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Research Article

Common Fixed Point for Self-Mappings Satisfying an Implicit Lipschitz-Type Condition in Kaleva-Seikkala's Type Fuzzy Metric Spaces

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We first introduce the new real function class $\mathscr{F}$ satisfying an implicit Lipschitz-type condition. Then, by using $\mathscr{F}$-type real functions, some common fixed point theorems for a pair of self-mappings satisfying an implicit Lipschitz-type condition in fuzzy metric spaces (in the sense of Kaleva and Seikkala) are established. As applications, we obtain the corresponding common fixed point theorems in metric spaces. Also, some examples are given, which show that there exist mappings which satisfy the conditions in this paper but cannot satisfy the general contractive type conditions.

1. Introduction

In 1984, Kaleva and Seikkala [1] introduced the concept of a fuzzy metric space by setting the distance between two points to be a nonnegative fuzzy real number and studied some of its properties. From then on, some important results for single-valued and multivalued mappings in fuzzy metric spaces, such as coincidence theorems, various fixed point theorems, and so forth, were stated in subsequent work (see [1–11], etc.). Recently, Zhang [12, 13] established some new common fixed point theorems for generalized contractive type mappings in metric spaces and for Lipschitz-type mappings in cone metric spaces. These theorems extended the original contractive type conditions. Moreover, various real function classes satisfying an implicit relation were introduced in [10, 14–23], and some common fixed point theorems for composite mappings satisfying an implicit relation were established in metric spaces and fuzzy metric spaces, respectively.

It is well known that the fuzzy metric space is an important generalization of the ordinary metric space (see [1]). Inspired by [13–23], we establish some common fixed point theorems for new contractive type mappings in fuzzy metric spaces in this paper. In Section 3, we first introduce the new real function class $\mathscr{F}$ satisfying an implicit Lipschitz-type condition. Then, in Section 4, by using $\mathscr{F}$-type real functions, some common fixed point theorems for a pair of self-mappings satisfying an implicit Lipschitz-type condition in fuzzy metric spaces are established. In Section 5, as their applications, we obtain the corresponding common fixed point theorems in metric spaces. Also, some examples are given, which show that there exist mappings which satisfy the conditions in this paper but cannot satisfy the general contractive type conditions.

2. Preliminaries and Lemmas

Throughout this paper, let $\mathbb{Z}^+$ be the set of all positive integers, $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}^+ = [0, +\infty)$. For the details of fuzzy real number, we refer the reader to Kaleva and Seikkala [1], Dubois and Prade [24], and Bag and Samanta [25].

Definition 1 (cf. Dubois and Prade [24]). A mapping $\eta : \mathbb{R} \to [0, 1]$ is called a fuzzy real number or fuzzy interval, whose α-level set is denoted by $[\eta]_\alpha = \{t \in \mathbb{R} : \eta(t) \geqslant \alpha\}$, if it satisfies two axioms.

(1) There exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.

(2) $[\eta]_\alpha = [\lambda_\alpha, \rho_\alpha]$ is a closed interval of $\mathbb{R}$ for each $\alpha \in (0, 1]$, where $-\infty < \lambda_\alpha \leq \rho_\alpha < +\infty$.

The set of all such fuzzy real numbers is denoted by G. If $\eta \in G$ and $\eta(t) = 0$ whenever $t < 0$, then η is called a nonnegative fuzzy real number, and by G^+ we mean the set of all nonnegative fuzzy real numbers. If $\lambda_\alpha = -\infty$ and $\rho_\alpha = +\infty$ are admissible, then, for the sake of clarity, η is called a generalized fuzzy real number. The sets of all generalized fuzzy real numbers or all generalized nonnegative fuzzy real numbers are denoted by G_∞ and G_∞^+, respectively. In that case, if $\lambda_\alpha = -\infty$, for instance, then $[\lambda_\alpha, \rho_\alpha]$ means the interval $(-\infty, \rho_\alpha]$.

The notation $\overline{0}$ stands for the fuzzy number satisfying $\overline{0}(0) = 1$ and $\overline{0}(t) = 0$ if $t \neq 0$. Clearly, $\overline{0} \in G^+$. $\mathbb{R}$ can be embedded in G: if $a \in \mathbb{R}$, then $\overline{a} \in G$ satisfies $\overline{a}(t) = \overline{0}(t-a)$.

Lemma 2 (Xiao et al. [8]). *Let $\eta \in G$, $\alpha \in (0,1]$, and $[\eta]_\alpha = [\lambda_\alpha, \rho_\alpha]$. Then*

(1) $\lim_{t\to-\infty}\eta(t) = 0 = \lim_{t\to+\infty}\eta(t)$.

(2) *$\eta(t)$ is a left continuous and nonincreasing function for $t \in (\lambda_1, +\infty)$.*

(3) *ρ_α is a left continuous and nonincreasing function for $\alpha \in (0,1]$.*

Definition 3 (cf. Kaleva and Seikkala [1]). Suppose that X is a nonempty set and that d is a mapping from $X \times X$ into G^+. Let $L, R : [0,1]\times[0,1] \to [0,1]$ be two symmetric and nondecreasing functions such that $L(0,0) = 0$ and $R(1,1) = 1$. For $\alpha \in (0,1]$ and $x, y \in X$, define the mapping

$$[d(x,y)]_\alpha = [\lambda_\alpha(x,y), \rho_\alpha(x,y)]. \tag{1}$$

The quadruple (X, d, L, R) is called a fuzzy metric space (briefly, FMS), and d is called a fuzzy metric, if

(FM-1) $d(x,y) = \overline{0}$ if and only if $x = y$;

(FM-2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(FM-3) for all $x, y, z \in X$:

(FM-3L) $d(x,y)(s+t) \geqslant L(d(x,z)(s), d(z,y)(t))$, whenever $s \leqslant \lambda_1(x,z), t \leqslant \lambda_1(z,y)$ and $s+t \leqslant \lambda_1(x,y)$;

(FM-3R) $d(x,y)(s+t) \leqslant R(d(x,z)(s), d(z,y)(t))$, whenever $s \geqslant \lambda_1(x,z)$, $t \geqslant \lambda_1(z,y)$ and $s+t \geqslant \lambda_1(x,y)$.

If d is a mapping from $X\times X$ into G_∞^+ and (X,d,L,R) satisfies (FM-1)–(FM-3), then (X,d,L,R) is called a generalized fuzzy metric space (briefly, GFMS).

From Lemma 2 and Definition 3, we obtain the following consequences.

Lemma 4. *Let (X,d,L,R) be a FMS, $[d(x,y)]_\alpha = [\lambda_\alpha(x,y), \rho_\alpha(x,y)]$ for $\alpha \in (0,1]$, where $x, y \in X$ are any two fixed elements. Then*

(1) $\lim_{t\to-\infty}d(x,y)(t) = 0 = \lim_{t\to+\infty}d(x,y)(t)$.

(2) *$d(x,y)(t)$ is a left continuous and nonincreasing function for $t \in (\lambda_1(x,y), +\infty)$.*

(3) *$\rho_\alpha(x,y)$ is a left continuous and nonincreasing function for $\alpha \in (0,1]$.*

Lemma 5 (Xiao et al. [8]). *Let (X,d,L,R) be a FMS, and suppose that*

(R-1) $R \leqslant \max$;

(R-2) $\lim_{a\to0^+}R(a,a) = 0$.

Then (R-1) ⇒(R-2).

Lemma 6. *Let (X,d,L,R) be a FMS. Then*

(1) *(R-1) ⇒ for each $\alpha \in (0,1]$, $\rho_\alpha(x,y) \leqslant \rho_\alpha(x,z) + \rho_\alpha(z,y)$ for all $x,y,z \in X$ (cf. [4, 5]).*

(2) *(R-2) ⇒ for each $\alpha \in (0,1]$ there exists $\mu = \mu(\alpha) \in (0,\alpha]$ such that $\rho_\alpha(x,y) \leqslant \rho_\mu(x,z) + \rho_\mu(z,y)$ for all $x,y,z \in X$ (cf. [5, 6, 8]).*

Lemma 7 (Kaleva and Seikkala [1]). *Let (X,d,L,R) be a FMS with (R-2). Then the family $\{U(\varepsilon,\alpha) : \varepsilon > 0, \alpha \in (0,1]\}$ of sets $U(\varepsilon,\alpha) = \{(x,y) \in X\times X : \rho_\alpha(x,y) < \varepsilon\}$ forms a basis for a Hausdorff uniformity on $X \times X$. Moreover, the sets*

$$N_x(\varepsilon,\alpha) = \{y \in X : \rho_\alpha(x,y) < \varepsilon\} \tag{2}$$

form a basis for a Hausdorff topology on X and this topology is metrizable.

According to Lemma 7, convergence in a FMS (X,d,L,R) can be defined by sequences. A sequence $\{x_n\}$ in X is said to be convergent to x (we write $x_n \to x$ or $\lim_{n\to\infty}x_n = x$) if $\lim_{n\to\infty}d(x_n,x) = \overline{0}$; that is, $\lim_{n\to\infty}\rho_\alpha(x_n,x) = 0$ for each $\alpha \in (0,1]$; $\{x_n\}$ is called a Cauchy sequence in X if $\lim_{m,n\to\infty}d(x_m,x_n) = \overline{0}$; equivalently, for any given $\varepsilon > 0$ and $\alpha \in (0,1]$, there exists $N = N(\varepsilon,\alpha) \in \mathbb{Z}^+$ such that $\rho_\alpha(x_m,x_n) < \varepsilon$, whenever $m,n \geqslant N$; (X,d,L,R) is said to be complete, if each Cauchy sequence in X is convergent to some point in X.

Lemma 8 (Kaleva and Seikkala [1]). *Let (X,d,L,R) be a FMS with $R \leqslant \max$. Then for each $\alpha \in (0,1]$, $\rho_\alpha(x,y)$ is continuous at $(x,y) \in X\times X$.*

3. The Real Functions Satisfying an Implicit Lipschitz-Type Condition

Definition 9. A lower semicontinuous function $F : \mathbb{R}_6^+ \to \mathbb{R}$ is called a real function satisfying an implicit Lipschitz-type condition, if the following conditions are satisfied.

($\mathcal{F}$-1) $F(t_1,\ldots,t_6)$ is nonincreasing in t_5, t_6.

($\mathcal{F}$-2) There exist $A, B > 0$ with $AB < 1$ such that for all $u, v \geqslant 0$, we have

($\mathcal{F}$-2a) $u \leqslant Av$, whenever $F(u,v,v,u,v+u,0) \leqslant 0$,

($\mathcal{F}$-2b) $u \leqslant Bv$, whenever $F(u,v,u,v,0,v+u) \leqslant 0$.

($\mathcal{F}$-3) For all $u \geqslant 0$ with $F(u,u,0,0,u,u) \leqslant 0$, we have $u = 0$.

We denote by $\mathcal{F}$ the collection of all real functions $F : \mathbb{R}_6^+ \to \mathbb{R}$ satisfying an implicit Lipschitz-type condition.

Remark 10. Let $F \in \mathcal{F}$; if $F(u,0,0,u,u,0) \leqslant 0$ or $F(u,0,u,0,0,u) \leqslant 0$, then by condition ($\mathcal{F}$-2) of Definition 9, we have $u = 0$.

The following examples show that the collection $\mathcal{F}$ is a largish class of real functions.

Example 11. Let $a \in (0,1/2)$. The function $F_1 : \mathbb{R}_6^+ \to \mathbb{R}$ is defined by

$$F_1(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a(\max\{t_2,t_3,t_4,t_5,t_6\}); \quad (3)$$

then $F_1 \in \mathcal{F}$.

In fact, it is easy to see that F_1 is continuous. Also, ($\mathcal{F}$-1) and ($\mathcal{F}$-3) are easy to check. For any $u, v \geqslant 0$, if $F_1(u,v,v,u,v+u,0) \leqslant 0$ or $F_1(u,v,u,v,0,v+u) \leqslant 0$, then we have $u \leqslant (a/(1-a))v$, or $u \leqslant (a/(1-a))v$ respectively. Taking $A = B = a/(1-a)$, we have $AB = (a/(1-a))^2 < 1$; that is, ($\mathcal{F}$-2) holds. Hence $F_1 \in \mathcal{F}$.

Example 12. Let $a,b,c,d > 0$ with $a+b+c \in (0,1)$ and $a+d \in (0,1)$. The function $F_2 : \mathbb{R}_6^+ \to \mathbb{R}$ is defined by

$$F_2(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5t_6; \quad (4)$$

then $F_2 \in \mathcal{F}$.

Obviously, F_2 is continuous, and ($\mathcal{F}$-1) and ($\mathcal{F}$-3) are easy to check. For any $u, v \geqslant 0$, if $F_2(u,v,v,u,v+u,0) \leqslant 0$ or $F_2(u,v,u,v,0,v+u) \leqslant 0$, then we have $u^2 - u(av+bv+cu) \leqslant 0$ or $u^2 - u(av+bu+cv) \leqslant 0$, respectively, which implies that $u \leqslant ((a+b)/(1-c))v$ or $u \leqslant ((a+c)/(1-b))v$. Let $A = (a+b)/(1-c)$ and $B = (a+c)/(1-b)$. By $a+b+c \in (0,1)$, we have $A < 1$ and $B < 1$; that is, $AB < 1$. Thus ($\mathcal{F}$-2) holds. Hence $F_2 \in \mathcal{F}$.

Example 13. Let $a \in (0,1/25)$. The function $F_3 : \mathbb{R}_6^+ \to \mathbb{R}$ is defined by

$$F_3(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^3 - a\left(t_2^3 + \cdots + t_6^3\right); \quad (5)$$

then $F_3 \in \mathcal{F}$.

Obviously, F_3 is continuous, and ($\mathcal{F}$-1) and ($\mathcal{F}$-3) are easy to check.

For any $u, v \geqslant 0$, if $F_3(u,v,v,u,v+u,0) \leqslant 0$, then we have $u^3 - a(2v^3 + u^3 + (v+u)^3) \leqslant 0$, which implies that

$$\begin{aligned}(1-a)u^3 &\leqslant 2av^3 + a(v+u)^3 \\ &\leqslant 3a(v+u)^3 \Longrightarrow u \leqslant \frac{v\sqrt[3]{3a/(1-a)}}{\left(1-\sqrt[3]{3a/(1-a)}\right)}.\end{aligned} \quad (6)$$

Similarly, if $F_3(u,v,u,v,0,v+u) \leqslant 0$, then we also have $u \leqslant v\sqrt[3]{3a/(1-a)}/(1-\sqrt[3]{3a/(1-a)})$. Let $A = B = \sqrt[3]{3a/(1-a)}/(1-\sqrt[3]{3a/(1-a)})$. Note that $a \in (0,1/25)$; we have $3a/(1-a) \in (0,1/8) \Rightarrow \sqrt[3]{3a/(1-a)} \in (0,1/2)$; that is, $A = B \in (0,1)$; thus $AB < 1$. Therefore ($\mathcal{F}$-2) holds. Hence $F_3 \in \mathcal{F}$.

Example 14. Let $\phi_1,\ldots,\phi_5$: $\mathbb{R}^+ \to [0,1)$ be five continuous functions satisfying the following conditions.

(i) $\phi_1(t) + \phi_4(t) + \phi_5(t) < 1$ for all $t \in \mathbb{R}^+$,

(ii) There exist $a > 0$, $b > 0$, $C > 0$, $D > 0$ with $CD < ab$, such that

$$\begin{aligned}&\inf_{t\geqslant 0}\{1-\phi_2(t)-\phi_5(t)\} = a,\\ &\inf_{t\geqslant 0}\{1-\phi_3(t)-\phi_4(t)\} = b,\\ &\sup_{t\geqslant 0}\{\phi_1(t)+\phi_2(t)+\phi_4(t)\} = C,\\ &\sup_{t\geqslant 0}\{\phi_1(t)+\phi_3(t)+\phi_5(t)\} = D.\end{aligned} \quad (7)$$

We define the function $F_4 : \mathbb{R}_6^+ \to \mathbb{R}$ as follows:

$$\begin{aligned}F_4&(t_1,t_2,t_3,t_4,t_5,t_6)\\ &= t_1 - (\phi_1(t_2)t_2 + \phi_2(t_2)t_3 + \phi_3(t_2)t_4\\ &\quad + \phi_4(t_2)t_5 + \phi_5(t_2)t_6).\end{aligned} \quad (8)$$

Then $F_4 \in \mathcal{F}$.

In fact, it is easy to see that F_4 is continuous, and ($\mathcal{F}$-1) is satisfied.

For any $u, v \geqslant 0$, if $F_4(u,v,v,u,v+u,0) \leqslant 0$, then we have $u - (\phi_1(v)v + \phi_2(v)v + \phi_3(v)u + \phi_4(v)(v+u)) \leqslant 0$, which implies that $u \leqslant ((\phi_1(v)+\phi_2(v)+\phi_4(v))/(1-\phi_3(v)-\phi_4(v)))v$. From condition (ii), we obtain $u \leqslant (C/b)v$. Similarly, if $F_4(u,v,u,v,0,v+u) \leqslant 0$, then we have $u \leqslant ((\phi_1(v)+\phi_3(v)+\phi_5(v))/(1-\phi_2(v)-\phi_5(v)))v \leqslant D/av$. Let $A = C/b, B = D/a$. Note that $CD < ab$; we have $AB < 1$; that is, ($\mathcal{F}$-2) holds.

Furthermore, for any $u \geqslant 0$, if $F_4(u,u,0,0,u,u) \leqslant 0$, then we have $u - (\phi_1(u)u + \phi_4(u)u + \phi_5(u)u) \leqslant 0$, which implies that $(1-\phi_1(u)-\phi_4(u)-\phi_5(u))u \leqslant 0$. Note that $\phi_1(t)+\phi_4(t)+\phi_5(t) < 1$ for all $t \in \mathbb{R}^+$; we have $u = 0$. This shows that ($\mathcal{F}$-3) holds. Hence $F_4 \in \mathcal{F}$.

Example 15. Let the function $F_5 : \mathbb{R}_6^+ \to \mathbb{R}$ be defined by

$$\begin{aligned}F_5&(t_1,t_2,t_3,t_4,t_5,t_6)\\ &= t_1 - \left(\frac{t_2+1}{20t_2+21}t_2 + \frac{t_2+1}{20t_2+21}t_3 + \frac{2t_2+1}{5t_2+6}t_4\right.\\ &\quad \left. + \frac{t_2+1}{40t_2+42}t_5 + \frac{t_2+1}{2t_2+3}t_6\right);\end{aligned} \quad (9)$$

then $F_5 \in \mathcal{F}$.

In fact, in Example 14, taking $\phi_1(t) = (t+1)/(20t+21)$, $\phi_2(t) = (t+1)/(20t+21)$, $\phi_3(t) = (2t+1)/(5t+6)$, $\phi_4(t) = (t+1)/(40t+42)$, and $\phi_5(t) = (t+1)/(2t+3)$, we obtain five continuous functions $\phi_1, \ldots, \phi_5$ from $\mathbb{R}^+$ into $[0,1)$ satisfying the following conditions:

$$\begin{aligned}
&\frac{17}{42} \leqslant \phi_1(t) + \phi_4(t) + \phi_5(t) < \frac{23}{40} < 1, \\
&\inf_{t\geqslant 0}\{1 - \phi_2(t) - \phi_5(t)\} = \frac{9}{20}, \\
&\inf_{t\geqslant 0}\{1 - \phi_3(t) - \phi_4(t)\} = \frac{23}{40}, \\
&\sup_{t\geqslant 0}\{\phi_1(t) + \phi_2(t) + \phi_4(t)\} = \frac{1}{8}, \\
&\sup_{t\geqslant 0}\{\phi_1(t) + \phi_3(t) + \phi_5(t)\} = \frac{19}{20}.
\end{aligned} \tag{10}$$

It is evident that $1/8 \cdot 19/20 = 19/160 < 207/800 = 9/20 \cdot 23/40$, and so all conditions of Example 14 are satisfied. Therefore, $F_5 \in \mathscr{F}$.

Example 16. Let $a, b, c, d, e \geqslant 0$ with $a+d+e < 1$, $c+d < 1$ and $b+e < 1$. There exists $\delta > 0$ such that $a+b+c+d+e = 1+\delta$ and $(c-b)(e-d) > 2\delta$. We define the function $F_6 : \mathbb{R}_6^+ \to \mathbb{R}$ as follows:

$$F_6(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - (at_2 + bt_3 + ct_4 + dt_5 + et_6); \tag{11}$$

then $F_6 \in \mathscr{F}$.

Obviously, in Example 14, taking $\phi_1(t) = a$, $\phi_2(t) = b$, $\phi_3(t) = c$, $\phi_4(t) = d$, and $\phi_5(t) = e$, we obtain five continuous functions $\phi_1, \ldots, \phi_5$ from $\mathbb{R}^+$ into $[0,1)$. Moreover, by $(c-b)(e-d) > 2\delta$, we have $e \neq d$. Hence, $0 < a+d+e < 1$; that is, condition (i) of Example 14 holds.

Furthermore, $0 < 1-b-e$, $0 < 1-c-d$ is obvious. Note that $a < 1$ and $(c-b)(e-d) > 2\delta$; it follows that

$$a(1+\delta) + be + cd < a + \delta + be + cd < a - \delta + bd + ce. \tag{12}$$

By $a+b+c+d+e = 1+\delta$, we obtain

$$\begin{aligned}
a(a+b+c+d+e) &+ be + cd \\
&+ bc + de < 1 - b - c - d \\
&- e + bd + ce + bc + de,
\end{aligned} \tag{13}$$

which implies that $(a+b+d)(a+c+e) < (1-c-d)(1-b-e)$. Hence, condition (ii) of Example 14 is satisfied. Thus, by Example 14, we have $F_6 \in \mathscr{F}$.

Remark 17. The numbers a, b, c, d, e, and δ in Example 16 really exist. For example, if we take $\delta = 1/40$, $a = 1/20$, $b = 1/20$, $c = 2/5$, $d = 1/40$, and $e = 1/2$, then $a+b+c+d+e = 1 + 1/40$, $a+d+e = 23/40 < 1$, $c+d = 17/40 < 1$, $b+e = 11/20 < 1$, and $(c-b)(e-d) = 133/800 > 1/20$; that is, the conditions of Example 16 are satisfied.

Example 18. Define the function $F_7 : \mathbb{R}_6^+ \to \mathbb{R}$ as follows:

$$F_7(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - (at_2 + bt_3 + ct_4 + dt_5 + et_6), \tag{14}$$

where a, b, c, d, and e are nonnegative real numbers, with $a+b+c+d+e = 1$ and either $c > b$, $e > d$ or $c < b$, $e < d$. Then $F_7 \in \mathscr{F}$.

In fact, if we take $\phi_1(t) = a, \phi_2(t) = b, \phi_3(t) = c, \phi_4(t) = d$, and $\phi_5(t) = e$, then condition (i) of Example 14 is obviously satisfied. Note that $c > b, e > d$; we have $(a+b+d)(a+c+e) = (1-c-e)(1-b-d) < (1-c-d)(1-b-e)$; that is, condition (ii) of Example 14 is satisfied. Similarly, we can prove the case of $c < b$, $e < d$. Therefore, by Example 14, $F_7 \in \mathscr{F}$.

4. Main Results

Theorem 19. *Let (X, d, L, R) be a complete FMS with $R \leqslant \max$ and let S and T be two self-mappings on (X, d, L, R). If there exists $F \in \mathscr{F}$ such that*

$$\begin{aligned}
u - F(u, v, w, s, t, g) &\geqslant \lambda_1(Sx, Ty), \\
d(Sx, Ty)(u - F(u, v, w, s, t, g)) & \\
\leqslant \max\{d(Sx, Ty)(u), &\, d(x, y)(v), d(x, Sx)(w), \\
d(y, Ty)(s), &\, d(x, Ty)(t), d(y, Sx)(g)\},
\end{aligned} \tag{15}$$

for all $x, y \in X$, whenever $u \geqslant \lambda_1(Sx, Ty)$, $v \geqslant \lambda_1(x, y)$, $w \geqslant \lambda_1(x, Sx)$, $s \geqslant \lambda_1(y, Ty)$, $t \geqslant \lambda_1(x, Ty)$, and $g \geqslant \lambda_1(y, Sx)$, then S and T have a unique common fixed point in X. Moreover, for any $x_0 \in X$, the iterative process $x_{2k+1} = Sx_{2k}$, $x_{2k+2} = Tx_{2k+1}$, $k = 0, 1, 2, \ldots$, converges to the fixed point.

Proof. Firstly, we use (15) to prove that the following inequality:

$$\begin{aligned}
F(\rho_\alpha(Sx, Ty), \rho_\alpha(x, y), \rho_\alpha(x, Sx), & \\
\rho_\alpha(y, Ty), \rho_\alpha(x, Ty), \rho_\alpha(y, Sx)) &\leqslant 0
\end{aligned} \tag{16}$$

holds for all $x, y \in X$ and $\alpha \in (0, 1]$.

In fact, for each $x, y \in X$ and $\alpha \in (0, 1]$, if we set $\rho_\alpha(Sx, Ty) = u$, $\rho_\alpha(x, y) = v$, $\rho_\alpha(x, Sx) = w$, $\rho_\alpha(y, Ty) = s$, $\rho_\alpha(x, Ty) = t$, $\rho_\alpha(y, Sx) = g$, then, for any $\varepsilon > 0$, it is obvious that $d(Sx, Ty)(u+\varepsilon) < \alpha$, $d(x, y)(v+\varepsilon) < \alpha$, $d(x, Sx)(w+\varepsilon) < \alpha$, $d(y, Ty)(s+\varepsilon) < \alpha$, $d(x, Ty)(t+\varepsilon) < \alpha$, $d(y, Sx)(g+\varepsilon) < \alpha$, and $u+\varepsilon \geqslant \lambda_1(Sx, Ty)$, $v+\varepsilon \geqslant \lambda_1(x, y)$, $w+\varepsilon \geqslant \lambda_1(x, Sx)$, $s+\varepsilon \geqslant \lambda_1(y, Ty)$, $t+\varepsilon \geqslant \lambda_1(x, Ty)$, $g+\varepsilon \geqslant \lambda_1(y, Sx)$. By (15), we have $u+\varepsilon - F(u+\varepsilon, v+\varepsilon, w+\varepsilon, s+\varepsilon, t+\varepsilon, g+\varepsilon) \geqslant \lambda_1(Sx, Ty)$ and $d(Sx, Ty)(u+\varepsilon - F(u+\varepsilon, v+\varepsilon, w+\varepsilon, s+\varepsilon, t+\varepsilon, g+\varepsilon)) < \alpha$, which imply that

$$\begin{aligned}
F(\rho_\alpha(Sx, Ty) + \varepsilon, \rho_\alpha(x, y) + \varepsilon, \rho_\alpha(x, Sx) + \varepsilon, \rho_\alpha(y, Ty) & \\
+ \varepsilon, \rho_\alpha(x, Ty) + \varepsilon, \rho_\alpha(y, Sx) + \varepsilon) &< \varepsilon;
\end{aligned} \tag{17}$$

then by the arbitrariness of ε and the lower semicontinuity of F, we have

$$\begin{aligned} & F(\rho_\alpha(Sx,Ty),\rho_\alpha(x,y),\rho_\alpha(x,Sx),\rho_\alpha(y,Ty), \\ &\quad \rho_\alpha(x,Ty),\rho_\alpha(y,Sx)) \\ &\quad \leqslant \liminf_{\varepsilon\to 0} F(\rho_\alpha(Sx,Ty)+\varepsilon,\rho_\alpha(x,y)+\varepsilon,\rho_\alpha(x,Sx)+\varepsilon, \\ &\qquad \rho_\alpha(y,Ty)+\varepsilon,\rho_\alpha(x,Ty)+\varepsilon,\rho_\alpha(y,Sx)+\varepsilon) \\ &\quad \leqslant 0, \end{aligned} \tag{18}$$

for each $x, y \in X$ and $\alpha \in (0,1]$; that is, the inequality (16) holds for all $x, y \in X$ and $\alpha \in (0,1]$.

For any $x_0 \in X$, we construct an iterative sequence $\{x_n\}$ in X as follows:

$$x_{2k+1} = Sx_{2k}, \quad x_{2k+2} = Tx_{2k+1}, \quad k = 0,1,2,\ldots. \tag{19}$$

For $k = 0,1,2,\ldots$, applying (16), we obtain for each $\alpha \in (0,1]$

$$\begin{aligned} & F(\rho_\alpha(Sx_{2k},Tx_{2k+1}),\rho_\alpha(x_{2k},x_{2k+1}),\rho_\alpha(x_{2k},Sx_{2k}), \\ &\quad \rho_\alpha(x_{2k+1},Tx_{2k+1}),\rho_\alpha(x_{2k},Tx_{2k+1}),\rho_\alpha(x_{2k+1},Sx_{2k})) \\ &\quad = F(\rho_\alpha(x_{2k+1},x_{2k+2}),\rho_\alpha(x_{2k},x_{2k+1}),\rho_\alpha(x_{2k},x_{2k+1}), \\ &\qquad \rho_\alpha(x_{2k+1},x_{2k+2}),\rho_\alpha(x_{2k},x_{2k+2}),0) \leqslant 0. \end{aligned} \tag{20}$$

By the known condition $R \leqslant \max$ and conclusion (1) of Lemma 6, we have $\rho_\alpha(x_{2k},x_{2k+2}) \leqslant \rho_\alpha(x_{2k}, x_{2k+1}) + \rho_\alpha(x_{2k+1},x_{2k+2})$. Note that F is nonincreasing in t_6; it is not difficult to see that

$$\begin{aligned} & F(\rho_\alpha(x_{2k+1},x_{2k+2}),\rho_\alpha(x_{2k},x_{2k+1}),\rho_\alpha(x_{2k},x_{2k+1}), \\ &\quad \rho_\alpha(x_{2k+1},x_{2k+2}),\rho_\alpha(x_{2k},x_{2k+1}) \\ &\quad +\rho_\alpha(x_{2k+1},x_{2k+2}),0) \leqslant 0, \quad \text{for each } \alpha \in (0,1]. \end{aligned} \tag{21}$$

Since $F \in \mathscr{F}$, there exists $A > 0$ such that

$$\rho_\alpha(x_{2k+1},x_{2k+2}) \leqslant A\rho_\alpha(x_{2k},x_{2k+1}), \quad \text{for each } \alpha \in (0,1]. \tag{22}$$

Similarly, for $k = 0,1,2,\ldots$, applying (16), we obtain for each $\alpha \in (0,1]$

$$\begin{aligned} & F(\rho_\alpha(Sx_{2k+2},Tx_{2k+1}),\rho_\alpha(x_{2k+2},x_{2k+1}), \\ &\quad \rho_\alpha(x_{2k+2},Sx_{2k+2}),\rho_\alpha(x_{2k+1},Tx_{2k+1}), \\ &\quad \rho_\alpha(x_{2k+2},Tx_{2k+1}),\rho_\alpha(x_{2k+1},Sx_{2k+2})) \\ &\quad = F(\rho_\alpha(x_{2k+3},x_{2k+2}),\rho_\alpha(x_{2k+2},x_{2k+1}), \\ &\qquad \rho_\alpha(x_{2k+2},x_{2k+3}),\rho_\alpha(x_{2k+1},x_{2k+2}),0, \\ &\qquad \rho_\alpha(x_{2k+1},x_{2k+3})) \leqslant 0. \end{aligned} \tag{23}$$

By $R \leqslant \max$ and (1) of Lemma 6, we have $\rho_\alpha(x_{2k+1},x_{2k+3}) \leqslant \rho_\alpha(x_{2k+1},x_{2k+2})+\rho_\alpha(x_{2k+2}, x_{2k+3})$. Note that F is nonincreasing in t_6; we obtain

$$\begin{aligned} & F(\rho_\alpha(x_{2k+3},x_{2k+2}),\rho_\alpha(x_{2k+2},x_{2k+1}),\rho_\alpha(x_{2k+2},x_{2k+3}), \\ &\quad \rho_\alpha(x_{2k+1},x_{2k+2}),0,\rho_\alpha(x_{2k+1},x_{2k+2}) \\ &\quad +\rho_\alpha(x_{2k+2},x_{2k+3})) \leqslant 0, \quad \text{for each } \alpha \in (0,1]. \end{aligned} \tag{24}$$

Since $F \in \mathscr{F}$, there exists $B > 0$ such that

$$\rho_\alpha(x_{2k+3},x_{2k+2}) \leqslant B\rho_\alpha(x_{2k+1},x_{2k+2}), \quad \text{for each } \alpha \in (0,1]. \tag{25}$$

Using inductive method, for $k = 0,1,2,\ldots$, we can obtain

$$\begin{aligned} \rho_\alpha(x_{2k+1},x_{2k+2}) &\leqslant A\rho_\alpha(x_{2k},x_{2k+1}) \\ &\leqslant AB\rho_\alpha(x_{2k-1},x_{2k}) \\ &\leqslant \cdots \leqslant (AB)^k A\rho_\alpha(x_0,x_1), \\ \rho_\alpha(x_{2k+2},x_{2k+3}) &\leqslant B\rho_\alpha(x_{2k+1},x_{2k+2}) \\ &\leqslant BA\rho_\alpha(x_{2k},x_{2k+1}) \\ &\leqslant \cdots \leqslant (AB)^{k+1}\rho_\alpha(x_0,x_1), \\ &\qquad \text{for each } \alpha \in (0,1]. \end{aligned} \tag{26}$$

Next, we prove that the sequence $\{x_n\}$ is a Cauchy sequence. For $k < p$, by $R \leqslant \max$ and conclusion (1) of Lemma 6, we have for each $\alpha \in (0,1]$

$$\begin{aligned} \rho_\alpha(x_{2k+1},x_{2p+1}) &\leqslant \rho_\alpha(x_{2k+1},x_{2k+2}) + \cdots + \rho_\alpha(x_{2p},x_{2p+1}) \\ &\leqslant \left(A\sum_{i=k}^{p-1}(AB)^i + \sum_{i=k+1}^{p}(AB)^i\right)\rho_\alpha(x_0,x_1) \\ &\leqslant \left(A\frac{(AB)^k}{1-AB} + \frac{(AB)^{k+1}}{1-AB}\right)\rho_\alpha(x_0,x_1) \\ &\leqslant M(AB)^k\rho_\alpha(x_0,x_1), \end{aligned} \tag{27}$$

where $M = (A + AB)/(1 - AB)$. By the similar reasoning process, we have for each $\alpha \in (0,1]$

$$\begin{aligned} \rho_\alpha(x_{2k},x_{2p+1}) &\leqslant M(AB)^k\rho_\alpha(x_0,x_1), \\ \rho_\alpha(x_{2k},x_{2p}) &\leqslant M(AB)^k\rho_\alpha(x_0,x_1), \\ \rho_\alpha(x_{2k+1},x_{2p}) &\leqslant M(AB)^k\rho_\alpha(x_0,x_1). \end{aligned} \tag{28}$$

Then there exists $k(n)$ with $(n-1)/2 \leqslant k \leqslant n/2$ for $0 < n < m$, such that for each $\alpha \in (0,1]$

$$\rho_\alpha(x_m,x_n) \leqslant M(AB)^k\rho_\alpha(x_0,x_1). \tag{29}$$

Since $0 < AB < 1$, it is evident that the sequence $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, we set $\lim_{n\to\infty} x_n = x_* \in X$. Applying (16), we have

$$\begin{aligned} &F(\rho_\alpha(Sx_*, Tx_{2n+1}), \rho_\alpha(x_*, x_{2n+1}), \rho_\alpha(x_*, Sx_*), \\ &\quad \rho_\alpha(x_{2n+1}, Tx_{2n+1}), \rho_\alpha(x_*, Tx_{2n+1}), \rho_\alpha(x_{2k+1}, Sx_*)) \\ &\quad = F(\rho_\alpha(Sx_*, x_{2n+2}), \rho_\alpha(x_*, x_{2n+1}), \rho_\alpha(x_*, Sx_*), \\ &\quad \rho_\alpha(x_{2n+1}, x_{2n+2}), \rho_\alpha(x_*, x_{2n+2}), \rho_\alpha(x_{2k+1}, Sx_*)) \\ &\quad \leqslant 0, \end{aligned} \tag{30}$$

for each $\alpha \in (0, 1]$. Let $n \to \infty$; by the lower semicontinuity of F and Lemma 8, we have

$$\begin{aligned} &F(\rho_\alpha(Sx_*, x_*), 0, \rho_\alpha(x_*, Sx_*), 0, 0, \rho_\alpha(x_*, Sx_*)) \\ &\quad \leqslant \liminf_{n\to\infty} F(\rho_\alpha(Sx_*, x_{2n+2}), \rho_\alpha(x_*, x_{2n+1}), \\ &\quad \rho_\alpha(x_*, Sx_*), \rho_\alpha(x_{2n+1}, x_{2n+2}), \\ &\quad \rho_\alpha(x_*, x_{2n+2}), \rho_\alpha(x_{2k+1}, Sx_*)) \leqslant 0, \end{aligned} \tag{31}$$

for each $\alpha \in (0, 1]$. By Remark 10, $\rho_\alpha(Sx_*, x_*) = 0$ for each $\alpha \in (0, 1]$, which implies that x_* is a fixed point of S.

Similarly, for each $\alpha \in (0, 1]$, we have

$$\begin{aligned} &F(\rho_\alpha(Sx_{2n}, Tx_*), \rho_\alpha(x_{2n}, x_*), \rho_\alpha(x_{2n}, Sx_{2n}), \\ &\quad \rho_\alpha(x_*, Tx_*), \rho_\alpha(x_{2n}, Tx_*), \rho_\alpha(x_*, Sx_{2n})) \\ &\quad = F(\rho_\alpha(x_{2n+1}, Tx_*), \rho_\alpha(x_{2n}, x_*), \rho_\alpha(x_{2n}, x_{2n+1}), \\ &\quad \rho_\alpha(x_*, Tx_*), \rho_\alpha(x_{2n}, Tx_*), \rho_\alpha(x_*, x_{2n+1})) \leqslant 0. \end{aligned} \tag{32}$$

Let $n \to \infty$; by the lower semicontinuity of F and Lemma 8, we have for each $\alpha \in (0, 1]$

$$F(\rho_\alpha(x_*, Tx_*), 0, 0, \rho_\alpha(x_*, Tx_*), \rho_\alpha(x_*, Tx_*), 0) \leqslant 0. \tag{33}$$

By Remark 10, $\rho_\alpha(x_*, Tx_*) = 0$ for each $\alpha \in (0, 1]$, which implies that x_* is also a fixed point of T. Thus x_* is a common fixed point of S, T.

Lastly, we prove the uniqueness of the common fixed point. If x^* is another common fixed point of S, T, then by (16), we have for each $\alpha \in (0, 1]$

$$\begin{aligned} &F(\rho_\alpha(Sx_*, Tx^*), \rho_\alpha(x_*, x^*), \rho_\alpha(x_*, Sx_*), \\ &\quad \rho_\alpha(x^*, Tx^*), \rho_\alpha(x_*, Tx^*), \rho_\alpha(x^*, Sx_*)) \\ &\quad = F(\rho_\alpha(x_*, x^*), \rho_\alpha(x_*, x^*), 0, 0, \\ &\quad \rho_\alpha(x_*, x^*), \rho_\alpha(x^*, x_*)) \leqslant 0. \end{aligned} \tag{34}$$

Note that $F \in \mathscr{F}$, and by ($\mathscr{F}$-3) of Definition 9, we obtain $\rho_\alpha(x^*, x_*) = 0$ for each $\alpha \in (0, 1]$; hence $x^* = x_*$. The uniqueness is proved and we complete the proof of the theorem. □

According to the proof of Theorem 19, we can easily obtain the following corollary.

Corollary 20. *Let (X, d, L, R) be a complete FMS with $R \leqslant \max$ and let S and T be two self-mappings on (X, d, L, R). If there exists $F \in \mathscr{F}$ such that (16) holds for all $x, y \in X$ and $\alpha \in (0, 1]$, then S and T have a unique common fixed point in X. Moreover, for any $x_0 \in X$, the iterative process $x_{2k+1} = Sx_{2k}$, $x_{2k+2} = Tx_{2k+1}$, $k = 0, 1, 2, \ldots$, converges to the fixed point.*

Theorem 21. *Let (X, d, L, R) be a complete FMS with $R \leqslant \max$. Let $\phi_1, \ldots, \phi_5 : \mathbb{R}^+ \to [0, 1)$ be five continuous functions which satisfy the following conditions:*

(i) *$\phi_1(t) + \phi_4(t) + \phi_5(t) < 1$ for all $t \in \mathbb{R}^+$,*

(ii) *$0 < \inf_{t\geqslant 0}\{1 - \phi_2(t) - \phi_5(t)\} = a$, $0 < \inf_{t\geqslant 0}\{1 - \phi_3(t) - \phi_4(t)\} = b$, $0 < \sup_{t\geqslant 0}\{\phi_1(t) + \phi_2(t) + \phi_4(t)\} = C$, $0 < \sup_{t\geqslant 0}\{\phi_1(t) + \phi_3(t) + \phi_5(t)\} = D$ and $CD < ab$.*

Let S and T be two self-mappings on (X, d, L, R) such that

$$\begin{aligned} \rho_\alpha(Sx, Ty) &\leqslant \phi_1(\rho_\alpha(x, y))\rho_\alpha(x, y) + \phi_2(\rho_\alpha(x, y)) \\ &\quad \times \rho_\alpha(x, Sx) + \phi_3(\rho_\alpha(x, y))\rho_\alpha(y, Ty) \\ &\quad + \phi_4(\rho_\alpha(x, y))\rho_\alpha(x, Ty) \\ &\quad + \phi_5(\rho_\alpha(x, y))\rho_\alpha(y, Sx), \end{aligned} \tag{35}$$

for all $x, y \in X$ and $\alpha \in (0, 1]$. Then S and T have a unique common fixed point in X. Moreover, for any $x_0 \in X$, the iterative process $x_{2k+1} = Sx_{2k}, x_{2k+2} = Tx_{2k+1}, k = 0, 1, 2, \ldots$, converges to the fixed point.

Proof. Taking $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - [\phi_1(t_2)t_2 + \phi_2(t_2)t_3 + \phi_3(t_2)t_4 + \phi_4(t_2)t_5 + \phi_5(t_2)t_6]$, from the known conditions (i) and (ii) and Example 14, we obtain $F = F_4 \in \mathscr{F}$. Furthermore, from (35) we can easily derive the inequality (16). Then by Corollary 20, the theorem is proved. □

Corollary 22. *Let (X, d, L, R) be a complete FMS with $R \leqslant \max$ and let S and T be two self-mappings on (X, d, L, R). If there exist $a, b, c, d, e \geqslant 0$ and $\delta > 0$ with $a+b+c+d+e = 1+\delta$, $a + d + e < 1$, $c + d < 1$, $b + e < 1$ and $(c - b)(e - d) > 2\delta$, such that*

$$\begin{aligned} \rho_\alpha(Sx, Ty) &\leqslant a\rho_\alpha(x, y) + b\rho_\alpha(x, Sx) \\ &\quad + c\rho_\alpha(y, Ty) + d\rho_\alpha(x, Ty) + e\rho_\alpha(y, Sx), \end{aligned} \tag{36}$$

for all $x, y \in X$ and $\alpha \in (0, 1]$, then S and T have a unique common fixed point in X. Moreover, for any $x_0 \in X$, the iterative process $x_{2k+1} = Sx_{2k}$, $x_{2k+2} = Tx_{2k+1}$, $k = 0, 1, 2, \ldots$, converges to the fixed point.

Proof. Taking $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - (at_2 + bt_3 + ct_4 + dt_5 + et_6)$, note that the known conditions and Example 16, we have $F = F_6 \in \mathscr{F}$. Furthermore, from (36) we can easily derive the inequality (16). Then by Corollary 20, the corollary is proved. □

5. Applications to the Ordinary Metric Spaces and Examples

In this section, we first establish some common fixed point theorems for a pair of self-mappings satisfying an implicit Lipschitz-type condition in complete metric spaces. After that, we give two examples, by which we can claim that our conclusions are really generalizations of the early results.

Let (X, ρ) be an ordinary metric space and

$$d(x, y)(t) = \begin{cases} 1, & t = \rho(x, y), \\ 0, & t \neq \rho(x, y), \end{cases} \quad \forall x, y \in X, t \in \mathbb{R}. \quad (37)$$

Then $(X, d, \min, \max)$ is a FMS (cf. [1, 9]). It is easy to see that (X, ρ) and $(X, d, \min, \max)$ are homeomorphic and $\rho(x, y) = \rho_\alpha(x, y)$ for all $\alpha \in (0, 1]$.

Theorem 23. *Let (X, ρ) be a complete metric space and let S and T be two self-mappings on (X, ρ). If there exists $F \in \mathcal{F}$ such that*

$$\begin{aligned} F(\rho(Sx, Ty), \rho(x, y), \rho(x, Sx), \\ \rho(y, Ty), \rho(x, Ty), \rho(y, Sx)) \leqslant 0, \end{aligned} \quad (38)$$

for all $x, y \in X$. Then S and T have a unique common fixed point in X. Moreover, for any $x_0 \in X$, the iterative process $x_{2k+1} = Sx_{2k}$, $x_{2k+2} = Tx_{2k+1}$, $k = 0, 1, 2, \ldots$, converges to the fixed point.

Proof. Note that the topology and completeness of (X, ρ) and the induced FMS $(X, d, \min, \max)$ are coincident, as well as $\rho(x, y) = \rho_\alpha(x, y)$ for all $\alpha \in (0, 1]$; it is not difficult to see that the inequality (16) holds as a result of (38). Moreover, the other conditions of Corollary 20 are satisfied; thus by Corollary 20, the theorem is proved. □

Applying the same method, we can obtain the following theorem and corollary by virtue of Theorem 21 and Corollary 22, respectively.

Theorem 24. *Let (X, ρ) be a complete metric space. Let $\phi_1, \ldots, \phi_5 : \mathbb{R}^+ \to [0, 1)$ be five continuous functions which satisfy the following conditions:*

(i) $\phi_1(t) + \phi_4(t) + \phi_5(t) < 1$ *for all* $t \in \mathbb{R}^+$,

(ii) $0 < \inf_{t \geqslant 0}\{1 - \phi_2(t) - \phi_5(t)\} = a, 0 < \inf_{t \geqslant 0}\{1 - \phi_3(t) - \phi_4(t)\} = b, 0 < \sup_{t \geqslant 0}\{\phi_1(t) + \phi_2(t) + \phi_4(t)\} = C, 0 < \sup_{t \geqslant 0}\{\phi_1(t) + \phi_3(t) + \phi_5(t)\} = D$ *and* $CD < ab$.

Let S and T be two self-mappings on (X, ρ) such that

$$\begin{aligned} \rho(Sx, Ty) \leqslant{} & \phi_1(\rho(x, y))\rho(x, y) \\ & + \phi_2(\rho(x, y))\rho(x, Sx) \\ & + \phi_3(\rho(x, y))\rho(y, Ty) \\ & + \phi_4(\rho(x, y))\rho(x, Ty) \\ & + \phi_5(\rho(x, y))\rho(y, Sx), \end{aligned} \quad (39)$$

for all $x, y \in X$. Then S and T have a unique common fixed point in X. Moreover, for any $x_0 \in X$, the iterative process $x_{2k+1} = Sx_{2k}$, $x_{2k+2} = Tx_{2k+1}$, $k = 0, 1, 2, \ldots$, converges to the fixed point.

Corollary 25. *Let (X, ρ) be a complete metric space and let S and T be two self-mappings on (X, ρ). If there exist $a, b, c, d, e \geqslant 0$, and $\delta > 0$ with $a + b + c + d + e = 1 + \delta$, $a + d + e < 1$, $c + d < 1$, $b + e < 1$, and $(c - b)(e - d) > 2\delta$, such that*

$$\begin{aligned} \rho(Sx, Ty) \leqslant{} & a\rho(x, y) + b\rho(x, Sx) \\ & + c\rho(y, Ty) + d\rho(x, Ty) + e\rho(y, Sx), \end{aligned} \quad (40)$$

for all $x, y \in X$, then S and T have a unique common fixed point in X. Moreover, for any $x_0 \in X$, the iterative process $x_{2k+1} = Sx_{2k}$, $x_{2k+2} = Tx_{2k+1}$, $k = 0, 1, 2, \ldots$, converges to the fixed point.

By Theorem 23 and Example 18, we can also obtain the following corollary.

Corollary 26. *Let (X, ρ) be a complete metric space and let S and T be two self-mappings on (X, ρ). If there exist $a, b, c, d, e \geqslant 0$ with $a + b + c + d + e = 1$ and either $c > b$, $e > d$ or $c < b$, $e < d$, such that*

$$\begin{aligned} \rho(Sx, Ty) \leqslant{} & a\rho(x, y) + b\rho(x, Sx) \\ & + c\rho(y, Ty) + d\rho(x, Ty) + e\rho(y, Sx), \end{aligned} \quad (41)$$

for all $x, y \in X$, then S and T have a unique common fixed point in X. Moreover, for any $x_0 \in X$, the iterative process $x_{2k+1} = Sx_{2k}$, $x_{2k+2} = Tx_{2k+1}$, $k = 0, 1, 2, \ldots$, converges to the fixed point.

Remark 27. In the conditions of Theorem 1 of [13], if we suppose the functions $A(x, y)$, $B(x, y)$, $C(x, y)$, $D(x, y)$, and $E(x, y)$ to be continuous, then the theorem is just Theorem 24 in this paper. Therefore, in some sense, Theorem 23 is the improvement and generalizations of Theorem 1 in [13]. Meanwhile, Corollaries 25 and 26 are just Corollaries 1 and 2 in [13], respectively.

If we assume $T = S$, then, by Corollary 26, we can obtain the following corollary.

Corollary 28. *Let (X, ρ) be a complete metric space and let S be a self-mapping on (X, ρ). If there exist $a, b, c, d, e \geqslant 0$ with $a + b + c + d + e = 1$ and either $c > b$, $e > d$ or $c < b$, $e < d$, such that*

$$\begin{aligned} \rho(Sx, Sy) \leqslant{} & a\rho(x, y) + b\rho(x, Sx) \\ & + c\rho(y, Sy) + d\rho(x, Sy) + e\rho(y, Sx), \end{aligned} \quad (42)$$

for all $x, y \in X$, then S has a unique fixed point in X. Moreover, for any $x_0 \in X$, the iterative process $x_{n+1} = Sx_n$, $n = 0, 1, 2, \ldots$, converges to the fixed point.

Corollary 29. *Let (X, ρ) be a complete metric space and let S be a self-mapping on (X, ρ). If there exist $A, B, C \in [0, 1)$ with $A + B + C < 1$, such that*

$$\rho(Sx, Sy) \leqslant A\rho(x, y) + B\rho(x, Sx) + C\rho(y, Sy), \quad (43)$$

for all $x, y \in X$, then S has a unique fixed point in X. Moreover, for any $x_0 \in X$, the iterative process $x_{n+1} = Sx_n$, $n = 0, 1, 2, \ldots$, converges to the fixed point.

Proof. If $B \geqslant C$, by $A + B + C < 1$, we can take $4\delta = 1 - (A + B + C)$, and let $a = A$, $b = B + \delta$, $c = C$, $d = 2\delta$, and $e = \delta$; then we have $a + b + c + d + e = 1$, $c < b$, $e < d$, and for $x, y \in X$,

$$\begin{aligned} \rho(Sx, Sy) &\leqslant A\rho(x, y) + B\rho(x, Sx) + C\rho(y, Sy) \\ &\leqslant a\rho(x, y) + b\rho(x, Sx) + c\rho(y, Sy) \\ &\quad + d\rho(x, Sy) + e\rho(y, Sx), \end{aligned} \tag{44}$$

which imply that the conditions of Corollary 28 are satisfied. Similarly, we can prove the case of $B \leqslant C$. Therefore, by Corollary 28, the corollary is proved. □

Remark 30. Corollary 29 is just the fixed point theorem for Reich-type contraction mappings in [26, 27]. Meanwhile, we can easily see that the conditions of Corollary 28 generalize the Reich-type contractive conditions in Corollary 29.

Remark 31. If we take $F = F_1$ in Theorem 23, then Theorem 23 is the generalizations of the fixed point theorem for mappings satisfying Ćirić-type [28] contractive conditions to the common fixed point theorem for a pair of self-mappings with $a \in (0, 1/2)$.

Now we give two examples, by which we can claim that the theorems in this paper are really the generalizations of the general contractive type fixed point theorems.

Example 32. Suppose $X = \{0, 1, 2\}$, ρ is an ordinary metric; then (X, ρ) is a complete metric space.

We define two mappings $S, T : X \to X$ as follows: $S(0) = S(1) = S(2) = 0$, $T(0) = T(1) = 0$, $T(2) = 1$; then

$$\begin{aligned} \rho(S(0), T(0)) &= \rho(S(0), T(1)) \\ &= \rho(S(1), T(0)) = \rho(S(1), T(1)) \\ &= \rho(S(2), T(0)) = \rho(S(2), T(1)) = 0, \\ \rho(S(0), T(2)) &= \rho(S(1), T(2)) = \rho(S(2), T(2)) = 1. \end{aligned} \tag{45}$$

We take $\delta = 1/40$, $a = 1/40$, $b = 1/40$, $c = 9/20$, $d = 1/40$, and $e = 1/2$; then $a+b+c+d+e = 1+\delta$, $a+d+e < 1$, $c+d < 1$, $b + e < 1$, and $(c - b)(e - d) > 2\delta$; that is, these numbers satisfy the conditions of Corollary 25. Moreover,

$$\begin{aligned} \rho(S(0), T(2)) &\leqslant \frac{1}{40}\rho(0, 2) + \frac{1}{40}\rho(0, S(0)) + \frac{9}{20}\rho(2, T(2)) \\ &\quad + \frac{1}{40}\rho(0, T(2)) + \frac{1}{2}\rho(2, S(0)) = \frac{61}{40}, \\ \rho(S(1), T(2)) &\leqslant \frac{1}{40}\rho(1, 2) + \frac{1}{40}\rho(1, S(1)) + \frac{9}{20}\rho(2, T(2)) \\ &\quad + \frac{1}{40}\rho(1, T(2)) + \frac{1}{2}\rho(2, S(1)) = \frac{3}{2}, \\ \rho(S(2), T(2)) &\leqslant \frac{1}{40}\rho(2, 2) + \frac{1}{40}\rho(2, S(2)) + \frac{9}{20}\rho(2, T(2)) \\ &\quad + \frac{1}{40}\rho(2, T(2)) + \frac{1}{2}\rho(2, S(2)) = \frac{61}{40}. \end{aligned} \tag{46}$$

Hence S, T satisfy the nonexpansive condition, and they have a unique common fixed point. But for any nonnegative real numbers α, β, γ with $0 < \alpha + 2\beta + 2\gamma < 1$, we have

$$\begin{aligned} &\alpha\rho(1, 2) + \beta(\rho(1, S(1)) + \rho(2, T(2))) \\ &\quad + \gamma(\rho(1, T(2)) + \rho(2, S(1))) \\ &= \alpha + 2\beta + 2\gamma < 1 = \rho(S(1), T(2)); \end{aligned} \tag{47}$$

thus S, T cannot satisfy the general contractive type condition $0 < \alpha + 2\beta + 2\gamma < 1$.

Remark 33. For the examples satisfying the conditions $a+b+c+d+e = 1$ and either $c > b$, $e > d$ or $c < b$, $e < d$ in Corollary 26, we refer the reader to Example 2 in [13].

Example 34. Suppose $X = \{0, 1, 2\}$; ρ is a discrete metric, then (X, ρ) is a complete metric space.

Likewise, we define the mappings $S, T : X \to X$ as follows: $S(0) = S(1) = S(2) = 0$, $T(0) = T(1) = 0$, $T(2) = 1$; then

$$\begin{aligned} \rho(S(0), T(0)) &= \rho(S(0), T(1)) \\ &= \rho(S(1), T(0)) = \rho(S(1), T(1)) \\ &= \rho(S(2), T(0)) = \rho(S(2), T(1)) = 0, \\ \rho(S(0), T(2)) &= \rho(S(1), T(2)) = \rho(S(2), T(2)) = 1. \end{aligned} \tag{48}$$

We take $\delta = 1/40$, $a = 1/40$, $b = 1/40$, $c = 9/20$, $d = 1/40$, $e = 1/2$, then $a+b+c+d+e = 1+\delta$, $a+d+e < 1$, $c+d < 1$, $b+e < 1$, and $(c-b)(e-d) > 2\delta$; that is, these numbers satisfy the conditions of Corollary 25. Moreover,

$$\begin{aligned} \rho(S(0), T(2)) &= \frac{1}{40}\rho(0, 2) + \frac{1}{40}\rho(0, S(0)) + \frac{9}{20}\rho(2, T(2)) \\ &\quad + \frac{1}{40}\rho(0, T(2)) + \frac{1}{2}\rho(2, S(0)) = 1, \\ \rho(S(1), T(2)) &= \frac{1}{40}\rho(1, 2) + \frac{1}{40}\rho(1, S(1)) + \frac{9}{20}\rho(2, T(2)) \\ &\quad + \frac{1}{40}\rho(1, T(2)) + \frac{1}{2}\rho(2, S(1)) = 1, \\ \rho(S(2), T(2)) &= \frac{1}{40}\rho(2, 2) + \frac{1}{40}\rho(2, S(2)) + \frac{9}{20}\rho(2, T(2)) \\ &\quad + \frac{1}{40}\rho(2, T(2)) + \frac{1}{2}\rho(2, S(2)) = 1. \end{aligned} \tag{49}$$

Hence S, T satisfy the nonexpansive condition, and they have a unique common fixed point. While for any nonnegative real numbers $\alpha, \beta, \gamma, \eta, \mu < 1$ with $\alpha + \beta + \gamma + \eta + \mu < 1$, we have

$$\begin{aligned} &\alpha\rho(1,2) + \beta\rho(1,S(1)) + \gamma\rho(2,T(2)) + \eta\rho(1,T(2)) \\ &\quad + \mu\rho(2,S(1)) = \alpha + \beta + \gamma + \mu < 1 = \rho(S(1),T(2)); \end{aligned} \tag{50}$$

thus S, T cannot satisfy the general contractive type condition $\alpha + \beta + \gamma + \eta + \mu < 1$.

6. Conclusion

In this paper, we studied the existence and uniqueness of fixed points for nonlinear contractions in fuzzy metric spaces in the sense of Kaleva-Seikkala. By virtue of a level-cut method, we established relationships between fuzzy metric and a family of quasi-metrics. Using them, we obtained some common fixed point theorems for a pair of self-mappings satisfying an implicit Lipschitz-type conditions in fuzzy metric spaces. Obviously, the present investigation enriches our knowledge of fixed points in fuzzy metric spaces. But, the work in this paper with respect to fixed point in fuzzy setting is based on $R \leqslant \max$. The question whether the discussion can be carried out under the weaker condition (R-2) deserves our further investigation.

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