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Research Article

Limits of Riemann Solutions to the Relativistic Euler Systems for Chaplygin Gas as Pressure Vanishes

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Vanishing pressure limits of Riemann solutions to relativistic Euler system for Chaplygin gas are identified and analyzed in detail. Unlike the polytropic or barotropic gas case, as the parameter decreases to a critical value, the two-shock solution converges firstly to a delta shock wave solution to the same system. It is shown that, as the parameter decreases, the strength of the delta shock increases. Then as the pressure vanishes ultimately, the solution is nothing but the delta shock wave solution to the zero pressure relativistic Euler system. Meanwhile, the two-rarefaction wave solution and the solution containing one-rarefaction wave and one-shock wave tend to the vacuum solution and the contact discontinuity solution to the zero pressure relativistic Euler system, respectively.

1. Introduction

The relativistic fluid dynamics plays a basic and significant role in many physics fields, such as astrophysics, cosmology, and nuclear physics [1]. The Euler system of conservation laws of energy and momentum in special relativity reads

$$\begin{aligned} &\left(\frac{(\rho c^2+p)v^2}{c^2(c^2-v^2)}+\rho\right)_t+\left(\frac{(\rho c^2+p)v}{c^2-v^2}\right)_x=0,\\ &\left(\frac{(\rho c^2+p)v}{c^2-v^2}\right)_t+\left(\frac{(\rho c^2+p)v^2}{c^2-v^2}+p\right)_x=0. \end{aligned} \tag{1}$$

Here ρ, p represent the proper mass-energy density and the pressure, respectively, constant c is the light speed, the sonic speed $\sqrt{p'(\rho)}$ should be not more than the light speed c, and v is the particle speed satisfying the relativistic constraint $v^2 < c^2$. In this paper, we consider the limit behavior of Riemann solutions of (1) for Chaplygin gas as pressure vanishes.

Formally, system (1) in the Newtonian limit reduces to the classical isentropic Euler equations for compressible fluids as $v/c \to 0$:

$$\begin{aligned} &\rho_t+(\rho v)_x=0,\\ &(\rho v)_t+\left(\rho v^2+p\right)_x=0. \end{aligned} \tag{2}$$

Thus system (1) can also be viewed as the relativistic generalization of system (2). System (2) and its generalized equation have been investigated intensively and widely as the typical system of nonlinear hyperbolic conservation laws [2–8]. One can refer to [9–11] for more systematic results on the systems of nonlinear hyperbolic conservation laws. For Chaplygin gas, Brenier obtained the Riemann solutions to system (2), in which there appears concentration phenomenon for some certain initial data [12]. That is, for Chaplygin gas, there exists a unique Riemann solution to system (2) involving the so-called delta shock wave in some cases. Chaplygin gas dynamics was widely studied recently, and there are some interesting and important results, especially for Riemann problems. We refer to [13, 14] and the references cited therein for more related results.

System (1) can be formally transformed into the following model as the pressure vanishes:

$$\begin{aligned} &\left(\frac{\rho}{c^2-v^2}\right)_t+\left(\frac{\rho v}{c^2-v^2}\right)_x=0,\\ &\left(\frac{\rho v}{c^2-v^2}\right)_t+\left(\frac{\rho v^2}{c^2-v^2}\right)_x=0. \end{aligned} \tag{3}$$

We call system (3) the zero pressure relativistic Euler system. In fact, it can be viewed as a relativistic version of the transport equations:

$$\begin{aligned} &\rho_t+(\rho v)_x=0,\\ &(\rho v)_t+\left(\rho v^2\right)_x=0, \end{aligned} \tag{4}$$

which can be used to describe the motion process of free particles sticking under collision in the low temperature and the information of large-scale structures in the universe [15, 16]. System (4) has been investigated extensively in the past two decades. In [17], Sheng and Zhang obtained the Riemann solutions of (4) involving delta shock wave or vacuum. The delta shock wave is characterized by the location, propagation speed, and weight which is the mass of concentrated particles. This shows that the delta shock can be regarded as the galaxies in the universe or the concentration of particles. We can also see [18–20] for the related results about the delta shock wave.

As for system (1), we recommend [21–23] for some results, in which the elementary nonlinear waves have been analyzed. The Riemann solutions and the BV weak solutions of Cauchy problem for system (1) under the equation of state $p=\sigma^2\rho$ (σ is the sound speed satisfying $0<\sigma<c$) were obtained analytically by Smoller and Temple [24]. Then in [25], Chen generalized their results with the general equation of state $p=p(\rho)$ satisfying $p'(\rho)>0$ and $p''(\rho)\geq 0$. Based on the results in [25], Hsu et al. [26] proved the existence of global weak solutions to (1) with a more realistic equation of state and L^∞ initial data containing vacuum state in the framework of compensated compactness. The existence of entropy solutions for problems without vacuum state was established by LeFloch and Yamazaki [27]. More results about entropy solutions to system (1) can be found in [28–31]. We also refer to [32] for a multidimensional piston problem and [33] for the blow-up of solutions. For Chaplygin gas, Cheng and Yang [34] considered the Riemann problem of system (1). The solutions are a bit different from those for polytropic gas. System (1) in this case is linearly degenerate. Thus there appear five kinds of solutions, in which four cases involve contact discontinuities and another contains delta shock for some certain initial data.

In the present paper, we focus on the limit of Riemann solutions to system

$$\begin{aligned} &\left(\frac{(\rho c^2+\varepsilon p)v^2}{c^2(c^2-v^2)}+\rho\right)_t+\left(\frac{(\rho c^2+\varepsilon p)v}{c^2-v^2}\right)_x=0,\\ &\left(\frac{(\rho c^2+\varepsilon p)v}{c^2-v^2}\right)_t+\left(\frac{(\rho c^2+\varepsilon p)v^2}{c^2-v^2}+\varepsilon p\right)_x=0, \end{aligned} \tag{5}$$

as pressure vanishes with the pressure function for Chaplygin gas:

$$p=-\frac{1}{\rho}. \tag{6}$$

It is clear to see that the Chaplygin equations can be mathematically expressed as an isentropic gas dynamics system with a negative pressure and can be used to depict some dark-energy models in cosmology [35].

The same problem to Euler system for isothermal case was carried out in [36]. Li proved that when temperature drops to zero, the solution containing two shock waves converges to the delta shock solution to transport equations (4) and the solution containing two rarefaction waves converges to the solution involving vacuum to system (4). Instead of isothermal case, Chen and Liu [37] considered the formation of delta shock and vacuum state of the Riemann solutions to Euler system for polytropic gas in which they took the equation of state as $P=\varepsilon p$ for $p=\rho^\gamma/\gamma$ ($\gamma>1$). Then, in [38, 39], Yin and Sheng extended the results above to system (1). In [40], Mitrović and Nedeljkov considered the generalized pressureless gas dynamics model with a scaled pressure term:

$$\begin{aligned} &\rho_t+(\rho g(u))_x=0,\\ &(\rho u)_t+(\rho u g(u)+\varepsilon p(\rho))_x=0, \end{aligned} \tag{7}$$

where $p=\kappa\rho^\gamma$ for $1<\gamma<3$ and g is a nondecreasing function. They extended the results in [37] to the system (7) and found that the delta shock wave appears as the limit of the solution involving two shock waves as ε goes to zero. We also refer to [41, 42] for the related results.

In this paper, we mainly describe the limit of Riemann solutions to system (5)-(6) as pressure vanishes. Unlike the cases we mentioned above, system (5)-(6) is linearly degenerate. That is, there appears delta shock in Riemann solutions. Thus, it is natural for us to guess that different structures or components of Riemann solutions in this case may directly cause some difference and interest during the process of vanishing pressure limit. This motivates us to do this work.

We will show that, as ε drops to a certain critical value $\varepsilon_1>0$ which only depends on the given Riemann initial data $(v_\pm,\rho_\pm)$, the solution involving two shock waves converges to a delta shock wave of the same system (5)-(6). When ε continues to decrease, we find that the strength of the delta shock wave increases. Eventually, when ε drops to zero, the delta shock wave solution is exactly the solution to system (3). Thus, we find that the process of delta shock wave formation is obviously different from those in [36, 37] and so forth. Meanwhile, any Riemann solution involving two rarefaction waves converges to the vacuum solution to system (3). Furthermore, the limit of the solution involving one rarefaction wave R_1 (or R_2) and one shock wave S_2 (or S_1) is just the contact discontinuity connecting the two constant states $(v_\pm,\rho_\pm)$.

The organization of this paper is as follows: in Sections 2 and 3, we give some results on the Riemann solutions to

system (5)-(6) and system (3). In Section 4, we study the limit of Riemann solutions involving two shocks to system (5)-(6) as pressure vanishes when $v_- > v_+$. In Section 5, we investigate the limit of solution containing two rarefaction waves to system (5)-(6) when $v_- < v_+$. In Section 6, we analyze the limit of solution when $v_- = v_+$.

2. Riemann Problems for System (5)-(6)

In this section, we mainly consider the solutions of (5)-(6) with initial data:

$$(v,\rho)(0,x) = (v_\pm, \rho_\pm), \quad (\pm x > 0). \tag{8}$$

As mentioned in the introduction, the process without the parameter ε has been done in [34] for system (1) of Chaplygin gas. However, we provide the Riemann solution of (5)-(6) for concreteness.

2.1. Elementary Waves and Riemann Problems. Noticing the relativistic constraint $|v| < c$ and $\sqrt{\varepsilon p'} \le c$, we see that the physically relevant region for solutions is

$$\Lambda = \left\{(v,\rho) \mid |v| < c, \rho \ge \frac{\sqrt{\varepsilon}}{c}\right\}, \tag{9}$$

which is obviously different from that for polytropic and barotropic gas. The eigenvalues of system (5) are

$$\lambda_1^\varepsilon = \frac{c^2\left(v - \sqrt{\varepsilon p'}\right)}{c^2 - v\sqrt{\varepsilon p'}}, \qquad \lambda_2^\varepsilon = \frac{c^2\left(v + \sqrt{\varepsilon p'}\right)}{c^2 + v\sqrt{\varepsilon p'}} \tag{10}$$

and the corresponding right eigenvectors are $r_1 = ((-1/(c^2 - v^2)), (\sqrt{\varepsilon p'}/(\varepsilon p + \rho c^2)))^T$, $r_2 = ((1/(c^2 - v^2)), (\sqrt{\varepsilon p'}/(\varepsilon p + \rho c^2)))^T$. Thus the fact of $\nabla\lambda_i^\varepsilon \cdot r_i = 0$ $(i = 1, 2)$ shows that both the characteristic fields are linearly degenerate. The Riemann problem (5) and (8) for $\xi = x/t$ can be reduced to

$$\begin{gathered}
-\xi\left(\frac{\left(\rho c^2 + \varepsilon p\right)v^2}{c^2\left(c^2 - v^2\right)} + \rho\right)_\xi + \left(\frac{\left(\rho c^2 + \varepsilon p\right)v}{c^2 - v^2}\right)_\xi = 0, \\
-\xi\left(\frac{\left(\rho c^2 + \varepsilon p\right)v}{c^2 - v^2}\right)_\xi + \left(\frac{\left(\rho c^2 + \varepsilon p\right)v^2}{c^2 - v^2} + \varepsilon p\right)_\xi = 0, \\
(v,\rho)(\pm\infty) = (v_\pm, \rho_\pm).
\end{gathered} \tag{11}$$

For smooth solutions, system (11) provides either the general solutions (constant states) $(v,\rho)(\xi) = \text{const}$ $(\rho > \sqrt{\varepsilon}/c)$ or the singular solution

$$\xi = \frac{c^2\left(\rho v \pm \sqrt{\varepsilon}\right)}{\rho c^2 \pm \sqrt{\varepsilon} v} = \frac{c^2\left(\rho_- v_- \pm \sqrt{\varepsilon}\right)}{\rho_- c^2 \pm \sqrt{\varepsilon} v_-}. \tag{12}$$

Given a state (v_-, ρ_-), the rarefaction wave curves in the phase plane are the sets of states that can be connected on the right by a 1-rarefaction or a 2-rarefaction wave in the form:

1-rarefaction wave curve $R_1(v_-, \rho_-)$:

$$\xi = \lambda_1^\varepsilon = \frac{c^2\left(\rho v - \sqrt{\varepsilon}\right)}{\rho c^2 - \sqrt{\varepsilon} v} = \frac{c^2\left(\rho_- v_- - \sqrt{\varepsilon}\right)}{\rho_- c^2 - \sqrt{\varepsilon} v_-}, \quad \rho < \rho_-; \tag{13}$$

2-rarefaction wave curve $R_2(v_-, \rho_-)$:

$$\xi = \lambda_2^\varepsilon = \frac{c^2\left(\rho v + \sqrt{\varepsilon}\right)}{\rho c^2 + \sqrt{\varepsilon} v} = \frac{c^2\left(\rho_- v_- + \sqrt{\varepsilon}\right)}{\rho_- c^2 + \sqrt{\varepsilon} v_-}, \quad \rho > \rho_-. \tag{14}$$

For a bounded discontinuity at $\xi = \sigma$, the Rankine-Hugoniot condition reads

$$\begin{gathered}
-\sigma\left[\frac{\left(\rho c^2 + \varepsilon p\right)v^2}{c^2\left(c^2 - v^2\right)} + \rho\right] + \left[\frac{\left(\rho c^2 + \varepsilon p\right)v}{c^2 - v^2}\right] = 0, \\
-\sigma\left[\frac{\left(\rho c^2 + \varepsilon p\right)v}{c^2 - v^2}\right] + \left[\frac{\left(\rho c^2 + \varepsilon p\right)v^2}{c^2 - v^2} + \varepsilon p\right] = 0,
\end{gathered} \tag{15}$$

where σ is the velocity of the discontinuity. The Lax entropy conditions imply that

$$\rho > \rho_- \ (\text{1-shock}), \qquad \rho < \rho_- \ (\text{2-shock}). \tag{16}$$

Given a state (v_-, ρ_-), the shock wave curves in the phase plane are the sets of states that can be connected on the right by a 1-shock or a 2-shock wave in the form

1-shock wave curve $S_1(v_-, \rho_-)$:

$$\sigma_1^\varepsilon = \frac{c^2\left(\rho v - \sqrt{\varepsilon}\right)}{\rho c^2 - \sqrt{\varepsilon} v} = \frac{c^2\left(\rho_- v_- - \sqrt{\varepsilon}\right)}{\rho_- c^2 - \sqrt{\varepsilon} v_-}, \quad \rho > \rho_-; \tag{17}$$

2-shock wave curve $S_2(v_-, \rho_-)$:

$$\sigma_2^\varepsilon = \frac{c^2\left(\rho v + \sqrt{\varepsilon}\right)}{\rho c^2 + \sqrt{\varepsilon} v} = \frac{c^2\left(\rho_- v_- + \sqrt{\varepsilon}\right)}{\rho_- c^2 + \sqrt{\varepsilon} v_-}, \quad \rho < \rho_-. \tag{18}$$

From (13)-(14) and (17)-(18), we can see that the rarefaction wave curves and the shock wave curves are coincident in the phase plane, which actually correspond to contact discontinuities. For convenience, we still call them rarefaction waves and shock waves, denoted by R and S, respectively, although each of them degenerates to a characteristic.

Given state (v_-, ρ_-), we draw the curves (13)-(14) and (17)-(18) for $\rho > \sqrt{\varepsilon}/c$ in the phase plane; see Figure 1. The curves S_1 and R_2 have asymptotic lines $v = v_1$ and $v = v_2$, respectively, where

$$v_1 = \frac{c^2\left(\rho_- v_- - \sqrt{\varepsilon}\right)}{\rho_- c^2 - \sqrt{\varepsilon} v_-}, \qquad v_2 = \frac{c^2\left(\rho_- v_- + \sqrt{\varepsilon}\right)}{\rho_- c^2 + \sqrt{\varepsilon} v_-}. \tag{19}$$

The curves S_2 and R_1 have singularity points $(-c, \sqrt{\varepsilon}/c)$ and $(c, \sqrt{\varepsilon}/c)$, respectively. Moreover, starting from the point (v_3, ρ_-) where v_3 will be shown below, we draw the contact discontinuity curve (12) with the positive sign, which have the

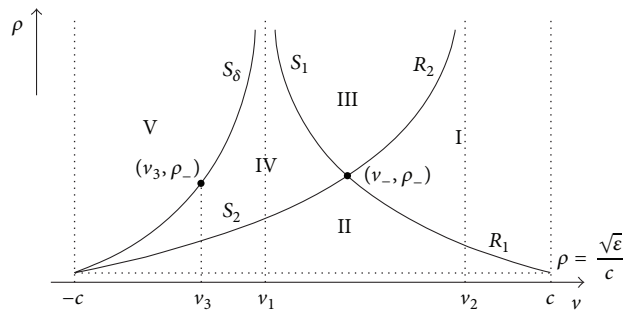


Figure 1: Wave curves in the phase plane.

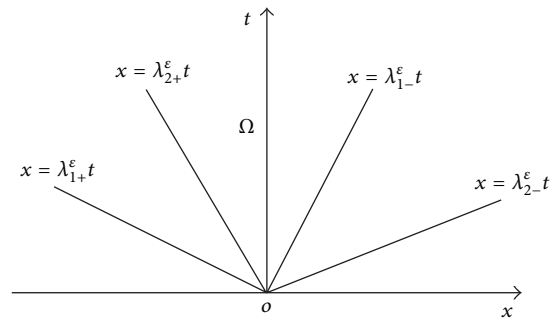


Figure 2: Characteristic lines for constant states $(v_\pm, \rho_\pm)$.

asymptotic line $v = v_1$ and the singularity point $(-c, \sqrt{\varepsilon}/c)$, with v_3 satisfying

$$\frac{c^2\left(\rho_- v_3 + \sqrt{\varepsilon}\right)}{\rho_- c^2 + \sqrt{\varepsilon} v_3} = \frac{c^2\left(\rho_- v_- - \sqrt{\varepsilon}\right)}{\rho_- c^2 - \sqrt{\varepsilon} v_-}. \tag{20}$$

Thus the region Λ can be divided into five regions I, II, III, IV, and V, as shown in Figure 1.

For any given right state (v_+, ρ_+), there exists a solution of (5)-(6) and (8) when $(v_+, \rho_+) \in (\mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV})(v_-, \rho_-)$ of which configurations are as follows:

$$\begin{aligned}
&(1)\ \left(v_+, \rho_+\right) \in \mathrm{I}\left(v_-, \rho_-\right) : R_1 + R_2,\\
&(2)\ \left(v_+, \rho_+\right) \in \mathrm{II}\left(v_-, \rho_-\right) : R_1 + S_2,\\
&(3)\ \left(v_+, \rho_+\right) \in \mathrm{III}\left(v_-, \rho_-\right) : S_1 + R_2,\\
&(4)\ \left(v_+, \rho_+\right) \in \mathrm{IV}\left(v_-, \rho_-\right) : S_1 + S_2.
\end{aligned} \tag{21}$$

For the remaining case $(v_+, \rho_+) \in \mathrm{V}$, we need to seek a nonclassical solution, which will be considered in the next subsection.

2.2. Delta Shock Wave Solution. In order to construct the unique global Riemann solution, we need to consider the case when $(v_+, \rho_+) \in \mathrm{V}$, in which we have

$$\begin{aligned}
\lambda_{1+}^{\varepsilon} &:= \frac{c^2\left(\rho_+ v_+ - \sqrt{\varepsilon}\right)}{\rho_+ c^2 - \sqrt{\varepsilon} v_+}\\
&< \lambda_{2+}^{\varepsilon} := \frac{c^2\left(\rho_+ v_+ + \sqrt{\varepsilon}\right)}{\rho_+ c^2 + \sqrt{\varepsilon} v_+}\\
&< \lambda_{1-}^{\varepsilon} := \frac{c^2\left(\rho_- v_- - \sqrt{\varepsilon}\right)}{\rho_- c^2 - \sqrt{\varepsilon} v_-}\\
&< \lambda_{2-}^{\varepsilon} := \frac{c^2\left(\rho_- v_- + \sqrt{\varepsilon}\right)}{\rho_- c^2 + \sqrt{\varepsilon} v_-}.
\end{aligned} \tag{22}$$

It means that characteristic lines from initial data will overlap in a domain Ω shown in Figure 2. Thus singularity must happen in Ω. The singularity is impossible to be a jump with finite amplitude; that is, there is no piecewise smooth and bounded solution. Hence, a weighted δ-measure solution should be constructed.

Now let us give the definition of a δ-shock wave type solution for system (5) with (6), in which we use the concept introduced in [43].

Suppose that $\Gamma = \{\gamma_i \mid i \in I\}$ is a graph in the closed upper half-plane $\{(x,t) \mid x \in \mathbb{R}, t \in [0,+\infty)\} \subset \mathbb{R}^2$ containing smooth arcs $\gamma_i, i \in I$, and I is a finite set. Let I_0 be a subset of I such that an arc γ_k for $k \in I_0$ starts from the points of the x-axis, and let $\Gamma_0 = \{x_k^0 \mid k \in I_0\}$ be the set of initial points of arc γ_k, $k \in I_0$. Let us consider δ-shock wave type initial data $(v^0(x), \rho^0(x))$, where

$$\rho^0(x) = \rho_0(x) + w^0 \delta\left(\Gamma_0\right), \tag{23}$$

v^0, $\rho_0 \in L^\infty(\mathbb{R};\mathbb{R})$, $w^0\delta(\Gamma_0) = \sum_{k\in I_0} w_k^0 \delta(x - x_k^0)$, and w_k^0 are constants for $k \in I_0$. Furthermore, the pressure $p = -1/\rho$ in (5) is a nonlinear term with respect to ρ, and $p^0(x,t)$ is defined by $-1/\rho_0$, where the delta measure does not contribute [12].

Definition 1. A pair of distributions $(v(x,t), \rho(x,t))$ and a graph Γ, where $\rho(x,t)$ and $p(x,t)$ have the form

$$\rho(x,t) = \overline{\rho}(x,t) + w^{\varepsilon}(x,t)\,\delta(\Gamma), \quad p(x,t) = -\frac{1}{\rho(x,t)}, \tag{24}$$

$v, \overline{\rho} \in L^\infty(\mathbb{R} \times \mathbb{R}_+; \mathbb{R})$, $w^\varepsilon(x,t)\delta(\Gamma) = \sum_{i\in I} w_i^\varepsilon(x,t)\delta(\gamma_i)$, and $w_i^\varepsilon(x,t) \in C(\Gamma)$ for $i \in I$, is called a generalized δ-shock wave type solution of system (5) with the initial data $(v^0(x), \rho^0(x))$ if the integral identities

$$\begin{aligned}
&\int_0^{+\infty}\int_{-\infty}^{+\infty}\left(\left(\frac{\left(\varepsilon p + \overline{\rho} c^2\right) v^2}{c^2\left(c^2 - v^2\right)} + \overline{\rho}\right)\phi_t\right.\\
&\qquad\left. + \left(\frac{\left(\varepsilon p + \overline{\rho} c^2\right) v}{c^2 - v^2}\right)\phi_x\right) dx\, dt\\
&\quad + \sum_{i\in I}\int_{\gamma_i} \frac{w_i^\varepsilon(x,t)\, c^2}{c^2 - v_\delta^2}\frac{\partial \phi}{\partial l} dl
\end{aligned}$$

$$
\begin{aligned}
&+\int_{-\infty}^{+\infty}\left(\frac{\left(\varepsilon p^{0}+\rho_{0}c^{2}\right)\left(v^{0}\right)^{2}}{c^{2}\left(c^{2}-\left(v^{0}\right)^{2}\right)}+\rho_{0}\right)\phi\left(x,0\right)dx\\
&+\sum_{k\in I_{0}}\frac{w_{k}^{0}c^{2}}{c^{2}-\left(v_{\delta}^{0}\left(x_{k}^{0}\right)\right)^{2}}\phi\left(x_{k}^{0},0\right)=0,\\
&\int_{0}^{+\infty}\int_{-\infty}^{+\infty}\left(\left(\frac{\left(\varepsilon p+\overline{\rho}c^{2}\right)v}{c^{2}-v^{2}}\right)\phi_{t}\right.\\
&\qquad\left.+\left(\frac{\left(\varepsilon p+\overline{\rho}c^{2}\right)v^{2}}{c^{2}-v^{2}}+\varepsilon p\right)\phi_{x}\right)dx\,dt\\
&+\sum_{i\in I}\int_{\gamma_{i}}\frac{w_{i}^{\varepsilon}\left(x,t\right)c^{2}v_{\delta}}{c^{2}-v_{\delta}^{2}}\frac{\partial\phi}{\partial l}dl\\
&+\int_{-\infty}^{+\infty}\left(\frac{\left(\varepsilon p^{0}+\rho_{0}c^{2}\right)v^{0}}{c^{2}-\left(v^{0}\right)^{2}}\right)(x)\,\phi\left(x,0\right)dx\\
&+\sum_{k\in I_{0}}\frac{w_{k}^{0}c^{2}v_{\delta}^{0}\left(x_{k}^{0}\right)}{c^{2}-\left(v_{\delta}^{0}\left(x_{k}^{0}\right)\right)^{2}}\phi\left(x_{k}^{0},0\right)=0
\end{aligned}
\tag{25}
$$

hold for any test functions $\phi(x,t)\in\mathscr{D}(\mathbb{R}\times\mathbb{R}_{+})$, where $\partial\phi/\partial l$ is the tangential derivative on the graph Γ, $\int_{\gamma_i}dl$ is a line integral along the arc γ_i, $v_\delta(x,t)$ is the velocity of the δ-shock wave, and $v_\delta^0(x_k^0)=v_\delta(x_k^0,0)$, $k\in I_0$.

Under the above definition, it is not difficult to get the following result.

Theorem 2. *When $(v_+,\rho_+)\in V$, for the Riemann problem* (5)-(6) *and* (8), *there is a δ-shock wave solution $(v(x,t),\rho(x,t))$ of the form*

$$
\begin{aligned}
v\left(x,t\right)&=v_{-}+\left[v\right]H\left(x-x\left(t\right)\right),\\
\rho\left(x,t\right)&=\rho_{-}+\left[\rho\right]H\left(x-x\left(t\right)\right)+w^{\varepsilon}\left(x,t\right)\delta\left(x-x\left(t\right)\right),
\end{aligned}
\tag{26}
$$

which satisfies the integral identities (25) *in the sense of Definition 1, where $\Gamma=\{(x,t)\mid x=x(t)=\sigma_\delta t, t\geq 0\}$, $\overline{\rho}(x,t)=\rho_-+[\rho]H(x-x(t))$,*

$$
\int_{\Gamma}w^{\varepsilon}\left(x,t\right)\frac{\partial\phi\left(x,t\right)}{\partial l}=\int_{0}^{\infty}w^{\varepsilon}\left(x\left(t\right),t\right)\frac{d\phi\left(x\left(t\right),t\right)}{dt},
\tag{27}
$$

and $H(x)$ is the Heaviside function.

The proof of this theorem is very similar to the Theorem 6 in [43], so we omit its proof.

Definition 3. Suppose that $\Omega\subset\mathbb{R}\times\mathbb{R}_{+}$ is a region cut by a smooth curve $\Gamma=\{(x,t)\mid x=x(t)\}$ into a left and right hand parts $\Omega_{\pm}=\{(x,t)\mid\pm(x-x(t))>0\}$, $(v(x,t),\rho(x,t))$ is a generalized δ-shock wave solution of system (5), and functions $\overline{\rho}(x,t)$, $v(x,t)$ are smooth in $\Omega_\pm$ and have one-side limits $\overline{\rho}_\pm, v_\pm$ on the curve Γ. Then the generalized Rankine-Hugoniot conditions for δ-shock wave are

$$
\begin{aligned}
\frac{dx\left(t\right)}{dt}&=\sigma_{\delta},\\
\frac{d}{dt}\left(\frac{w^{\varepsilon}\left(x\left(t\right),t\right)c^{2}}{c^{2}-\left(\sigma_{\delta}\right)^{2}}\right)&=\sigma_{\delta}\left[\frac{\left(\rho c^{2}+\varepsilon p\right)v^{2}}{c^{2}\left(c^{2}-v^{2}\right)}+\rho\right]\\
&\quad-\left[\frac{\left(\rho c^{2}+\varepsilon p\right)v}{c^{2}-v^{2}}\right],\\
\frac{d}{dt}&\left(\frac{w^{\varepsilon}\left(x\left(t\right),t\right)\sigma_{\delta}c^{2}}{c^{2}-\left(\sigma_{\delta}\right)^{2}}\right)\\
&=\sigma_{\delta}\left[\frac{\left(\rho c^{2}+\varepsilon p\right)v}{c^{2}-v^{2}}\right]-\left[\frac{\left(\rho c^{2}+\varepsilon p\right)v^{2}}{c^{2}-v^{2}}+\varepsilon p\right],
\end{aligned}
\tag{28}
$$

with initial data $w^\varepsilon(x(0),0)=0$.

To guarantee uniqueness, the generalized entropy condition

$$
\lambda_{2+}^{\varepsilon}\leq\sigma_{\delta}\leq\lambda_{1-}^{\varepsilon}
\tag{29}
$$

should be satisfied, which means that all the characteristic lines on both sides of the delta shock wave are not outcoming.

Denote

$$
\begin{gathered}
E=\left[\frac{\left(\rho c^{2}+\varepsilon p\right)v^{2}}{c^{2}\left(c^{2}-v^{2}\right)}+\rho\right],\qquad F=\left[\frac{\left(\rho c^{2}+\varepsilon p\right)v}{c^{2}-v^{2}}\right],\\
G=\left[\frac{\left(\rho c^{2}+\varepsilon p\right)v^{2}}{c^{2}-v^{2}}+\varepsilon p\right].
\end{gathered}
\tag{30}
$$

Thus from (28) and (29), we have, for $E\neq 0$, that

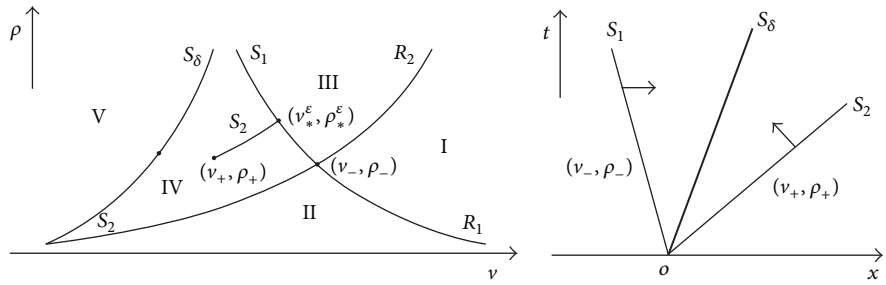
$$
\begin{aligned}
\sigma_{\delta}&=\frac{F+\sqrt{F^{2}-EG}}{E},\\
w^{\varepsilon}\left(x\left(t\right),t\right)&=\sqrt{F^{2}-EG}\left(1-\left(\frac{F-\sqrt{F^{2}-EG}}{cE}\right)^{2}\right)t,
\end{aligned}
\tag{31}
$$

and for $E=0$,

$$
\begin{aligned}
\sigma_{\delta}&=\frac{G}{2F},\\
w^{\varepsilon}\left(x\left(t\right),t\right)&=F\left(1-\left(\frac{G}{2Fc}\right)^{2}\right)t.
\end{aligned}
\tag{32}
$$

3. Riemann Problems for System (3)

In this section, we show some results briefly on Riemann problems to system (3) with initial data (8). The idea is similar with that in [17], so we omit the details here.



(a) The phase plane (b) The (x, t)-plane

Figure 3: Riemann solutions when $(v_+, \rho_+) \in \mathrm{IV}(v_-, \rho_-)$.

The system has a double eigenvalue $\lambda = v$ and only one right eigenvector $r = (1, 0)^\top$. System (3) is obviously linearly degenerate by $\nabla\lambda \cdot r = 0$. We seek the self-similar solution $(v, \rho)(t, x) = (v, \rho)(\xi)$, $\xi = x/t$, for which the Riemann problem can be transformed into the infinity boundary value problem:

$$\begin{gathered} -\xi\left(\frac{\rho}{c^2 - v^2}\right)_\xi + \left(\frac{\rho v}{c^2 - v^2}\right)_\xi = 0, \\ -\xi\left(\frac{\rho v}{c^2 - v^2}\right)_\xi + \left(\frac{\rho v^2}{c^2 - v^2}\right)_\xi = 0, \\ (v, \rho)(\pm\infty) = (v_\pm, \rho_\pm). \end{gathered} \tag{33}$$

For the case $v_- < v_+$, the solution consists of two contact discontinuities plus a vacuum state between them and can be expressed as

$$(v, \rho)(\xi) = \begin{cases} (v_-, \rho_-), & -\infty < \xi < v_-, \\ (v(\xi), 0), & v_- \le \xi \le v_+, \\ (v_+, \rho_+), & v_+ < \xi < +\infty, \end{cases} \tag{34}$$

where $v(\xi)$ is an arbitrary smooth function satisfying $v(v_-) = v_-$ and $v(v_+) = v_+$.

For the case $v_- > v_+$, singularity must happen. According to the procedure in Section 2, a δ-shock wave type solution can be constructed. We omit the details here and just give a description of the solution because of the similarity. With the definitions in Section 2, for the case $v_- > v_+$, one can construct a δ-shock wave type solution with this form

$$\begin{gathered} v = v_- + [v] H(x - x(t)), \\ \rho = \rho_- + [\rho] H(x - x(t)) + w(x(t), t)\, \delta(x - x(t)), \end{gathered} \tag{35}$$

where $x(t)$ and $w(x, t)$ should satisfy the generalized Rankine-Hugoniot condition:

$$\begin{gathered} \frac{dx}{dt} = v_\delta, \\ \frac{d}{dt}\left(\frac{w(x(t), t)}{c^2 - v_\delta^2}\right) = v_\delta\left[\frac{\rho}{c^2 - v^2}\right] - \left[\frac{\rho v}{c^2 - v^2}\right], \\ \frac{d}{dt}\left(\frac{w(x(t), t)\, v_\delta}{c^2 - v_\delta^2}\right) = v_\delta\left[\frac{\rho v}{c^2 - v^2}\right] - \left[\frac{\rho v^2}{c^2 - v^2}\right]. \end{gathered} \tag{36}$$

Here $[h] = h(t, x(t) + 0) - h(t, x(t) - 0)$ is the jump of h across discontinuity.

In order to ensure the uniqueness of the Riemann solution, the generalized entropy condition should be proposed as

$$v_+ < \frac{dx}{dt} < v_-, \tag{37}$$

which means that all the characteristic lines on either side of a delta shock run into the line of the delta shock in the (x, t)-plane; this is to say that a delta shock is an overcompressive shock. From (36) and (37), we determine that

$$\begin{gathered} v_\delta = \frac{v_+\sqrt{\rho_+/(c^2 - v_+^2)} + v_-\sqrt{\rho_-/(c^2 - v_-^2)}}{\sqrt{\rho_+/(c^2 - v_+^2)} + \sqrt{\rho_-/(c^2 - v_-^2)}}, \\ x(t) = v_\delta t, \\ w(x(t), t) = \sqrt{\frac{\rho_+}{c^2 - v_+^2}\frac{\rho_-}{c^2 - v_-^2}}\,(v_- - v_+)\left(c^2 - v_\delta^2\right) t. \end{gathered} \tag{38}$$

4. Limit of Riemann Solutions to (5)-(6) for $v_- > v_+$

In this section, we deal with the limit behavior of Riemann solutions to system (5)-(6) in the case $(v_+, \rho_+) \in \mathrm{IV} \cup \mathrm{V}(v_-, \rho_-)$ (see Figure 3(a)) as pressure vanishes. Let us assume that $v_- > v_+$, $\rho_\pm > \sqrt{\varepsilon}/c$ and then we divide our discussion into two parts. First, we identify the formation of delta shock wave in the case $(v_+, \rho_+) \in \mathrm{IV}(v_-, \rho_-)$ and compare them with the delta shock solution to (5)-(6). Then, we show how the strength and propagation speed of the delta shock wave change along with the values of ε, when ε becomes smaller and smaller and ultimately goes to zero. Here two critical values ε_0 and ε_1 for ε should be introduced, which will play very important roles in the following discussion.

Lemma 4. *If $v_- > v_+$, then there exist $\varepsilon_0, \varepsilon_1 > 0$ such that $(v_+, \rho_+) \in IV(v_-, \rho_-)$ when $\varepsilon_1 < \varepsilon < \varepsilon_0$; $(v_+, \rho_+) \in V(v_-, \rho_-)$ when $0 < \varepsilon < \varepsilon_1$.*

Proof. It follows from (17) and (18) that all possible states (v, ρ) which can be connected on the right to the left state

(v_-, ρ_-) by a 1-shock wave S_1 or a 2-shock wave S_2 should satisfy

$$S_1 : \frac{\rho v - \sqrt{\varepsilon}}{\rho c^2 - \sqrt{\varepsilon} v} = \frac{\rho_- v_- - \sqrt{\varepsilon}}{\rho_- c^2 - \sqrt{\varepsilon} v_-}, \quad \rho > \rho_-,$$
$$S_2 : \frac{\rho v + \sqrt{\varepsilon}}{\rho c^2 + \sqrt{\varepsilon} v} = \frac{\rho_- v_- + \sqrt{\varepsilon}}{\rho_- c^2 + \sqrt{\varepsilon} v_-}, \quad \rho < \rho_-. \tag{39}$$

The states that can be connected on the right to the left state (v_-, ρ_-) directly by a delta shock wave S_δ should satisfy

$$S_\delta : \frac{\rho v + \sqrt{\varepsilon}}{\rho c^2 + \sqrt{\varepsilon} v} = \frac{\rho_- v_- - \sqrt{\varepsilon}}{\rho_- c^2 - \sqrt{\varepsilon} v_-}. \tag{40}$$

If $\rho_+ \neq \rho_-$ and $(v_+, \rho_+) \in \mathrm{IV} \cup \mathrm{V}(v_-, \rho_-)$, then we observe that both $(\rho v - \sqrt{\varepsilon})/(\rho c^2 - \sqrt{\varepsilon} v)$ and $(\rho v + \sqrt{\varepsilon})/(\rho c^2 + \sqrt{\varepsilon} v)$ are monotone increasing with v. Also we can see intuitively from Figure 3(a) together with (39) that ε should satisfy

$$\frac{\rho_+ v_+ - \sqrt{\varepsilon}}{\rho_+ c^2 - \sqrt{\varepsilon} v_+} < \frac{\rho_- v_- - \sqrt{\varepsilon}}{\rho_- c^2 - \sqrt{\varepsilon} v_-}, \quad \text{if } \rho_+ > \rho_-,$$
$$\frac{\rho_+ v_+ + \sqrt{\varepsilon}}{\rho_+ c^2 + \sqrt{\varepsilon} v_+} < \frac{\rho_- v_- + \sqrt{\varepsilon}}{\rho_- c^2 + \sqrt{\varepsilon} v_-}, \quad \text{if } \rho_+ < \rho_-, \tag{41}$$

which imply that

$$(v_- - v_+)\varepsilon + |\rho_- - \rho_+| (c^2 - v_+ v_-) \sqrt{\varepsilon} - c^2 \rho_+ \rho_- (v_- - v_+) < 0. \tag{42}$$

Thus we can take

$$\sqrt{\varepsilon_0} = \frac{|\rho_- - \rho_+| (v_+ v_- - c^2)}{2(v_- - v_+)} + \frac{\sqrt{(\rho_- - \rho_+)^2 (c^2 - v_+ v_-)^2 + 4c^2 \rho_+ \rho_- (v_- - v_+)^2}}{2(v_- - v_+)}. \tag{43}$$

Obviously we have $(v_+, \rho_+) \in \mathrm{IV} \cup \mathrm{V}(v_-, \rho_-)$ if $\varepsilon < \varepsilon_0$.

Furthermore, if $\rho_+ \neq \rho_-$ and $(v_+, \rho_+) \in \mathrm{V}(v_-, \rho_-)$, then we can also see intuitively from Figure 3(a) together with (40) that ε should satisfy

$$\frac{\rho_+ v_+ + \sqrt{\varepsilon}}{\rho_+ c^2 + \sqrt{\varepsilon} v_+} < \frac{\rho_- v_- - \sqrt{\varepsilon}}{\rho_- c^2 - \sqrt{\varepsilon} v_-}, \tag{44}$$

which is equivalent to

$$(v_- - v_+)\varepsilon - (\rho_- + \rho_+)(c^2 - v_+ v_-) \sqrt{\varepsilon} + c^2 \rho_+ \rho_- (v_- - v_+) > 0. \tag{45}$$

Taking

$$\sqrt{\varepsilon_1} = \frac{(\rho_- + \rho_+)(c^2 - v_+ v_-)}{2(v_- - v_+)} - \frac{\sqrt{(\rho_- + \rho_+)^2 (c^2 - v_+ v_-)^2 - 4c^2 \rho_+ \rho_- (v_- - v_+)^2}}{2(v_- - v_+)}, \tag{46}$$

we have $(v_+, \rho_+) \in \mathrm{V}(v_-, \rho_-)$ if $\varepsilon < \varepsilon_1$. It is not difficult to show that $\varepsilon_1 < \varepsilon_0$.

In particular, if $\rho_+ = \rho_-$, then $(v_+, \rho_+) \in \mathrm{IV} \cup \mathrm{V}(v_-, \rho_-)$ for any $\varepsilon > 0$, and moreover $(v_+, \rho_+) \in \mathrm{V}(v_-, \rho_-)$ if

$$(v_- - v_+)\varepsilon - 2\rho_+ (c^2 - v_+ v_-) \sqrt{\varepsilon} + c^2 \rho_+^2 (v_- - v_+) > 0. \tag{47}$$

Thus we can take

$$\sqrt{\varepsilon_1} := \frac{c^2 - v_+ v_- - \sqrt{(c^2 - v_+^2)(c^2 - v_-^2)}}{v_- - v_+} \rho_+ \tag{48}$$

and arbitrary ε_0 which only needs to satisfy $\varepsilon_0 > \varepsilon_1$ in this special situation. □

Lemma 4 shows that the shock wave curves S_1 and S_2 become steeper when ε decreases. There is no delta shock wave in the Riemann solution of (5)-(6) for a fluid with strong pressure. As pressure decreases, delta shock wave occurs in the Riemann solution. Hence we divide our discussion into two parts according to different ranges of ε.

4.1. Formation of Delta Shocks. In this subsection, we discuss the situation $\varepsilon_1 < \varepsilon < \varepsilon_0$, namely $(v_+, \rho_+) \in \mathrm{IV}(v_-, \rho_-)$. When $v_- > v_+$, the Riemann solution to system (5)-(6) with initial data (8) consists of two constant states $(v_\pm, \rho_\pm)$, an intermediate state $(v_*^\varepsilon, \rho_*^\varepsilon)$ and two shock wave curves S_1, S_2 (see Figure 3(a)). Then, we have

$$S_1 : \sigma_1^\varepsilon = \frac{c^2 (\rho_*^\varepsilon v_*^\varepsilon - \sqrt{\varepsilon})}{\rho_*^\varepsilon c^2 - \sqrt{\varepsilon} v_*^\varepsilon} = \frac{c^2 (\rho_- v_- - \sqrt{\varepsilon})}{\rho_- c^2 - \sqrt{\varepsilon} v_-} = \lambda_{1-}^\varepsilon, \quad \rho_*^\varepsilon > \rho_-;$$
$$S_2 : \sigma_2^\varepsilon = \frac{c^2 (\rho_*^\varepsilon v_*^\varepsilon + \sqrt{\varepsilon})}{\rho_*^\varepsilon c^2 + \sqrt{\varepsilon} v_*^\varepsilon} = \frac{c^2 (\rho_+ v_+ + \sqrt{\varepsilon})}{\rho_+ c^2 + \sqrt{\varepsilon} v_+} = \lambda_{2+}^\varepsilon, \quad \rho_*^\varepsilon > \rho_+. \tag{49}$$

Equations (49) can be easily transformed into the following form:

$$S_1 : \sigma_1^\varepsilon = \lambda_{1-}^\varepsilon, \quad v_*^\varepsilon = \frac{c^2 (\rho_*^\varepsilon \lambda_{1-}^\varepsilon + \sqrt{\varepsilon})}{\rho_*^\varepsilon c^2 + \sqrt{\varepsilon} \lambda_{1-}^\varepsilon}, \quad \rho_*^\varepsilon > \rho_-; \tag{50}$$

$$S_2 : \sigma_2^\varepsilon = \lambda_{2+}^\varepsilon, \quad v_*^\varepsilon = \frac{c^2 (\rho_*^\varepsilon \lambda_{2+}^\varepsilon - \sqrt{\varepsilon})}{\rho_*^\varepsilon c^2 - \sqrt{\varepsilon} \lambda_{2+}^\varepsilon}, \quad \rho_*^\varepsilon > \rho_+. \tag{51}$$

It is not difficult to derive from (50) and (51) that

$$\frac{\lambda_{1-}^\varepsilon - \lambda_{2+}^\varepsilon}{c^2} \frac{\varepsilon}{(\rho_*^\varepsilon)^2} + 2\left(1 - \frac{\lambda_{1-}^\varepsilon \lambda_{2+}^\varepsilon}{c^2}\right) \frac{\sqrt{\varepsilon}}{\rho_*^\varepsilon} + \lambda_{1-}^\varepsilon - \lambda_{2+}^\varepsilon = 0. \tag{52}$$

For given $\rho_\pm > 0$, taking limit $\varepsilon \to \varepsilon_1$ in (52), we have

$$\lim_{\varepsilon \to \varepsilon_1} \left(1 - \frac{\lambda_{1-}^\varepsilon \lambda_{2+}^\varepsilon}{c^2}\right) \frac{\sqrt{\varepsilon}}{\rho_*^\varepsilon} = 0, \tag{53}$$

in which we use the fact that $\lambda_{1-}^{\varepsilon_1} = \lambda_{2+}^{\varepsilon_1}$. Noticing that $\lim_{\varepsilon\to\varepsilon_1}\lambda_{1-}^{\varepsilon}, \lambda_{2+}^{\varepsilon} \neq c$, we obtain

$$\lim_{\varepsilon\to\varepsilon_1}\frac{\sqrt{\varepsilon}}{\rho_*^{\varepsilon}} = 0. \tag{54}$$

It follows from Lemma 4 that $\sqrt{\varepsilon_1} > 0$ is a constant if two constant states $(v_\pm, \rho_\pm)$ are given. Hence, we deduce that

$$\lim_{\varepsilon\to\varepsilon_1}\rho_*^{\varepsilon} = \infty. \tag{55}$$

Thus we have the following result.

Lemma 5. *Set $\sigma = \lambda_{1-}^{\varepsilon_1} = \lambda_{2+}^{\varepsilon_1}$. Then, one has*

$$\lim_{\varepsilon\to\varepsilon_1} v_*^{\varepsilon} = \lim_{\varepsilon\to\varepsilon_1}\sigma_1^{\varepsilon} = \lim_{\varepsilon\to\varepsilon_1}\sigma_2^{\varepsilon} = \sigma, \tag{56}$$

$$\begin{aligned}\lim_{\varepsilon\to\varepsilon_1}&\int_{\sigma_1^{\varepsilon}t}^{\sigma_2^{\varepsilon}t}\frac{\rho_*^{\varepsilon}c^2}{c^2-\left(v_*^{\varepsilon}\right)^2}dx\\ &= \left(\sigma\left[\frac{\left(\varepsilon_1 p+\rho c^2\right)v^2}{c^2\left(c^2-v^2\right)}+\rho\right]-\left[\frac{\left(\varepsilon_1 p+\rho c^2\right)v}{c^2-v^2}\right]\right)t.\end{aligned} \tag{57}$$

Proof. Letting $\varepsilon \to \varepsilon_1$ in (50), we have

$$\lim_{\varepsilon\to\varepsilon_1} v_*^{\varepsilon} = \lim_{\varepsilon\to\varepsilon_1}\frac{c^2\left(\rho_*^{\varepsilon}\lambda_{1-}^{\varepsilon}+\sqrt{\varepsilon}\right)}{\rho_*^{\varepsilon}c^2+\sqrt{\varepsilon}\lambda_{1-}^{\varepsilon}}. \tag{58}$$

Combining (54) and (58), one can obtain

$$\lim_{\varepsilon\to\varepsilon_1} v_*^{\varepsilon} = \lim_{\varepsilon\to\varepsilon_1}\frac{c^2\left(\lambda_{1-}^{\varepsilon}+\sqrt{\varepsilon}/\rho_*^{\varepsilon}\right)}{c^2+\lambda_{1-}^{\varepsilon}\sqrt{\varepsilon}/\rho_*^{\varepsilon}} = \sigma. \tag{59}$$

From (50) and (51), we see that

$$\lim_{\varepsilon\to\varepsilon_1}\sigma_1^{\varepsilon} = \lim_{\varepsilon\to\varepsilon_1}\sigma_2^{\varepsilon} = \sigma. \tag{60}$$

Thus from (60) the two shocks S_1 and S_2 will coalesce together when ε arrives at ε_1. Using the Rankine-Hugoniot condition (15) for both S_1 and S_2, we have

$$\begin{aligned}&\sigma_1^{\varepsilon}\left(\frac{\rho_*^{\varepsilon}}{c^2-\left(v_*^{\varepsilon}\right)^2}\left(c^2-\frac{\varepsilon\left(v_*^{\varepsilon}\right)^2}{\left(\rho_*^{\varepsilon}\right)^2c^2}\right)-\frac{\rho_-}{c^2-v_-^2}\left(c^2-\frac{\varepsilon v_-^2}{\rho_-^2c^2}\right)\right)\\ &\quad=\frac{\rho_*^{\varepsilon}v_*^{\varepsilon}}{c^2-\left(v_*^{\varepsilon}\right)^2}\left(c^2-\frac{\varepsilon}{\left(\rho_*^{\varepsilon}\right)^2}\right)-\frac{\rho_-v_-}{c^2-v_-^2}\left(c^2-\frac{\varepsilon}{\rho_-^2}\right),\\ &\sigma_2^{\varepsilon}\left(\frac{\rho_+}{c^2-v_+^2}\left(c^2-\frac{\varepsilon v_+^2}{\rho_+^2c^2}\right)-\frac{\rho_*^{\varepsilon}}{c^2-\left(v_*^{\varepsilon}\right)^2}\left(c^2-\frac{\varepsilon\left(v_*^{\varepsilon}\right)^2}{\left(\rho_*^{\varepsilon}\right)^2c^2}\right)\right)\\ &\quad=\frac{\rho_+v_+}{c^2-v_+^2}\left(c^2-\frac{\varepsilon}{\rho_+^2}\right)-\frac{\rho_*^{\varepsilon}v_*^{\varepsilon}}{c^2-\left(v_*^{\varepsilon}\right)^2}\left(c^2-\frac{\varepsilon}{\left(\rho_*^{\varepsilon}\right)^2}\right),\end{aligned} \tag{61}$$

which yields

$$\begin{aligned}&\frac{\rho_*^{\varepsilon}}{c^2-\left(v_*^{\varepsilon}\right)^2}\left(c^2-\frac{\varepsilon\left(v_*^{\varepsilon}\right)^2}{\left(\rho_*^{\varepsilon}\right)^2c^2}\right)\left(\sigma_1^{\varepsilon}-\sigma_2^{\varepsilon}\right)\\ &\quad+\frac{\rho_+\sigma_2^{\varepsilon}}{c^2-v_+^2}\left(c^2-\frac{\varepsilon v_+^2}{\rho_+^2c^2}\right)-\frac{\rho_-\sigma_1^{\varepsilon}}{c^2-v_-^2}\left(c^2-\frac{\varepsilon v_-^2}{\rho_-^2c^2}\right)\\ &\quad=\frac{\rho_+v_+}{c^2-v_+^2}\left(c^2-\frac{\varepsilon}{\rho_+^2}\right)-\frac{\rho_-v_-}{c^2-v_-^2}\left(c^2-\frac{\varepsilon}{\rho_-^2}\right).\end{aligned} \tag{62}$$

Letting $\varepsilon \to \varepsilon_1$, we have

$$\begin{aligned}&\lim_{\varepsilon\to\varepsilon_1}\left(\sigma_2^{\varepsilon}-\sigma_1^{\varepsilon}\right)\frac{\rho_*^{\varepsilon}c^2}{c^2-\left(v_*^{\varepsilon}\right)^2}\\ &\quad=\sigma\left[\frac{\rho}{c^2-v^2}\left(c^2-\frac{\varepsilon_1v^2}{\rho^2c^2}\right)\right]-\left[\frac{\rho v}{c^2-v^2}\left(c^2-\frac{\varepsilon_1}{\rho^2}\right)\right]\\ &\quad=\sigma\left[\frac{\left(\varepsilon_1p+\rho c^2\right)v^2}{c^2\left(c^2-v^2\right)}+\rho\right]-\left[\frac{\left(\varepsilon_1p+\rho c^2\right)v}{c^2-v^2}\right].\end{aligned} \tag{63}$$

According to

$$\lim_{\varepsilon\to\varepsilon_1}\int_{\sigma_1^{\varepsilon}t}^{\sigma_2^{\varepsilon}t}\frac{\rho_*^{\varepsilon}c^2}{c^2-\left(v_*^{\varepsilon}\right)^2}dx = \lim_{\varepsilon\to\varepsilon_1}\left(\sigma_2^{\varepsilon}-\sigma_1^{\varepsilon}\right)\frac{\rho_*^{\varepsilon}c^2}{c^2-\left(v_*^{\varepsilon}\right)^2}t. \tag{64}$$

Equation (57) can be easily obtained. □

It can be concluded from Lemma 5 that the two shock waves S_1 and S_2 will coincide when ε tends to ε_1 (see Figure 3(b)). We give the following result which provides a very nice depiction of the limit in the case $v_- > v_+$. This theorem is similar to that in [37].

Theorem 6. *Let $v_- > v_+$. For each fixed $\varepsilon \in (\varepsilon_1, \varepsilon_0)$, assume that $(v^{\varepsilon}, \rho^{\varepsilon})$ is a solution containing two shocks S_1 and S_2 of* (5)-(6) *with initial data* (8), *constructed in Section 3. Then, $(v^{\varepsilon}, \rho^{\varepsilon})$ converges in the sense of distributions as $\varepsilon \to \varepsilon_1$, and the limit functions $\rho c^2/(c^2-v^2)$ and $\rho vc^2/(c^2-v^2)$ are the sums of a step function and a δ-measure with weights*

$$\begin{aligned}&\left(\sigma\left[\frac{\left(\varepsilon_1p+\rho c^2\right)v^2}{c^2\left(c^2-v^2\right)}+\rho\right]-\left[\frac{\left(\varepsilon_1p+\rho c^2\right)v}{c^2-v^2}\right]\right)t,\\ &\left(\sigma\left[\frac{\left(\varepsilon_1p+\rho c^2\right)v}{c^2-v^2}\right]-\left[\frac{\left(\varepsilon_1p+\rho c^2\right)v^2}{c^2-v^2}+\varepsilon_1p\right]\right)t,\end{aligned} \tag{65}$$

respectively, which form a delta shock solution of (5)-(6) *when $\varepsilon = \varepsilon_1$.*

Proof. Let $\xi = x/t$. Then for each fixed $\varepsilon > 0$, the Riemann solution is determined by

$$\left(v^{\varepsilon}, \rho^{\varepsilon}\right)(\xi) = \begin{cases}\left(v_-, \rho_-\right), & \text{as } \xi < \sigma_1^{\varepsilon},\\ \left(v_*^{\varepsilon}, \rho_*^{\varepsilon}\right), & \text{as } \sigma_1^{\varepsilon} < \xi < \sigma_2^{\varepsilon},\\ \left(v_+, \rho_+\right), & \text{as } \xi > \sigma_2^{\varepsilon},\end{cases} \tag{66}$$

which satisfies the following weak formulations:

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{\rho^{\varepsilon}c^{2}}{c^{2}-(v^{\varepsilon})^{2}}\psi(\xi)\,d\xi+\int_{-\infty}^{\infty}\frac{\rho^{\varepsilon}c^{4}+\varepsilon p(\rho^{\varepsilon})(v^{\varepsilon})^{2}}{c^{2}\left(c^{2}-(v^{\varepsilon})^{2}\right)}\xi\psi'(\xi)\,d\xi\\&\quad-\int_{-\infty}^{\infty}\frac{\left(\rho^{\varepsilon}c^{2}+\varepsilon p(\rho^{\varepsilon})\right)v^{\varepsilon}}{c^{2}-(v^{\varepsilon})^{2}}\psi'(\xi)\,d\xi\\&\quad+\int_{-\infty}^{\infty}\frac{\varepsilon p(\rho^{\varepsilon})(v^{\varepsilon})^{2}}{c^{2}\left(c^{2}-(v^{\varepsilon})^{2}\right)}\psi(\xi)\,d\xi=0,\end{aligned}\tag{67}$$

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{\rho^{\varepsilon}c^{2}v^{\varepsilon}}{c^{2}-(v^{\varepsilon})^{2}}\psi(\xi)\,d\xi+\int_{-\infty}^{\infty}\frac{\left(\rho^{\varepsilon}c^{2}+\varepsilon p(\rho^{\varepsilon})\right)v^{\varepsilon}}{c^{2}-(v^{\varepsilon})^{2}}\xi\psi'(\xi)\,d\xi\\&\quad-\int_{-\infty}^{\infty}\frac{\left(\rho^{\varepsilon}(v^{\varepsilon})^{2}+\varepsilon p(\rho^{\varepsilon})\right)c^{2}}{c^{2}-(v^{\varepsilon})^{2}}\psi'(\xi)\,d\xi\\&\quad+\int_{-\infty}^{\infty}\frac{\varepsilon p(\rho^{\varepsilon})v^{\varepsilon}}{c^{2}-(v^{\varepsilon})^{2}}\psi(\xi)\,d\xi=0\end{aligned}\tag{68}$$

for any test function $\psi\in C_0^{\infty}(-\infty,\infty)$. The second integral on the left side of (67) can be decomposed into

$$\left\{\int_{-\infty}^{\sigma_1^{\varepsilon}}+\int_{\sigma_1^{\varepsilon}}^{\sigma_2^{\varepsilon}}+\int_{\sigma_2^{\varepsilon}}^{\infty}\right\}\frac{\rho^{\varepsilon}c^{4}+\varepsilon p(\rho^{\varepsilon})(v^{\varepsilon})^{2}}{c^{2}\left(c^{2}-(v^{\varepsilon})^{2}\right)}\xi\psi'(\xi)\,d\xi,\tag{69}$$

which equals

$$\begin{aligned}&\frac{\rho_-c^{4}+\varepsilon p_-v_-^{2}}{c^{2}(c^{2}-v_-^{2})}\sigma_1^{\varepsilon}\psi(\sigma_1^{\varepsilon})-\frac{\rho_+c^{4}+\varepsilon p_+v_+^{2}}{c^{2}(c^{2}-v_+^{2})}\sigma_2^{\varepsilon}\psi(\sigma_2^{\varepsilon})\\&\quad-\frac{\rho_-c^{4}+\varepsilon p_-v_-^{2}}{c^{2}(c^{2}-v_-^{2})}\int_{-\infty}^{\sigma_1^{\varepsilon}}\psi(\xi)\,d\xi\\&\quad-\frac{\rho_+c^{4}+\varepsilon p_+v_+^{2}}{c^{2}(c^{2}-v_+^{2})}\int_{\sigma_2^{\varepsilon}}^{\infty}\psi(\xi)\,d\xi\\&\quad-\frac{\rho_*^{\varepsilon}c^{4}+\varepsilon p_*^{\varepsilon}v_*^{\varepsilon}}{c^{2}\left(c^{2}-(v_*^{\varepsilon})^{2}\right)}\int_{\sigma_1^{\varepsilon}}^{\sigma_2^{\varepsilon}}\psi(\xi)\,d\xi\\&\quad+\frac{\rho_*^{\varepsilon}c^{4}+\varepsilon p_*^{\varepsilon}v_*^{\varepsilon}}{c^{2}\left(c^{2}-(v_*^{\varepsilon})^{2}\right)}\left(\sigma_2^{\varepsilon}\psi(\sigma_2^{\varepsilon})-\sigma_1^{\varepsilon}\psi(\sigma_1^{\varepsilon})\right).\end{aligned}\tag{70}$$

The third term on the left side of (67) can be calculated by

$$\begin{aligned}&-\left\{\int_{-\infty}^{\sigma_1^{\varepsilon}}+\int_{\sigma_1^{\varepsilon}}^{\sigma_2^{\varepsilon}}+\int_{\sigma_2^{\varepsilon}}^{\infty}\right\}\frac{\left(\rho^{\varepsilon}c^{2}+\varepsilon p(\rho^{\varepsilon})\right)v^{\varepsilon}}{c^{2}-(v^{\varepsilon})^{2}}\psi'(\xi)\,d\xi\\&\quad=-\frac{\left(\rho_-c^{2}+\varepsilon p_-\right)v_-}{c^{2}-v_-^{2}}\psi(\sigma_1^{\varepsilon})\\&\qquad+\frac{\left(\rho_+c^{2}+\varepsilon p_+\right)v_+}{c^{2}-v_+^{2}}\psi(\sigma_2^{\varepsilon})\\&\qquad-\frac{\left(\rho_*^{\varepsilon}c^{2}+\varepsilon p_*^{\varepsilon}\right)v_*^{\varepsilon}}{c^{2}-(v_*^{\varepsilon})^{2}}\left(\psi(\sigma_2^{\varepsilon})-\psi(\sigma_1^{\varepsilon})\right),\end{aligned}\tag{71}$$

and the last term in (67) yields

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{\varepsilon p(\rho^{\varepsilon})(v^{\varepsilon})^{2}}{c^{2}\left(c^{2}-(v^{\varepsilon})^{2}\right)}\psi(\xi)\,d\xi\\&\quad=\left\{\int_{-\infty}^{\sigma_1^{\varepsilon}}+\int_{\sigma_1^{\varepsilon}}^{\sigma_2^{\varepsilon}}+\int_{\sigma_2^{\varepsilon}}^{\infty}\right\}\frac{\varepsilon p(\rho^{\varepsilon})(v^{\varepsilon})^{2}}{c^{2}\left(c^{2}-(v^{\varepsilon})^{2}\right)}\psi(\xi)\,d\xi\\&\quad=\frac{\varepsilon p_-v_-^{2}}{c^{2}(c^{2}-v_-^{2})}\int_{-\infty}^{\sigma_1^{\varepsilon}}\psi(\xi)\,d\xi\\&\qquad+\frac{\varepsilon p_+v_+^{2}}{c^{2}(c^{2}-v_+^{2})}\int_{\sigma_2^{\varepsilon}}^{\infty}\psi(\xi)\,d\xi\\&\qquad+\frac{\varepsilon p_*^{\varepsilon}(v_*^{\varepsilon})^{2}}{c^{2}\left(c^{2}-(v_*^{\varepsilon})^{2}\right)}\int_{\sigma_1^{\varepsilon}}^{\sigma_2^{\varepsilon}}\psi(\xi)\,d\xi.\end{aligned}\tag{72}$$

Combining expressions (67), (70)–(72), we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{\rho^{\varepsilon}c^{2}}{c^{2}-(v^{\varepsilon})^{2}}\psi(\xi)\,d\xi\\&\quad=-\frac{\rho_-c^{4}+\varepsilon p_-v_-^{2}}{c^{2}(c^{2}-v_-^{2})}\sigma_1^{\varepsilon}\psi(\sigma_1^{\varepsilon})+\frac{\rho_+c^{4}+\varepsilon p_+v_+^{2}}{c^{2}(c^{2}-v_+^{2})}\sigma_2^{\varepsilon}\psi(\sigma_2^{\varepsilon})\\&\qquad+\frac{\rho_-c^{2}}{c^{2}-v_-^{2}}\int_{-\infty}^{\sigma_1^{\varepsilon}}\psi(\xi)\,d\xi+\frac{\rho_+c^{2}}{c^{2}-v_+^{2}}\int_{\sigma_2^{\varepsilon}}^{\infty}\psi(\xi)\,d\xi\\&\qquad+\frac{\rho_*^{\varepsilon}c^{2}}{c^{2}-(v_*^{\varepsilon})^{2}}\int_{\sigma_1^{\varepsilon}}^{\sigma_2^{\varepsilon}}\psi(\xi)\,d\xi\\&\qquad-\frac{\rho_*^{\varepsilon}c^{4}+\varepsilon p_*^{\varepsilon}v_*^{\varepsilon}}{c^{2}(c^{2}-v_*^{\varepsilon})}\left(\sigma_2^{\varepsilon}\psi(\sigma_2^{\varepsilon})-\sigma_1^{\varepsilon}\psi(\sigma_1^{\varepsilon})\right)\\&\qquad+\frac{\left(\rho_-c^{2}+\varepsilon p_-\right)v_-}{c^{2}-v_-^{2}}\psi(\sigma_1^{\varepsilon})-\frac{\left(\rho_+c^{2}+\varepsilon p_+\right)v_+}{c^{2}-v_+^{2}}\psi(\sigma_2^{\varepsilon})\\&\qquad+\frac{\left(\rho_*^{\varepsilon}c^{2}+\varepsilon p_*^{\varepsilon}\right)v_*^{\varepsilon}}{c^{2}-(v_*^{\varepsilon})^{2}}\left(\psi(\sigma_2^{\varepsilon})-\psi(\sigma_1^{\varepsilon})\right).\end{aligned}\tag{73}$$

Taking the limit $\varepsilon \to \varepsilon_1$ in (73), observing that $\psi \in C_0^\infty(-\infty,\infty)$ and $p(\rho_*^\varepsilon)$ is bounded, together with the fact $\lim_{\varepsilon\to\varepsilon_1} v_*^\varepsilon = \lim_{\varepsilon\to\varepsilon_1}\sigma_1^\varepsilon = \lim_{\varepsilon\to\varepsilon_1}\sigma_2^\varepsilon = \sigma$, we have

$$\begin{aligned}\lim_{\varepsilon\to\varepsilon_1}\int_{-\infty}^{\infty}&\left(\frac{\rho^\varepsilon c^2}{c^2-(v^\varepsilon)^2}(\xi)-\frac{\rho_0 c^2}{c^2-v_0^2}(\xi-\sigma)\right)\psi(\xi)\,d\xi\\ &=\left(\sigma\left[\frac{\left(\varepsilon_1 p+\rho c^2\right)v^2}{c^2\left(c^2-v^2\right)}+\rho\right]\right.\\ &\qquad\left.-\left[\frac{\left(\varepsilon_1 p+\rho c^2\right)v}{c^2-v^2}\right]\right)\psi(\sigma),\end{aligned}\tag{74}$$

where $a_0(\xi) = a_- + [a]H(\xi-\sigma)$ and H is the Heaviside function. With the same reason as above, the limit of the second and the last integral on the left side of equality (68) as $\varepsilon \to \varepsilon_1$ is

$$\begin{aligned}&-\int_{-\infty}^{\sigma}\frac{\rho_- v_- c^2}{c^2-v_-^2}\psi(\xi)\,d\xi-\int_{\sigma}^{\infty}\frac{\rho_+ v_+ c^2}{c^2-v_+^2}\psi(\xi)\,d\xi\\ &\quad-\sigma\psi(\sigma)\left[\frac{\left(\rho c^2+\varepsilon_1 p\right)v}{c^2-v^2}\right].\end{aligned}\tag{75}$$

As done to (71), due to the fact that $p(\rho_*^\varepsilon)$ is bounded and $\lim_{\varepsilon\to\varepsilon_1}\sigma_1^\varepsilon = \lim_{\varepsilon\to\varepsilon_1}\sigma_2^\varepsilon = \sigma$, the third integral on the left side of equality (68) converges to

$$\psi(\sigma)\left[\frac{\left(\rho v^2+\varepsilon_1 p\right)c^2}{c^2-v^2}\right]\tag{76}$$

as $\varepsilon \to \epsilon_1$. Then it follows from (68), (75), and (76) that

$$\begin{aligned}\lim_{\varepsilon\to\varepsilon_1}\int_{-\infty}^{\infty}&\left(\frac{\rho^\varepsilon v^\varepsilon c^2}{c^2-(v^\varepsilon)^2}(\xi)-\frac{\rho_0 v_0 c^2}{c^2-v_0^2}(\xi-\sigma)\right)\psi(\xi)\,d\xi\\ &=\left(\sigma\left[\frac{\left(\varepsilon_1 p+\rho c^2\right)v}{c^2-v^2}\right]\right.\\ &\qquad\left.-\left[\frac{\left(\varepsilon_1 p+\rho c^2\right)v^2}{c^2-v^2}+\varepsilon_1 p\right]\right)\psi(\sigma).\end{aligned}\tag{77}$$

Finally, we study the limits of $\rho^\varepsilon c^2/(c^2-(v^\varepsilon)^2)$ and $\rho^\varepsilon v^\varepsilon c^2/(c^2-(v^\varepsilon)^2)$ as $\varepsilon \to \varepsilon_1$ by tracing the time-dependence of weights of the δ-measure. Let $\phi(x,t) \in C_0^\infty((-\infty,\infty)\times[0,\infty))$ and set $\tilde\phi(\xi,t) := \phi(\xi t, t)$. Then we have

$$\begin{aligned}\lim_{\varepsilon\to\varepsilon_1}\int_0^\infty\int_{-\infty}^{\infty}&\frac{\rho^\varepsilon c^2}{c^2-(v^\varepsilon)^2}\left(\frac{x}{t}\right)\phi(x,t)\,dx\,dt\\ &=\lim_{\varepsilon\to\varepsilon_1}\int_0^\infty t\left(\int_{-\infty}^{\infty}\frac{\rho^\varepsilon c^2}{c^2-(v^\varepsilon)^2}(\xi)\,\tilde\phi(\xi,t)\,d\xi\right)dt.\end{aligned}\tag{78}$$

On the other hand,

$$\begin{aligned}\lim_{\varepsilon\to\varepsilon_1}\int_{-\infty}^{\infty}&\frac{\rho^\varepsilon c^2}{c^2-(v^\varepsilon)^2}(\xi)\,\tilde\phi(\xi,t)\,d\xi\\ &=\int_{-\infty}^{\infty}\frac{\rho_0 c^2}{c^2-v_0^2}(\xi-\sigma)\,\tilde\phi(\xi,t)\,d\xi\\ &\quad+\left(\sigma\left[\frac{\left(\varepsilon_1 p+\rho c^2\right)v^2}{c^2\left(c^2-v^2\right)}+\rho\right]\right.\\ &\qquad\left.-\left[\frac{\left(\varepsilon_1 p+\rho c^2\right)v}{c^2-v^2}\right]\right)\tilde\phi(\xi,t)\\ &=t^{-1}\int_{-\infty}^{\infty}\frac{\rho_0 c^2}{c^2-v_0^2}(x-\sigma t)\,\phi(x,t)\,dx\\ &\quad+\left(\sigma\left[\frac{\left(\varepsilon_1 p+\rho c^2\right)v^2}{c^2\left(c^2-v^2\right)}+\rho\right]\right.\\ &\qquad\left.-\left[\frac{\left(\varepsilon_1 p+\rho c^2\right)v}{c^2-v^2}\right]\right)\phi(\sigma t,t).\end{aligned}\tag{79}$$

Combining (78) and (79) together, we obtain

$$\begin{aligned}\lim_{\varepsilon\to\varepsilon_1}\int_0^\infty\int_{-\infty}^{\infty}&\frac{\rho^\varepsilon c^2}{c^2-(v^\varepsilon)^2}\left(\frac{x}{t}\right)\phi(x,t)\,dx\,dt\\ &=\int_0^\infty\int_{-\infty}^{\infty}\frac{\rho_0 c^2}{c^2-v_0^2}(x-\sigma t)\,\phi(x,t)\,dx\,dt\\ &\quad+\int_0^\infty t\left(\sigma\left[\frac{\left(\varepsilon_1 p+\rho c^2\right)v^2}{c^2\left(c^2-v^2\right)}+\rho\right]\right.\\ &\qquad\left.-\left[\frac{\left(\varepsilon_1 p+\rho c^2\right)v}{c^2-v^2}\right]\right)\phi(\sigma t,t)\,dt.\end{aligned}\tag{80}$$

With the same reason as before, we deduce that

$$\begin{aligned}\lim_{\varepsilon\to\varepsilon_1}\int_0^\infty\int_{-\infty}^{\infty}&\frac{\rho^\varepsilon v^\varepsilon c^2}{c^2-(v^\varepsilon)^2}\left(\frac{x}{t}\right)\phi(x,t)\,dx\,dt\\ &=\int_0^\infty\int_{-\infty}^{\infty}\frac{\rho_0 v_0 c^2}{c^2-v_0^2}(x-\sigma t)\,\phi(x,t)\,dx\,dt\\ &\quad+\int_0^\infty t\left(\sigma\left[\frac{\left(\varepsilon_1 p+\rho c^2\right)v}{c^2-v^2}\right]\right.\\ &\qquad\left.-\left[\frac{\left(\varepsilon_1 p+\rho c^2\right)v^2}{c^2-v^2}+\varepsilon_1 p\right]\right)\phi(\sigma t,t)\,dt.\end{aligned}\tag{81}$$

Thus the results have been obtained. □

Comparing the above results with (28), we can see that the quantities σ_δ, $\omega^\varepsilon(t)$ and the limits of v_*^ε, σ_1^ε, and σ_2^ε should be consistent with (31) as proposed for the Riemann solutions to (5)-(6) for $E \neq 0$ when we take $\varepsilon = \varepsilon_1$. If $E = 0$, then the assert is obviously true. Thus, as $\varepsilon \to \varepsilon_1$ it uniquely determines that the limit of the Riemann solutions to system (5)-(6) in the case $(v_+, \rho_+) \in \mathrm{IV}(v_-, \rho_-)$ is just the delta shock solution of (5)-(6) in the case $(v_+, \rho_+) \in S_\delta$, where the curve S_δ is actually the boundary between the regions $\mathrm{IV}(v_-, \rho_-)$ and $\mathrm{V}(v_-, \rho_-)$.

4.2. Limit Behavior of Delta Shock Wave. In this subsection, we continue to discuss the situation $0 < \varepsilon < \varepsilon_1$, in which $(v_+, \rho_+) \in \mathrm{V}(v_-, \rho_-)$ with $v_- > v_+$ and $\rho_\pm > \sqrt{\varepsilon}/c$. In this case, the Riemann solution to system (5)-(6) contains a delta shock wave besides two constant states $(v_\pm, \rho_\pm)$. We want to observe the behavior of strength and propagation speed of the delta shock wave when ε decreases and finally tends to zero.

If $0 < \varepsilon < \varepsilon_1$, the Riemann solution to system (5)-(6) contains a delta shock wave whose strength and propagation speed can be expressed by (31) for $E \neq 0$. Rewrite σ_δ in (31) as

$$\begin{aligned}\sigma_\delta = \Bigg(& \frac{\rho_+ v_+}{c^2 - v_+^2}\left(c^2 - \frac{\varepsilon}{\rho_+^2}\right) \\ & - \frac{\rho_- v_-}{c^2 - v_-^2}\left(c^2 - \frac{\varepsilon}{\rho_-^2}\right) \\ & + \sqrt{\frac{\rho_+\rho_-\alpha(\varepsilon)}{(c^2 - v_+^2)(c^2 - v_-^2)}}\Bigg) \\ & \times \left(\frac{\rho_+}{c^2 - v_+^2}\left(c^2 - \frac{\varepsilon v_+^2}{\rho_+^2 c^2}\right)\right. \\ & \left. - \frac{\rho_-}{c^2 - v_-^2}\left(c^2 - \frac{\varepsilon v_-^2}{\rho_-^2 c^2}\right)\right)^{-1},\end{aligned} \tag{82}$$

where

$$\begin{aligned}\alpha(\varepsilon) = {} & (v_- - v_+)^2\left(c^2 - \frac{\varepsilon}{\rho_+\rho_-}\right)^2 \\ & - \varepsilon\left(\frac{1}{\rho_+} - \frac{1}{\rho_-}\right)^2 (c^2 - v_+ v_-)^2.\end{aligned} \tag{83}$$

It can be derived from (31), (82), and (83) that

$$\frac{d}{d\varepsilon}\left(\frac{w(t)c^2}{c^2 - \sigma_\delta^2}\right) < 0, \tag{84}$$

which means the strength of the delta shock wave increases when ε decreases. Furthermore, taking the limit $\varepsilon \to 0$ in (82) leads to

$$\lim_{\varepsilon \to 0} \sigma_\delta = \frac{v_+\sqrt{\rho_+/(c^2 - v_+^2)} + v_-\sqrt{\rho_-/(c^2 - v_-^2)}}{\sqrt{\rho_+/(c^2 - v_+^2)} + \sqrt{\rho_-/(c^2 - v_-^2)}}. \tag{85}$$

Thus one can easily deduce from (31) and (85) that

$$\lim_{\varepsilon \to 0} w^\varepsilon(t) = \sqrt{\frac{\rho_+}{c^2 - v_+^2}\frac{\rho_-}{c^2 - v_-^2}}\,(v_- - v_+)\left(c^2 - v_\delta^2\right)t. \tag{86}$$

Thus, the limit values of σ_δ and $w^\varepsilon(t)$ are identical with (38). For the special case $E = 0$, the same result can be derived from (32).

From the above discussion, we can see that the limit of the strength and propagation speed of the delta shock wave in the Riemann solution to system (5)-(6) is in accordance with that of system (4) with the same initial data (8). That is to say, the delta shock solution to system (5)-(6) converges to the delta shock solution to system (4) as pressure vanishes.

Combining the results of the above two subsections, we conclude that the two shock waves of the Riemann solution to system (5)-(6) become steeper as ε decreases and then coincide with a delta shock wave at a certain critical value ε_1. As ε continues to drop, the strength of this delta shock becomes stronger and stronger. In the end, as ε goes to zero, the delta shock solution is nothing but the Riemann solution to the zero-pressure relativistic Euler system (4).

5. Limit of Riemann Solutions to (5)-(6) for $v_- < v_+$

In this section, we will show the limit behavior of rarefaction waves in the Riemann solutions to (5)-(6) and (8) when ε tends to zero in the case $(v_+, \rho_+) \in \mathrm{I}(v_-, \rho_-)$ with $v_- < v_+$ and $\rho_\pm > \sqrt{\varepsilon}/c$. See Figure 4(a).

Lemma 7. *If $v_- < v_+$, then there exists $\varepsilon_0 > 0$ such that $(v_+, \rho_+) \in I(v_-, \rho_-)$ for $0 < \varepsilon < \varepsilon_0$.*

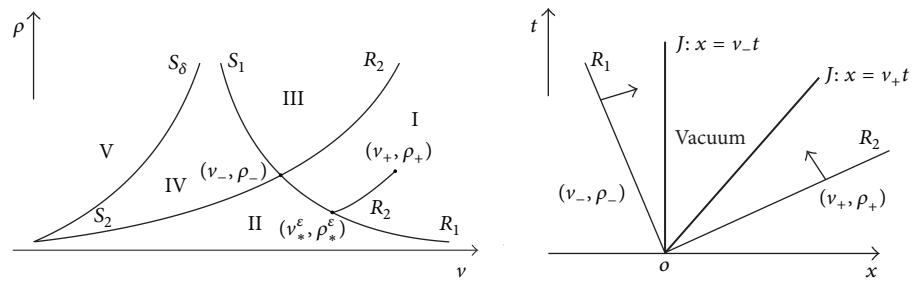
Proof. It follows from (13) and (14) that all possible states (v, ρ) which can be connected on the right side to the left state (v_-, ρ_-) by a 1-rarefaction wave R_1 or a 2-rarefaction wave R_2 should satisfy

$$\begin{aligned} R_1 : \frac{\rho v - \sqrt{\varepsilon}}{\rho c^2 - \sqrt{\varepsilon} v} &= \frac{\rho_- v_- - \sqrt{\varepsilon}}{\rho_- c^2 - \sqrt{\varepsilon} v_-}, \quad \rho < \rho_-, \\ R_2 : \frac{\rho v + \sqrt{\varepsilon}}{\rho c^2 + \sqrt{\varepsilon} v} &= \frac{\rho_- v_- + \sqrt{\varepsilon}}{\rho_- c^2 + \sqrt{\varepsilon} v_-}, \quad \rho > \rho_-. \end{aligned} \tag{87}$$

Particularly, if $\rho_+ = \rho_-$, then $(v_+, \rho_+) \in \mathrm{I}(v_-, \rho_-)$ for any $\varepsilon > 0$. Thus we can take arbitrary $\varepsilon_0 > 0$ in this special situation.

Let $\rho_+ \neq \rho_-$ and $(v_+, \rho_+) \in \mathrm{I}(v_-, \rho_-)$. Observing that $(\rho v - \sqrt{\varepsilon})/(\rho c^2 - \sqrt{\varepsilon} v)$ and $(\rho v + \sqrt{\varepsilon})/(\rho c^2 + \sqrt{\varepsilon} v)$ are both monotone increasing with v, we can see intuitively from Figure 4(a) together with (87) that ε satisfies

$$\begin{aligned} \frac{\rho_+ v_+ - \sqrt{\varepsilon}}{\rho_+ c^2 - \sqrt{\varepsilon} v_+} &> \frac{\rho_- v_- - \sqrt{\varepsilon}}{\rho_- c^2 - \sqrt{\varepsilon} v_-}, \quad \text{if } \rho_+ < \rho_-, \\ \frac{\rho_+ v_+ + \sqrt{\varepsilon}}{\rho_+ c^2 + \sqrt{\varepsilon} v_+} &> \frac{\rho_- v_- + \sqrt{\varepsilon}}{\rho_- c^2 + \sqrt{\varepsilon} v_-}, \quad \text{if } \rho_+ > \rho_-, \end{aligned} \tag{88}$$



(a) The phase plane (b) The (x, t)-plane

Figure 4: Riemann solutions when $(v_+, \rho_+) \in \mathrm{I}(v_-, \rho_-)$.

which imply that

$$(v_+ - v_-)\varepsilon + |\rho_- - \rho_+|(c^2 - v_+ v_-)\sqrt{\varepsilon} - c^2 \rho_+ \rho_- (v_+ - v_-) < 0. \tag{89}$$

Thus we can take

$$\sqrt{\varepsilon_0} := \frac{|\rho_- - \rho_+|(v_+ v_- - c^2)}{2(v_+ - v_-)} + \frac{\sqrt{(\rho_- - \rho_+)^2 (c^2 - v_+ v_-)^2 + 4c^2 \rho_+ \rho_- (v_+ - v_-)^2}}{2(v_+ - v_-)}. \tag{90}$$

Obviously we have $(v_+, \rho_+) \in \mathrm{I}(v_-, \rho_-)$ if $\varepsilon < \varepsilon_0$. □

Lemma 7 implies that the rarefaction wave curves R_1 and R_2 become steeper when ε goes to zero. As $v_- < v_+$, by Lemma 7, for any given $\varepsilon \in (0, \varepsilon_0)$, the Riemann solution to system (5)-(6) with initial data (8) consists of two constant states $(v_\pm, \rho_\pm)$, an intermediate state $(v_*^\varepsilon, \rho_*^\varepsilon)$, and two rarefaction curves R_1, R_2. See Figure 4(a). Then, it follows from (13) and (14) that

$$\begin{aligned} R_1 :\ \lambda_1^\varepsilon &= \frac{c^2(\rho v - \sqrt{\varepsilon})}{\rho c^2 - \sqrt{\varepsilon} v} = \frac{c^2(\rho_- v_- - \sqrt{\varepsilon})}{\rho_- c^2 - \sqrt{\varepsilon} v_-} = \lambda_{1-}^\varepsilon, \\ &\qquad \rho_*^\varepsilon \le \rho \le \rho_-, \\ R_2 :\ \lambda_2^\varepsilon &= \frac{c^2(\rho v + \sqrt{\varepsilon})}{\rho c^2 + \sqrt{\varepsilon} v} = \frac{c^2(\rho_+ v_+ + \sqrt{\varepsilon})}{\rho_+ c^2 + \sqrt{\varepsilon} v_+} = \lambda_{2+}^\varepsilon, \\ &\qquad \rho_*^\varepsilon \le \rho \le \rho_+. \end{aligned} \tag{91}$$

From (91), we can derive

$$(\lambda_{2+}^\varepsilon - \lambda_{1-}^\varepsilon)\frac{\varepsilon}{(\rho_*^\varepsilon)^2} + 2(\lambda_{1-}^\varepsilon \lambda_{2+}^\varepsilon - c^2)\frac{\sqrt{\varepsilon}}{\rho_*^\varepsilon} + (\lambda_{2+}^\varepsilon - \lambda_{1-}^\varepsilon)c^2 = 0. \tag{92}$$

So ρ_*^ε can be expressed as

$$\rho_*^\varepsilon = \frac{\sqrt{\varepsilon}(\lambda_{2+}^\varepsilon - \lambda_{1-}^\varepsilon)}{c^2 - \lambda_{1-}^\varepsilon \lambda_{2+}^\varepsilon + \sqrt{(c^2 - (\lambda_{1-}^\varepsilon)^2)(c^2 - (\lambda_{2+}^\varepsilon)^2)}}. \tag{93}$$

It is observed that the propagation speed of R_1 increases with the decrease of ε, while the propagation speed of R_2 decreases with the decrease of ε. Thus from $\lim_{\varepsilon \to 0} \lambda_{1-}^\varepsilon = v_-$, $\lim_{\varepsilon \to 0} \lambda_{2+}^\varepsilon = v_+$, (93) yields

$$\lim_{\varepsilon \to 0} \rho_*^\varepsilon = 0, \qquad \lim_{\varepsilon \to 0} \lambda_1^\varepsilon = v_-, \qquad \lim_{\varepsilon \to 0} \lambda_2^\varepsilon = v_+. \tag{94}$$

The above identities assert that as ε drops to zero, ρ_*^ε vanishes and two rarefaction waves R_1 and R_2 become two contact discontinuities connecting the constant states $(v_\pm, \rho_\pm)$ and the vacuum ($\rho_* = 0$), with the speeds v_- and v_+, respectively; see Figure 4(b). From the above discussion, we can summarize our results as follows.

Theorem 8. *In the case of $v_- < v_+$, as ε drops to zero, the Riemann solution of* (5)-(6) *with initial data* (8) *converges to a vacuum solution, which is exactly the corresponding Riemann solution to system* (4) *with the same initial data.*

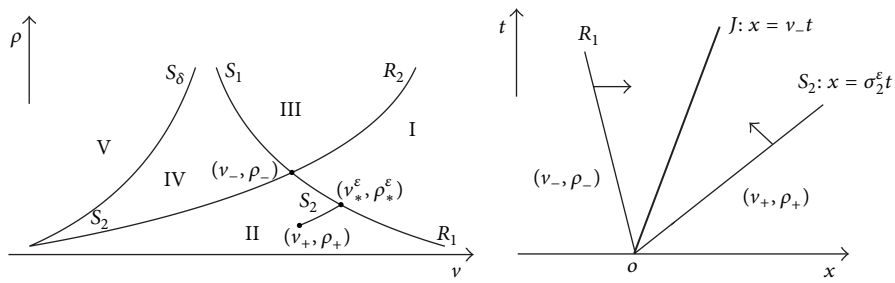
6. Limit of Riemann Solutions to (5)-(6) for $v_- = v_+$

In this section, we consider the limit of Riemann solutions to system (5)-(6) with initial data (8) when $v_- = v_+$. In this case, the Riemann solutions contain a 1-rarefaction wave R_1 and a 2-shock wave S_2 for $\rho_+ < \rho_-$ or a 1-shock wave S_1 and a 2-rarefaction wave R_2 for $\rho_+ > \rho_-$. Particularly, if $\rho_+ = \rho_-$, the solution is a constant state (v_-, ρ_-).

For the case $\rho_+ < \rho_-$, we have $(v_+, \rho_+) \in \mathrm{II}(v_-, \rho_-)$ obviously, as shown in Figure 5(a), the Riemann solution of (5)-(6) has the construction as

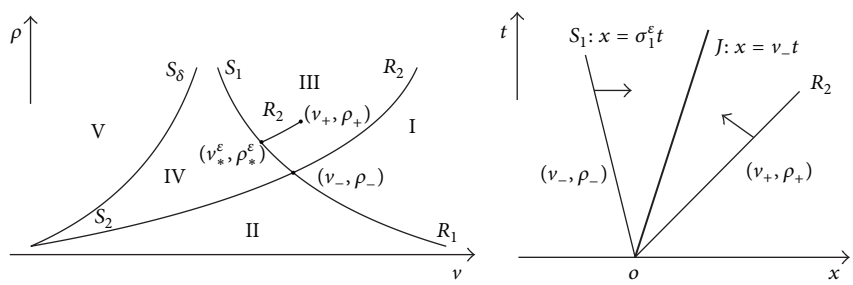
$$(v^\varepsilon, \rho^\varepsilon)(\xi) = \begin{cases} (v_-, \rho_-), & -\infty < \xi \le \lambda_{1-}^\varepsilon, \\ R_1, & \lambda_{1-}^\varepsilon \le \xi \le \lambda_{1*}^\varepsilon, \\ (v_*^\varepsilon, \rho_*^\varepsilon), & \lambda_{1*}^\varepsilon \le \xi < \sigma_2^\varepsilon, \\ (v_+, \rho_+), & \sigma_2^\varepsilon < \xi < \infty, \end{cases} \tag{95}$$

where σ_2^ε is the propagation speed of S_2 and R_1 consists of the states (v, ρ) satisfying (13).



(a) The phase plane (b) The (x, t)-plane

Figure 5: Riemann solutions when $(v_+, \rho_+) \in \mathrm{II}(v_-, \rho_-)$.



(a) The phase plane (b) The (x, t)-plane

Figure 6: Riemann solutions when $(v_+, \rho_+) \in \mathrm{III}(v_-, \rho_-)$.

By (13) and (18), the intermediate state $(v_*^\varepsilon, \rho_*^\varepsilon)$ between R_1 and S_2 satisfies

$$\frac{\rho_*^\varepsilon v_*^\varepsilon - \sqrt{\varepsilon}}{\rho_*^\varepsilon c^2 - \sqrt{\varepsilon} v_*^\varepsilon} = \frac{\rho_- v_- - \sqrt{\varepsilon}}{\rho_- c^2 - \sqrt{\varepsilon} v_-}, \tag{96}$$

$$\frac{\rho_*^\varepsilon v_*^\varepsilon + \sqrt{\varepsilon}}{\rho_*^\varepsilon c^2 + \sqrt{\varepsilon} v_*^\varepsilon} = \frac{\rho_+ v_+ + \sqrt{\varepsilon}}{\rho_+ c^2 + \sqrt{\varepsilon} v_+}, \tag{97}$$

with $\rho_+ < \rho_*^\varepsilon < \rho_-$. Letting $\varepsilon \to 0$ in (96), one can immediately get $\lim_{\varepsilon \to 0} v_*^\varepsilon = v_-$, noticing the fact that ρ_*^ε is bounded. Meanwhile, we have

$$\begin{aligned} \lim_{\varepsilon \to 0} \sigma_2^\varepsilon &= \lim_{\varepsilon \to 0} \frac{c^2 (\rho_+ v_+ + \sqrt{\varepsilon})}{\rho_+ c^2 + \sqrt{\varepsilon} v_+} = v_-, \\ \lim_{\varepsilon \to 0} \lambda_{1*}^\varepsilon &= \lim_{\varepsilon \to 0} \frac{c^2 (\rho_*^\varepsilon v_*^\varepsilon - \sqrt{\varepsilon})}{\rho_*^\varepsilon c^2 - \sqrt{\varepsilon} v_*^\varepsilon} = v_-, \\ \lim_{\varepsilon \to 0} \lambda_{1-}^\varepsilon &= \lim_{\varepsilon \to 0} \frac{c^2 (\rho_- v_- - \sqrt{\varepsilon})}{\rho_- c^2 - \sqrt{\varepsilon} v_-} = v_-. \end{aligned} \tag{98}$$

From identities (98), as ε goes to zero, we conclude that the rarefaction wave R_1 and shock wave S_2 converge to one contact discontinuity with the propagation speed v_-, which connects the constant states $(v_\pm, \rho_\pm)$; see Figure 5(b).

For the case $\rho_+ > \rho_-$, $(v_+, \rho_+) \in \mathrm{III}(v_-, \rho_-)$, as shown in Figure 6(a), the Riemann solution of (5)-(6) has the construction as

$$(v^\varepsilon, \rho^\varepsilon)(\xi) = \begin{cases} (v_-, \rho_-), & -\infty < \xi < \sigma_1^\varepsilon, \\ (v_*^\varepsilon, \rho_*^\varepsilon), & \sigma_1^\varepsilon < \xi \le \lambda_{2*}^\varepsilon, \\ R_2, & \lambda_{2*}^\varepsilon \le \xi \le \lambda_{2+}^\varepsilon, \\ (v_+, \rho_+), & \lambda_{2+}^\varepsilon \le \xi < \infty, \end{cases} \tag{99}$$

where σ_1^ε is the propagation speed of S_1 and R_2 consists of the states (v, ρ) satisfying (14). By (14) and (17), the intermediate state $(v_*^\varepsilon, \rho_*^\varepsilon)$ between R_2 and S_1 satisfies (96) and (97) with $\rho_- < \rho_*^\varepsilon < \rho_+$. Similarly, we have

$$\lim_{\varepsilon \to 0} \lambda_{2*}^\varepsilon = \lim_{\varepsilon \to 0} \lambda_{2+}^\varepsilon = \lim_{\varepsilon \to 0} \sigma_1^\varepsilon = v_-. \tag{100}$$

Thus as ε goes to zero, the rarefaction wave R_2 and shock wave S_1 converge to a contact discontinuity with the propagation speed v_-, which connects the constant states $(v_\pm, \rho_\pm)$; see Figure 6(b). From the above discussion, we can summarize our results as follows.

Theorem 9. *In the case of $v_- = v_+$, as ε drops to zero, the Riemann solution of* (5)-(6) *with initial data* (8) *converges to a contact discontinuity connecting the constant states $(v_\pm, \rho_\pm)$,*

which is exactly the corresponding Riemann solution to system (4) *with the same initial data.*

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