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Research Article

Some Identities on the Generalized q-Bernoulli, q-Euler, and q-Genocchi Polynomials

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Mahmudov (2012, 2013) introduced and investigated some q-extensions of the q-Bernoulli polynomials $\mathscr{B}_{n,q}^{(\alpha)}(x,y)$ of order α, the q-Euler polynomials $\mathscr{E}_{n,q}^{(\alpha)}(x,y)$ of order α, and the q-Genocchi polynomials $\mathscr{G}_{n,q}^{(\alpha)}(x,y)$ of order α. In this paper, we give some identities for $\mathscr{B}_{n,q}^{(\alpha)}(x,y)$, $\mathscr{G}_{n,q}^{(\alpha)}(x,y)$, and $\mathscr{E}_{n,q}^{(\alpha)}(x,y)$ and the recurrence relations between these polynomials. This is an analogous result to the q-extension of the Srivastava-Pintér addition theorem in Mahmudov (2013).

1. Introduction, Definitions, and Notations

Throughout this paper, we always make use of the following notation: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{C}$ denotes the set of complex numbers. The q-numbers and q-factorial are defined by

$$[a]_q = \frac{1-q^a}{1-q}, \quad q \neq 1, \qquad [n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q, \tag{1}$$

respectively, where $[0]_q! = 1$, $n \in \mathbb{N}$, and $a \in \mathbb{C}$. The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q:q)_n}{(q:q)_{n-k}(q:q)_k}, \tag{2}$$

where $(q:q)_n = (1-q)\cdots(1-q^n)$. The q-analogue of the function $(x+y)_q^n$ is defined by

$$(x+y)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(k(k-1))/2} x^{n-k} y^k. \tag{3}$$

The q-binomial formula is known as

$$\begin{aligned}(n;q)_a &= (1-a)_q^n \\ &= \prod_{j=0}^{n-1}\left(1-q^j a\right) \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(k(k-1))/2}(-1)^k a^k.\end{aligned} \tag{4}$$

The q-exponential functions are given by

$$\begin{aligned} e_q(z) &= \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \\ &= \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)q^k z)}, \quad 0<|q|<1, \\ &\qquad |z| < \frac{1}{|1-q|},\end{aligned}$$

$$E_q(z)=\sum_{n=0}^{\infty}q^{(n(n-1))/2}\frac{z^n}{[n]_q!}=\prod_{k=0}^{\infty}\left(1+(1-q)q^kz\right),\quad 0<|q|<1,\ z\in\mathbb{C}. \tag{5}$$

From these forms, we easily see that $e_q(z)E_q(-z)=1$. Moreover, $D_qe_q(z)=e_q(z)$ and $D_qE_q(z)=E_q(qz)$, where D_q is defined by

$$D_qf(z)=\frac{f(qz)-f(z)}{qz-z},\quad 0<|q|<1,\ 0\neq z\in\mathbb{C}. \tag{6}$$

The previous q-standard notation can be found in [1, 2]. Carlitz firstly extended the classical Bernoulli numbers and polynomials and Euler numbers and polynomials [3, 4]. There are numerous recent investigations on this subject by many other authors. Among them are Cenkci et al. [5, 6], Choi et al. [1], Cheon [7], Kim [8], Kurt [9], Kurt [10], Luo and Srivastava [11–13], Srivastava et al. [14, 15], Natalini and Bernardini [16], Tremblay et al. [17, 18], Gaboury and Kurt [19], Mahmudov [2, 20, 21], Araci et al. [22], and Kupershmidt [23].

Mahmudov defined and studied the properties of the following generalized q-Bernoulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y)$ of order α and q-Euler polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x,y)$ of order α as follows [2].

Let $q\in\mathbb{C}$, $\alpha\in\mathbb{N}$, and $0<|q|<1$. The q-Bernoulli numbers $\mathcal{B}_{n,q}^{(\alpha)}$ and polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y)$ in x and y of order α are defined by means of the generating functions:

$$\sum_{n=0}^{\infty}\mathcal{B}_{n,q}^{(\alpha)}\frac{t^n}{[n]_q!}=\left(\frac{t}{e_q(t)-1}\right)^{\alpha},\quad |t|<2\pi, \tag{7}$$

$$\sum_{n=0}^{\infty}\mathcal{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}=\left(\frac{t}{e_q(t)-1}\right)^{\alpha}e_q(tx)E_q(ty),\quad |t|<2\pi. \tag{8}$$

The q-Euler numbers $\mathcal{E}_{n,q}^{(\alpha)}$ and polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x,y)$ in x and y of order α are defined by means of the generating functions:

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,q}^{(\alpha)}\frac{t^n}{[n]_q!}=\left(\frac{2}{e_q(t)+1}\right)^{\alpha},\quad |t|<\pi, \tag{9}$$

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}=\left(\frac{2}{e_q(t)+1}\right)^{\alpha}e_q(tx)E_q(ty),\quad |t|<\pi. \tag{10}$$

The q-Genocchi numbers $\mathcal{G}_{n,q}^{(\alpha)}$ and polynomials $\mathcal{G}_{n,q}^{(\alpha)}(x,y)$ in x and y of order α are defined by means of the generating functions:

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(\alpha)}\frac{t^n}{[n]_q!}=\left(\frac{2t}{e_q(t)+1}\right)^{\alpha},\quad |t|<\pi, \tag{11}$$

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}=\left(\frac{2t}{e_q(t)+1}\right)^{\alpha}e_q(tx)E_q(ty),\quad |t|<\pi. \tag{12}$$

The familiar q-Stirling numbers $S_{2,q}(n,k)$ of the second kind are defined by

$$\frac{\left(e_q(t)-1\right)^k}{[k]_q!}=\sum_{n=0}^{\infty}S_{2,q}(n,k)\frac{t^n}{[n]_q!}. \tag{13}$$

It is obvious that

$$\begin{gathered}\mathcal{B}_{n,q}^{(1)}(x,y):=\mathcal{B}_{n,q}(x,y),\qquad \mathcal{E}_{n,q}^{(1)}(x,y):=\mathcal{E}_{n,q}(x,y),\\ \mathcal{G}_{n,q}^{(1)}(x,y):=\mathcal{G}_{n,q}(x,y),\qquad \mathcal{B}_{n,q}(0,0):=\mathcal{B}_{n,q},\\ \mathcal{E}_{n,q}(0,0):=\mathcal{E}_{n,q},\qquad \mathcal{G}_{n,q}(0,0):=\mathcal{G}_{n,q},\\ \mathcal{B}_{n,q}^{(\alpha)}=\mathcal{B}_{n,q}^{(\alpha)}(0,0),\\ \lim_{q\to1^-}\mathcal{B}_{n,q}^{(\alpha)}(x,y)=\mathcal{B}_{n}^{(\alpha)}(x+y),\\ \lim_{q\to1^-}\mathcal{B}_{n,q}^{(\alpha)}=\mathcal{B}_{n}^{(\alpha)},\qquad \mathcal{E}_{n,q}^{(\alpha)}=\mathcal{E}_{n,q}^{(\alpha)}(0,0),\\ \lim_{q\to1^-}\mathcal{E}_{n,q}^{(\alpha)}(x,y)=\mathcal{E}_{n}^{(\alpha)}(x+y),\qquad \lim_{q\to1^-}\mathcal{E}_{n,q}^{(\alpha)}=\mathcal{E}_{n}^{(\alpha)},\\ \mathcal{G}_{n,q}^{(\alpha)}=\mathcal{G}_{n,q}^{(\alpha)}(0,0),\qquad \lim_{q\to1^-}\mathcal{G}_{n,q}^{(\alpha)}(x,y)=\mathcal{G}_{n}^{(\alpha)}(x+y),\\ \lim_{q\to1^-}\mathcal{G}_{n,q}^{(\alpha)}=\mathcal{G}_{n}^{(\alpha)}.\end{gathered} \tag{14}$$

From (8) and (10), it is easy to check that

$$\begin{aligned}\mathcal{B}_{n,q}^{(\alpha)}(x,y)&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathcal{B}_{n-k,q}(x,0)\,\mathcal{B}_{k,q}^{(\alpha-1)}(0,y),\\ \mathcal{E}_{n,q}^{(\alpha)}(x,y)&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathcal{E}_{n-k,q}^{(\alpha-1)}(x,0)\,\mathcal{E}_{k,q}(0,y).\end{aligned} \tag{15}$$

In this work, we give some identities for the q-Bernoulli polynomials. Also, we give some relations between the q-Bernoulli polynomials and q-Euler polynomials and the q-Genocchi polynomials and q-Bernoulli polynomials. Furthermore, we give a different form of the analogue of the Srivastava-Pintér addition theorem. More precisely, we prove the following theorems.

Theorem 1. *There are the following relations between the q-Bernoulli polynomials and q-Stirling numbers of the second kind:*

$$\mathscr{B}_{n,q}^{(\alpha)}(x,y)=\frac{[k]_q![n]_q!}{[n+k]_q!}\times\sum_{l=0}^{n+k}\begin{bmatrix} n+k\\ l\end{bmatrix}_q\mathscr{B}_{l,q}^{(\alpha+k)}(x,y)\times S_{2,q}(n+k-l,k),\tag{16}$$

$$\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\mathscr{B}_{n-k,q}^{(\alpha)}(x,y)[\alpha]_q!S_{2,q}(k,\alpha)=\sum_{l=0}^{n-\alpha}\begin{bmatrix} n-\alpha\\ l\end{bmatrix}_q\frac{[n]_q!}{[n-\alpha]_q!}x^{n-\alpha-l}y^lq^{\binom{l}{2}},\tag{17}$$

where $q\in\mathbb{C}$, $\alpha,n\in\mathbb{N}$, *and* $0<|q|<1$.

Theorem 2. *The q-Stirling numbers of the second kind satisfy the following relations:*

$$\mathscr{E}_{n,q}^{(\alpha)}(x,y)=\sum_{j=0}^{\infty}\binom{-\alpha}{j}\frac{1}{2^j}[j]_q!\times\sum_{p=0}^{n}\begin{bmatrix} n\\ p\end{bmatrix}_qS_{2,q}(n-p,j)\times\sum_{l=0}^{p}\begin{bmatrix} p\\ l\end{bmatrix}_qx^{p-l}y^lq^{\binom{l}{2}},\tag{18}$$

$$\mathscr{B}_{n,q}^{(\alpha)}=[\alpha]_q!\sum_{j=0}^{\infty}\binom{-\alpha}{j}\times\sum_{k=0}^{j}\binom{j}{k}[k]_q!\frac{S_{2,q}(n+k,k)}{[n+k]_q!}[k]_q!(-1)^{j-k},$$

$$\mathscr{B}_{n,q}^{(-\alpha)}(x,y)=[\alpha]_q!\sum_{m=0}^{n+\alpha}\begin{bmatrix} n+\alpha\\ m\end{bmatrix}_qS_{2,q}(m,\alpha)\times(x+y)_q^{n+\alpha-m}\frac{[n]_q!}{[n+\alpha]_q!},\tag{19}$$

where $q\in\mathbb{C}$, $\alpha,n\in\mathbb{N}$, *and* $0<|q|<1$.

Theorem 3. *The generalized q-Euler polynomials satisfy the following relation:*

$$\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\mathscr{E}_{k,q}(x,y)=2(x+y)_q^n-\mathscr{E}_{n,q}(x,y),\tag{20}$$

where $q\in\mathbb{C}$, α, $n\in\mathbb{N}$, *and* $0<|q|<1$.

Theorem 4. *The polynomials* $B_{n,q}(x,y)$ *and* $\mathscr{G}_{n,q}(x,y)$ *satisfy the following difference relationships:*

$$\mathscr{B}_{n,q}(x,y)=\sum_{\substack{l=0\\ l\neq n}}^{n+1}\begin{bmatrix} n+1\\ l\end{bmatrix}_q\frac{1}{[n+1]_q}\mathscr{G}_{l,q}(x,y)\mathscr{B}_{n+1-l,q},\tag{21}$$

$$\mathscr{G}_{n,q}(x,y)=-2\sum_{\substack{l=0\\ l\neq n}}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q\frac{1}{[l+1]_q}\mathscr{G}_{l+1,q}\mathscr{B}_{n-l,q}(x,y),\tag{22}$$

where $q\in\mathbb{C}$, $\alpha,n\in\mathbb{N}$, *and* $0<|q|<1$.

Theorem 5. *There is the following relation between the generalized q-Euler polynomials and generalized q-Bernoulli polynomials:*

$$\mathscr{E}_{n,q}^{(\alpha)}(x,y)=\left\{\sum_{s=0}^{n+1}\begin{bmatrix} n+1\\ s\end{bmatrix}_q\sum_{l=0}^{s}\begin{bmatrix} s\\ l\end{bmatrix}_q\mathscr{B}_{s-l,q}(mx,0)-\sum_{l=0}^{n+1}\begin{bmatrix} n+1\\ l\end{bmatrix}_q\mathscr{B}_{n+1-l,q}(mx,0)\right\}\times\frac{m}{[n+1]_q!}\mathscr{E}_{l,q}^{(\alpha)}(0,y)m^{l-n-1},\tag{23}$$

where $q\in\mathbb{C}$, $\alpha,n\in\mathbb{N}$, *and* $0<|q|<1$.

2. Proof of the Theorems

Lemma 6. *The generalized q-Bernoulli polynomials, q-Euler polynomials, and q-Genocchi polynomials satisfy the following relations:*

$$\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\mathscr{B}_{k,q}^{(\alpha)}(x,y)\mathscr{B}_{n-k,q}^{(-\alpha)}=(x+y)_q^n,$$

$$\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\mathscr{B}_{k,q}^{(\alpha)}(0,y)\mathscr{B}_{n-k,q}^{(-\alpha)}=q^{(n(n-1))/2}y^n,$$

$$\mathscr{B}_{n,q}^{(\alpha)}(x,y)=\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q\mathscr{B}_{n-l,q}^{(\alpha)}(0,y)\times\sum_{k=0}^{l}\begin{bmatrix} l\\ k\end{bmatrix}_q\mathscr{E}_{k,q}^{(\alpha)}(x,0)\mathscr{E}_{l-k,q}^{(-\alpha)},$$

$$\mathscr{E}_{n,q}^{(\alpha)}(x,y)=\sum_{l=0}^{n}\begin{bmatrix}n\\ l\end{bmatrix}_q \mathscr{E}_{n-l,q}^{(\alpha)}(0,y)$$
$$\times\sum_{k=0}^{l}\begin{bmatrix}l\\ k\end{bmatrix}_q \mathscr{B}_{k,q}^{(\alpha)}(x,0)\,\mathscr{B}_{l-k,q}^{(-\alpha)},$$
$$\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}_q \mathscr{G}_{k,q}(x,y)+\mathscr{G}_{n,q}(x,y)$$
$$=2[n]_q(x+y)_q^{n-1},$$
$$\mathscr{G}_{n,q}^{(\alpha-\beta)}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}_q \mathscr{G}_{k,q}^{(\alpha)}(x,0)\,\mathscr{G}_{n-k,q}^{(-\beta)}(0,y). \tag{24}$$

Proof. The proof of this lemma can be found from (7)–(12). □

Proof of Theorem 1. By (8) and (13) we have

$$\sum_{n=0}^{\infty}\mathscr{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}$$
$$=\left(\frac{t}{e_q(t)-1}\right)^{\alpha}e_q(tx)\,E_q(ty)$$
$$\times\frac{[k]_q!}{\left(e_q(t)-1\right)^k}\frac{\left(e_q(t)-1\right)^k}{[k]_q!}$$
$$=[k]_q!\frac{t^{\alpha}}{\left(e_q(t)-1\right)^{\alpha+k}}e_q(tx)\,E_q(ty)$$
$$\times\sum_{m=0}^{\infty}S_{2,q}(m,k)\frac{t^m}{[m]_q!}$$
$$=[k]_q!t^{-k}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{bmatrix}n\\ l\end{bmatrix}_q \mathscr{B}_{l,q}^{(\alpha+k)} \tag{25}$$
$$\times(x,y)\,S_{2,q}(n-l,k)\frac{t^n}{[n]_q!}$$
$$=[k]_q!\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{bmatrix}n\\ l\end{bmatrix}_q \mathscr{B}_{l,q}^{(\alpha+k)}$$
$$\times(x,y)\,S_{2,q}(n-l,k)\frac{t^{n-k}}{[n]_q!}$$
$$=[k]_q!\sum_{n=-k}^{\infty}\sum_{l=0}^{n+k}\begin{bmatrix}n+k\\ l\end{bmatrix}_q \mathscr{B}_{l,q}^{(\alpha+k)}$$
$$\times(x,y)\,S_{2,q}(n+k-l,k)\frac{t^{n-k}}{[n]_q!}.$$

Equating the coefficients of $(t^n/[n]_q!)$, we obtain (16).
Similarly, we have (17). □

Proof of Theorem 2. Combining (10) and (13), we obtain

$$\left(\frac{2}{e_q(t)+1}\right)^{\alpha}=\left(1+\frac{e_q(t)-1}{2}\right)^{(-\alpha)}$$
$$=\sum_{j=0}^{\infty}\binom{-\alpha}{j}\left(\frac{e_q(t)-1}{2}\right)^{(j)},$$
$$\sum_{n=0}^{\infty}\mathscr{E}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}$$
$$=\sum_{j=0}^{\infty}\binom{-\alpha}{j}\left(\frac{e_q(t)-1}{2}\right)^{(j)}e_q(tx)\,E_q(ty)$$
$$=\sum_{j=0}^{\infty}\binom{-\alpha}{j}\frac{1}{2^j}[j]_q!\sum_{n=0}^{\infty}S_{2,q}(n,j)\frac{t^n}{[n]_q!}$$
$$\times\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}_q x^{n-k}y^kq^{\binom{k}{2}}\frac{t^n}{[n]_q!}$$
$$=\sum_{j=0}^{\infty}\binom{-\alpha}{j}\frac{1}{2^j}$$
$$\times\sum_{n=0}^{\infty}\sum_{p=0}^{n}\begin{bmatrix}n\\ p\end{bmatrix}_q [j]_q!S_{2,q}(n-p,j)$$
$$\times\sum_{l=0}^{p}\begin{bmatrix}p\\ l\end{bmatrix}_q x^{p-l}y^lq^{\binom{l}{2}}\frac{t^n}{[n]_q!}. \tag{26}$$

Comparing the coefficients of $(t^n/[n]_q!)$, we find (18). Similarly, we have (19). □

Proof of Theorem 3. It is obvious that

$$\frac{-2}{\left(e_q(t)+1\right)e_q(t)}=\frac{2}{\left(e_q(t)+1\right)}-\frac{2}{e_q(t)}. \tag{27}$$

We write it as

$$\frac{-2}{e_q(t)+1}\frac{e_q(tx)\,E_q(ty)}{e_q(t)}=\frac{2}{e_q(t)+1}e_q(tx)\,E_q(ty)$$
$$-\frac{2}{e_q(t)}e_q(tx)\,E_q(ty),$$
$$\frac{-2}{e_q(t)+1}e_q(tx)\,E_q(ty)=\frac{2}{e_q(t)+1}e_q(tx)\,E_q(ty)$$
$$-2e_q(tx)\,E_q(ty)$$

$$
\begin{aligned}
&-\sum_{n=0}^{\infty}\mathscr{E}_{n,q}(x,y)\frac{t^n}{[n]_q!}\\
&=\sum_{n=0}^{\infty}\mathscr{E}_{n,q}(x,y)\frac{t^n}{[n]_q!}\\
&\quad\times\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}-2\sum_{n=0}^{\infty}(x+y)_q^n\frac{t^n}{[n]_q!}.
\end{aligned}
\tag{28}
$$

Using the Cauchy product and comparing the coefficients of $(t^n/[n]_q!)$, we have

$$
\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{E}_{k,q}(x,y)=2(x+y)_q^n-\mathscr{E}_{k,q}(x,y). \tag{29}
$$

□

Finally, we consider the interesting relationships between the q-Bernoulli polynomials and q-Genocchi polynomials and the q-Euler polynomials and q-Bernoulli polynomials. These relations are q-analogues to the Srivastava-Pintér addition theorems.

Proof of Theorem 4. It follows immediately that

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\mathscr{B}_{n,q}(x,y)\frac{t^n}{[n]_q!}\\
&=\frac{1}{2}\frac{2te_q(tx)E_q(ty)}{e_q(t)+1}\\
&\quad+\frac{1}{t}\left(\frac{t}{e_q(t)-1}\right)\frac{2t}{e_q(t)+1}e_q(tx)E_q(ty)\\
&=\frac{1}{2}\sum_{n=0}^{\infty}\mathscr{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}+\frac{1}{t}\\
&\quad\times\sum_{n=0}^{\infty}\mathscr{B}_{n,q}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathscr{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}\\
&=\frac{1}{2}\sum_{n=0}^{\infty}\mathscr{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}\\
&\quad+\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q\frac{1}{[n]_q}\mathscr{G}_{l,q}(x,y)\\
&\qquad\times\mathscr{B}_{n-l,q}\frac{t^{n-1}}{[n-1]_q!}\\
&=\frac{1}{2}\sum_{n=0}^{\infty}\mathscr{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}\\
&\quad+\sum_{n=0}^{\infty}\left(-\frac{1}{2}\mathscr{G}_{n,q}(x,y)\right.\\
&\quad+\sum_{l=0}^{n+1}\begin{bmatrix} n+1 \\ l \end{bmatrix}_q\frac{1}{[n+1]_q}\\
&\qquad\left.\times\mathscr{G}_{l,q}(x,y)\mathscr{B}_{n+1-l,q}\right)\frac{t^n}{[n]_q!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{\substack{l=0\\ l\neq n}}^{n+1}\begin{bmatrix} n+1 \\ l \end{bmatrix}_q\frac{1}{[n+1]_q}\right.\\
&\qquad\left.\times\mathscr{G}_{l,q}(x,y)\mathscr{B}_{n+1-l,q}\right)\frac{t^n}{[n]_q!}.
\end{aligned}
\tag{30}
$$

Equating the coefficients of $(t^n/[n]_q!)$, we have (21).

In a similar fashion, (12) yields

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\mathscr{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}\\
&=\frac{1}{t}\left(\frac{2t}{e_q(t)+1}\left(e_q(t)-1\right)\right)\left(\frac{te_q(tx)E_q(ty)}{e_q(t)-1}\right)\\
&=\frac{1}{t}\left(2t-2\frac{2t}{e_q(t)+1}\right)\left(\frac{t}{e_q(t)-1}e_q(tx)E_q(ty)\right)\\
&=\frac{1}{t}\left(2t-2\sum_{n=0}^{\infty}\mathscr{G}_{n,q}\frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty}\mathscr{B}_{n,q}(x,y)\frac{t^n}{[n]_q!}\right)\\
&=\frac{1}{t}\left(-2\sum_{l=0}^{\infty}\frac{1}{[l+1]_q!}\mathscr{G}_{l+1,q}\frac{t^{l+1}}{[l]_q!}\right)\left(\sum_{n=0}^{\infty}\mathscr{B}_{n,q}(x,y)\frac{t^n}{[n]_q!}\right)\\
&=\sum_{n=1}^{\infty}\left(-2\sum_{\substack{l=0\\ l\neq n}}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q\frac{\mathscr{G}_{l+1,q}}{[l+1]_q}\mathscr{B}_{n-l,q}(x,y)\right)\frac{t^n}{[n]_q!}.
\end{aligned}
\tag{31}
$$

Comparing the coefficients of $(t^n/[n]_q!)$, we have (22). □

Proof of Theorem 5. By (10), we write

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\mathscr{E}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}\\
&=\left(\frac{2}{e_q(t)+1}\right)^{\alpha}\\
&\quad\times E_q(ty)\frac{e_q(t/m)-1}{(t/m)}\frac{(t/m)}{e_q(t/m)\ -1}e_q((t/m)mx)
\end{aligned}
$$

$$\begin{aligned}
&= \frac{m}{t}\left\{\sum_{n=0}^{\infty}\mathscr{E}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}\right.\\
&\quad\times\sum_{n=0}^{\infty}\mathscr{B}_{n,q}(mx,0)\frac{t^n}{m^n[n]_q!}\\
&\quad\times\sum_{n=0}^{\infty}\frac{t^n}{m^n[n]_q!}-\sum_{n=0}^{\infty}\mathscr{E}_{n,q}^{(\alpha)}(0,y)\frac{t^n}{[n]_q!}\\
&\quad\left.\times\sum_{n=0}^{\infty}\mathscr{B}_{n,q}(mx,0)\frac{t^n}{m^n[n]_q!}\right\}\\
&= m\sum_{n=-1}^{\infty}\frac{1}{[n+1]_q}\\
&\quad\times\left\{\sum_{s=0}^{n+1}\begin{bmatrix}n+1\\ s\end{bmatrix}_q\sum_{l=0}^{s}\begin{bmatrix}s\\ l\end{bmatrix}_q\mathscr{B}_{s-l,q}(mx,0)\right.\\
&\qquad\left.-\sum_{l=0}^{n+1}\begin{bmatrix}n+1\\ l\end{bmatrix}_q\mathscr{B}_{n+1-l,q}(mx,0)\right\}\\
&\quad\times\frac{m}{[n+1]_q!}\mathscr{E}_{l,q}^{(\alpha)}(0,y)\,m^{l-n-1}\frac{t^n}{[n]_q!}.
\end{aligned}\tag{32}$$

By equating the coefficients of $(t^n/[n]_q!)$, we get the theorem. □

Remark 7. There are many different relationships which are analogues to the Srivastava-Pintér addition theorems at these polynomials.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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