{y} \\
&- \gamma_{\Psi}\Psi_{yxy}\Psi_{y} - \gamma_{\Psi y}\Psi_{y}\Psi_{yx} - \gamma_{\Psi}\Psi_{yy}\Psi_{yx} \\
&- \gamma_{\Psi}\Psi_{y}\Psi_{yxy} - \gamma_{\Psi\Psi y}\Psi_{x}\Psi_{y}\Psi_{y} - \gamma_{\Psi\Psi}\Psi_{xy}\Psi_{y}\Psi_{y} \\
&- \gamma_{\Psi\Psi}\Psi_{x}\Psi_{yy}\Psi_{y} - \gamma_{\Psi\Psi}\Psi_{x}\Psi_{y}\Psi_{yy} - \gamma_{\Psi u_e y}(u_e)_{x}\Psi_{y}\Psi_{y} \\
&- \gamma_{\Psi u_e}(u_e)_{xy}\Psi_{y}\Psi_{y} - \gamma_{\Psi u_e}(u_e)_{x}\Psi_{yy}\Psi_{y} \\
&- \gamma_{\Psi u_e}(u_e)_{x}\Psi_{y}\Psi_{yy} - \gamma_{\Psi T y}T_{x}\Psi_{y}\Psi_{y} - \gamma_{\Psi T}T_{xy}\Psi_{y}\Psi_{y} \\
&- \gamma_{\Psi T}T_{x}\Psi_{yy}\Psi_{y} - \gamma_{\Psi T}T_{x}\Psi_{y}\Psi_{yy} - \gamma_{Txy}T_{y}\Psi_{y} \\
&- \gamma_{Tx}T_{yy}\Psi_{y} - \gamma_{Tx}T_{y}\Psi_{yy} - \gamma_{Ty}T_{yx}\Psi_{y} - \gamma_{T}T_{yxy}\Psi_{y} \\
&- \gamma_{T}T_{yx}\Psi_{yy} - \gamma_{Ty}T_{y}\Psi_{yx} - \gamma_{T}T_{yy}\Psi_{yx} \\
&- \gamma_{T}T_{y}\Psi_{yxy} - \gamma_{T\Psi y}\Psi_{x}T_{y}\Psi_{y} - \gamma_{T\Psi}\Psi_{xy}T_{y}\Psi_{y} \\
&- \gamma_{T\Psi}\Psi_{x}T_{yy}\Psi_{y} - \gamma_{T\Psi}\Psi_{x}T_{y}\Psi_{yy} - \gamma_{Tu_e y}(u_e)_{x}T_{y}\Psi_{y} \\
&- \gamma_{Tu_e}(u_e)_{xy}T_{y}\Psi_{y} - \gamma_{Tu_e}(u_e)_{x}T_{yy}\Psi_{y} \\
&- \gamma_{Tu_e}(u_e)_{x}T_{yy}\Psi_{y} - \gamma_{TTy}T_{x}T_{y}T_{y} - \gamma_{TT}T_{xy}T_{y}T_{y} \\
&- \gamma_{TT}T_{x}T_{yy}T_{y} - \gamma_{TT}T_{x}T_{y}T_{yy} - \zeta_{y}\Psi_{yyx} - \zeta_{\Psi}\Psi_{y}\Psi_{yyx} \\
&- \zeta_{u_e}(u_e)_{y}\Psi_{yyx} - \zeta_{T}T_{y}\Psi_{yyx} - \gamma_{y}\Psi_{yyy} \\
&- \gamma_{\Psi}\Psi_{y}\Psi_{yyy} - \gamma_{u_e}(u_e)_{y}\Psi_{yyy} - \gamma_{T}T_{y}\Psi_{yyy}\Big] = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\Upsilon_{x}\Psi_{y} + \Upsilon_{\Psi}\Psi_{x}\Psi_{y} + \Upsilon_{u_e}(u_e)_{x}\Psi_{y} \\
&\quad + \Upsilon_{T}T_{x}\Psi_{y} - \zeta_{x}T_{x}\Psi_{y} - \zeta_{\Psi}T_{x}\Psi_{x}\Psi_{y} \\
&\quad - \zeta_{u_e}(u_e)_{x}\Psi_{x}\Psi_{y} - \zeta_{T}T_{x}T_{x}\Psi_{y} - \gamma_{x}T_{y}\Psi_{y} \\
&\quad - \gamma_{\Psi}T_{y}\Psi_{x}\Psi_{y} - \gamma_{u_e}(u_e)_{x}T_{y}\Psi_{y} \\
&\quad - \gamma_{T}T_{x}T_{y}\Psi_{y} - \Upsilon_{y}\Psi_{x} - \Upsilon_{\Psi}\Psi_{y}\Psi_{x} \\
&\quad - \Upsilon_{u_e}(u_e)_{y}\Psi_{x} - \Upsilon_{T}T_{y}\Psi_{x} + \zeta_{y}T_{x}\Psi_{x} \\
&\quad + \zeta_{\Psi}T_{x}\Psi_{y}\Psi_{x} + \zeta_{u_e}(u_e)_{y}T_{x}\Psi_{x} \\
&\quad + \zeta_{T}T_{y}T_{x}\Psi_{x} + \gamma_{y}T_{y}\Psi_{x} + \gamma_{\Psi}T_{y}\Psi_{y}\Psi_{x} \\
&\quad + \zeta_{u_e}(u_e)_{y}T_{y}\Psi_{x} + \zeta_{T}T_{y}T_{y}\Psi_{x} - \Phi_{x}T_{y} \\
&\quad - \Phi_{\Psi}T_{y}\Psi_{x} - \Phi_{u_e}T_{y}(u_e)_{x} - \Phi_{T}T_{y}T_{x} \\
&\quad + \zeta_{x}\Psi_{x}T_{y} + \zeta_{\Psi}\Psi_{x}\Psi_{x}T_{y} + \zeta_{u_e}(u_e)_{x}\Psi_{x}T_{y} \\
&\quad + \zeta_{T}T_{x}\Psi_{x}T_{y} + \gamma_{x}\Psi_{y}T_{y} + \gamma_{\Psi}\Psi_{x}\Psi_{y}T_{y} \\
&\quad + \gamma_{u_e}(u_e)_{x}\Psi_{x}T_{y} + \gamma_{T}T_{x}\Psi_{y}T_{y} + \Phi_{y}T_{x} \\
&\quad + \Phi_{\Psi}T_{x}\Psi_{y} + \Phi_{u_e}T_{x}(u_e)_{y} + \Phi_{T}T_{x}T_{y} \\
&\quad - \zeta_{y}\Psi_{x}T_{x} - \zeta_{\Psi}\Psi_{x}\Psi_{y}T_{x} - \zeta_{u_e}(u_e)_{y}\Psi_{x}T_{x} \\
&\quad - \zeta_{T}T_{y}\Psi_{x}T_{x} - \gamma_{y}\Psi_{y}T_{x} - \gamma_{\Psi}\Psi_{y}\Psi_{y}T_{x} \\
&\quad - \gamma_{u_e}(u_e)_{y}\Psi_{y}T_{x} - \gamma_{T}T_{y}\Psi_{y}T_{x} \\
&\quad - \alpha\Big[\Upsilon_{yy} + \Upsilon_{\Psi y}\Psi_{y} + \Upsilon_{\Psi}\Psi_{yy} + \Upsilon_{u_e y}(u_e)_{y} \\
&\qquad + \Upsilon_{u_e}(u_e)_{yy} + \Upsilon_{Ty}T_{y} + \Upsilon_{T}T_{yy} - \zeta_{yy}T_{x} \\
&\qquad - \zeta_{y}T_{xy} - \zeta_{\Psi y}T_{x}\Psi_{y} - \zeta_{\Psi}T_{xy}\Psi_{y} \\
&\qquad - \zeta_{\Psi}T_{x}\Psi_{yy} - \zeta_{u_e y}(u_e)_{y}T_{x} - \zeta_{u_e}(u_e)_{yy}T_{x} \\
&\qquad - \zeta_{u_e}(u_e)_{y}T_{xy} - \zeta_{Ty}T_{y}T_{x} - \zeta_{T}T_{yy}T_{x} \\
&\qquad - \zeta_{T}T_{y}T_{xy} - \gamma_{yy}T_{y} - \gamma_{y}T_{yy} - \gamma_{\Psi y}T_{y}\Psi_{y} \\
&\qquad - \gamma_{\Psi}T_{yy}\Psi_{y} - \gamma_{\Psi}T_{y}\Psi_{yy} - \zeta_{u_e y}(u_e)_{y}T_{y} \\
&\qquad - \zeta_{u_e}(u_e)_{yy}T_{y} - \zeta_{u_e}(u_e)_{y}T_{yy} - \zeta_{Ty}T_{y}T_{y} \\
&\qquad - \zeta_{T}T_{yy}T_{y} - \zeta_{T}T_{y}T_{yy} - \zeta_{y}T_{yx} - \zeta_{\Psi}T_{yx}\Psi_{y} \\
&\qquad - \zeta_{u_e}(u_e)_{y}T_{yx} - \zeta_{T}T_{y}T_{yx} - \gamma_{y}T_{yy} \\
&\qquad - \gamma_{\Psi}T_{yy}\Psi_{y} - \gamma_{u_e}(u_e)_{y}T_{yy} - \gamma_{T}T_{y}T_{yy}\Big] = 0.
\end{aligned}
\tag{A.1}
$$

List of Symbols

a, b, and c: Constants
g: Acceleration due to gravity (ms^{-2})
f: Dimensionless stream function
Gr_x: Local Grashof number
Pr: Prandtl number
Re_x: Local Reynolds number
T: Fluid temperature (K)
T_∞: Ambient temperature (K)
$T_{w(x)}$: Temperature of the stretching surface (K)
u, v: Velocity components along the x and y directions, respectively,
$u_{e(x)}$: Velocity of external flow (ms^{-1})
$u_{w(x)}$: Velocity of the stretching surface (ms^{-1})
x, y: Cartesian coordinates along the surface and normal to it, respectively, (m).

Greek Symbols

α: Thermal diffusivity ($m^2\ s^{-1}$)
β: Thermal expansion coefficient (K^{-1})
η: Pseudo-similarity variable
θ: Dimensionless temperature
λ: Buoyancy parameter
ν: Kinematic viscosity ($m^2\ s^{-1}$)
Ψ: Stream function.

Subscripts

w: Condition at the stretching sheet
∞: Condition at infinity.

References

[1] M. V. Karwe and Y. Jaluria, "Numerical simulation of thermal transport associated with a continuously moving flat sheet in materials processing," *Journal of Heat Transfer*, vol. 113, no. 3, pp. 612–619, 1991.

[2] B. C. Sakiadis, "Boundary layer behaviour on continuous solid surfaces—ii. the boundary layer on a continuous flat surface," *AIChE Journal*, vol. 7, pp. 221–225, 1961.

[3] L. J. Crane, "Flow past a stretching plate," *Zeitschrift für Angewandte Mathematik und Physik ZAMP*, vol. 21, no. 4, pp. 645–647, 1970.

[4] P. Carragher and L. J. Crane, "Heat transfer on a continuous stretching sheet," *Journal of Applied Mathematics and Mechanics*, vol. 62, pp. 564–565, 1982.

[5] E. M. A. Elbashbeshy and M. A. A. Bazid, "Effect of temperature-dependent viscosity on heat transfer over a continuous moving surface," *Journal of Physics D*, vol. 33, no. 21, pp. 2716–2721, 2000.

[6] P. S. Gupta and A. S. Gupta, "Heat and mass transfer on a stretching sheet with suction or blowing," *The Canadian Journal of Chemical Engineering*, vol. 55, pp. 744–746, 1977.

[7] E. Magyari and B. Keller, "Heat and mass transfer in the boundary layers on an exponentially stretching continuous surface," *Journal of Physics D*, vol. 32, no. 5, pp. 577–585, 1999.

[8] E. Magyari and B. Keller, "Exact solutions for self-similar boundary-layer flows induced by permeable stretching walls," *European Journal of Mechanics B*, vol. 19, no. 1, pp. 109–122, 2000.

[9] E. Magyari, M. E. Ali, and B. Keller, "Heat and mass transfer characteristics of the self-similar boundary-layer flows induced by continuous surfaces stretched with rapidly decreasing velocities," *Heat and Mass Transfer*, vol. 38, no. 1, pp. 65–74, 2002.

[10] S. J. Liao and I. Pop, "On explicit analytic solutions of boundary-layer equations about flows in a porous medium or for a stretching wall," *International Journal of Heat and Mass Transfer*, vol. 47, pp. 75–85, 2004.

[11] R. Nazar, N. Amin, D. Filip, and I. Pop, "Unsteady boundary layer flow in the region of the stagnation point on a stretching sheet," *International Journal of Engineering Science*, vol. 42, no. 11-12, pp. 1241–1253, 2004.

[12] I. Pop, "Mhd flow over asymmetric plane stagnation point," *Journal of Applied Mathematics and Mechanics*, vol. 63, pp. 580–581, 1983.

[13] H. I. Andersson, "MHD flow of a viscoelastic fluid past a stretching surface," *Acta Mechanica*, vol. 95, no. 1–4, pp. 227–230, 1992.

[14] H. S. Takhar and G. Nath, "Similarity solution of unsteady boundary layer equations with a magnetic fiield," *Meccanica*, vol. 32, no. 2, pp. 157–163, 1997.

[15] R. Nazar, N. Amin, and I. Pop, "Unsteady boundary layer flow due to a stretching surface in a rotating fluid," *Mechanics Research Communications*, vol. 31, no. 1, pp. 121–128, 2004.

[16] C. H. Chen, "Laminar mixed convection adjacent to vertical, continuously stretching sheets," *Heat and Mass Transfer*, vol. 33, no. 5-6, pp. 471–476, 1998.

[17] C. H. Chen, "Mixed convection cooling of a heated, continuously stretching surface," *Warme- und Stoffubertragung Zeitschrift*, vol. 36, no. 1, pp. 79–86, 2000.

[18] C. R. Lin and C. K. Chen, "Exact solution of heat transfer from a stretching surface with variable heat flux," *Heat and Mass Transfer*, vol. 33, no. 5-6, pp. 477–480, 1998.

[19] M. Ali and F. Al-Yousef, "Laminar mixed convection from a continuously moving vertical surface with suction or injection," *Heat and Mass Transfer*, vol. 33, no. 4, pp. 301–306, 1998.

[20] M. Ali and F. Al-Yousef, "Laminar mixed convection boundary layers induced by a linearly stretching permeable surface," *International Journal of Heat and Mass Transfer*, vol. 45, no. 21, pp. 4241–4250, 2002.

[21] M. E. Ali, "The buoyancy effects on the boundary layers induced by continuous surfaces stretched with rapidly decreasing velocities," *Heat and Mass Transfer*, vol. 40, no. 3-4, pp. 285–291, 2004.

[22] M. E. Ali, "The effect of variable viscosity on mixed convection heat transfer along a vertical moving surface," *International Journal of Thermal Sciences*, vol. 45, no. 1, pp. 60–69, 2006.

[23] E. M. Abo-Eldahab, "The effects of temperature-dependent fluid properties on free convective flow along a semi-infinite vertical plate by the presence of radiation," *Heat and Mass Transfer*, vol. 41, no. 2, pp. 163–169, 2004.

[24] A. Ishak, R. Nazar, and I. Pop, "Unsteady mixed convection boundary layer flow due to a stretching vertical surface," *Arabian Journal for Science and Engineering*, vol. 31, no. 2 B, pp. 165–182, 2006.

[25] T. R. Mahapatra and A. S. Gupta, "Unsteady mixed convection boundary layer flow due to a stretching vertical surface," *Arabian Journal for Science and Engineering*, vol. 38, pp. 517–521, 2002.

[26] T. R. Mahapatra and A. S. Gupta, "Stagnation-point flow of a viscoelastic fluid towards a stretching surface," *International Journal of Non-Linear Mechanics*, vol. 39, no. 5, pp. 811–820, 2004.

[27] F. Rabiei and F. Ismail, "Fifth-order improved runge-kutta method for solving ordinary differential equations," *Australian Journal of Basic and Applied Sciences*, vol. 6, pp. 97–105, 2012.

[28] A. Ishak, R. Nazar, and I. Pop, "Mixed convection boundary layers in the stagnation-point flow toward a stretching vertical sheet," *Meccanica*, vol. 41, no. 5, pp. 509–518, 2006.