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*Research Article*

# Perturbation Theory for Abstract Volterra Equations

**Marko Kostić**

*Faculty of Technical Sciences, University of Novi Sad, Trg Dositeja Obradovića 6, 21125 Novi Sad, Serbia*

Correspondence should be addressed to Marko Kostić; marco.s@verat.net

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We consider additive perturbation theorems for subgenerators of $(a,k)$-regularized $C$-resolvent families. A major part of our research is devoted to the study of perturbation properties of abstract time-fractional equations, primarily from their importance in modeling of various physical phenomena. We illustrate the results with several examples.

Dedicated to the memory of Slobodan Novaković

## 1. Introduction and Preliminaries

A recently introduced notion of an $(a,k)$-regularized $C$-resolvent family on a sequentially complete locally convex space $E$ plays an important role in the theory of abstract Volterra equations. A lot of effort has been directed towards characterizing spectral properties of subgenerators of $(a,k)$-regularized $C$-resolvent families, smoothing and duality properties, a generalized variation of parameters formula and subordination principles. The aim of this paper is to present a comprehensive survey of results about perturbation properties of abstract Volterra equations.

The paper is organized as follows. In the second section, we consider bounded perturbation theorems for subgenerators of $(a,k)$-regularized $C$-resolvent families. A new line of approach to bounded commuting perturbations of abstract time-fractional equations is developed in Theorem 5. Our analysis is inspired, on the one side, by the incompleteness of the study of bounded perturbations of integrated $C$-cosine functions and, on the other side, by the possibilities of extension of [1, Theorem 2.5.3] to fractional operator families. We consider an exponentially equicontinuous $(g_\alpha,k)$-regularized $C$-resolvent family $(R(t))_{t\geq 0}$ with a subgenerator $A$ ($g_\alpha(t)\equiv t^{\alpha-1}/\Gamma(\alpha)$, $t>0$, $\alpha>0$), a function $k_1(t)$ satisfying certain properties and an $A$-bounded perturbation $B$ such that $BA\subseteq AB$ and $BC=CB$. In order to prove the existence of perturbed $(g_\alpha,k_1)$-regularized $C$-resolvent family $(R_B(t))_{t\geq 0}$ with a subgenerator $A+B$, we employ the method that involves only direct computations and differs from those established in [2–12] in that we do not consider $(R_B(t))_{t\geq 0}$ as the unique solution of a corresponding integral equation. The main objective in Theorem 7 is to show that, under some additional conditions, the perturbed $(g_\alpha,k_1)$-regularized $C$-resolvent family $(R_B(t))_{t\geq 0}$ inherits analytical properties from $(R(t))_{t\geq 0}$. In case $\alpha=2$ and $B$ satisfies the aforementioned conditions, Corollary 8 produces significantly better results compared with [13, Theorem 10.1] and [5, Theorem 3.1]. This is important since Hieber [14] proved that the Laplacian $\Delta$ with maximal distributional domain generates an exponentially bounded $\alpha$-times integrated cosine function on $L^p(\mathbb{R}^n)$ ($1\leq p<\infty$, $n\in\mathbb{N}$) for any $\alpha\geq(n-1)|(1/2)-(1/p)|$. Notice also that Keyantuo and Warma proved in [15] a similar result for the Laplacian $\Delta$ on $L^p([0,\pi]^n)$, with Dirichlet or Neumann boundary conditions. In Corollary 11, we focus our attention to the case $k(t)=\mathscr{L}^{-1}(\lambda^{-\alpha}e^{-\varrho\lambda^\sigma})(t)$, $t\geq 0$ ($\alpha>1$, $\varrho>0$, $\sigma\in(0,1)$), which is important in the theory of ultradistribution semigroups of Gevrey type. As a special case of Corollary 11, we obtain that the class of tempered ultradistribution sines of $(p!^s)$-class ($\{p!^s\}$-class) is stable under bounded commuting perturbations ($s>1$); cf. [16], [17, Definition 13, Remark 15], [1, Section 3.5], [18], and the final part of the third section for more details. It is worthwhile to mention here the following fact: in order for the proof of Theorem 5 to work, one has to assume that the

considered $(g_\alpha, k)$-regularized $C$-resolvent family $(R(t))_{t\geq 0}$ is exponentially equicontinuous. It seems to be really difficult to prove an analogue of Theorem 5 in the context of local $(g_\alpha, k)$-regularized $C$-resolvent families (cf. [3, 7, 8, 13] and [1, Section 2.5, Theorem 3.5.17] for further information in this direction), which implies, however, that it is not clear whether the class of ultradistribution sines of $(p!^s)$-class ($\{p!^s\}$-class) retains the property stated above. In Theorems 13 and 14, Remark 15, and Corollary 17, we continue the researches of Arendt and Kellermann [2], Lizama and Sánchez [9], and Rhandi [4]. The local Hölder continuity with exponent $\sigma \in (0, 1]$ is the property stable under perturbations considered in these assertions, as explained in Remark 16.

The final part of the paper is devoted to the study of unbounded perturbation theorems. The main purpose of Theorems 20 and 21 is to generalize perturbation results of Kaiser and Weis [19]. The loss of regularity appearing in Theorem 20 is slightly reduced in Theorem 21 by assuming that the underlying Banach space $E$ has certain geometrical properties. As an application, we consider $(g_\alpha, g_{r+1})$-regularized resolvent families generated by higher order differential operators $(0 < \alpha \leq 2,\ r \geq 0)$. Perturbations of subgenerators of analytic $(a, k)$-regularized $C$-resolvent families are also analyzed in Theorem 24, which might be surprising in the case $C \neq I$. The above result is applied to abstract time-fractional equations considered in [20, 21] and to differential operators in the spaces of Hölder continuous functions (von Wahl [22]). Possible applications of Corollary 8 and Theorem 7 can be also made to coercive differential operators considered by Li et al. [23, Section 4] and by the author [24]. In the remainder of the third section, we reconsider and slightly improve results of Arendt and Batty [25] and Desch et al. [26] on rank-1 perturbations. Before we collect the material needed later on, we would like to draw the attention to paper [27] of Xiao et al. for the analysis of time-dependent perturbations of abstract Volterra equations. The results obtained in [27] can be straightforwardly generalized to the class of $(a, C)$-regularized resolvent families, and it is not the intention in this paper to go into further details (cf. also [28–30] and the review paper [31] for time-dependent perturbations).

Henceforth, $E$ denotes a Hausdorff sequentially complete locally convex space, SCLCS for short, and the abbreviation $\circledast$ stands for the fundamental system of seminorms which defines the topology of $E$; if $E$ is a Banach space, then $\|x\|$ denotes the norm of an element $x \in E$. If $F$ is a SCLCS, then we denote by $L(E, F)$ the space of all continuous linear mappings from $E$ into $F$; $L(E) := L(E, E)$. We assume that $A$ is a closed linear operator acting on $E$ and that (with the exception of assertions concerning rank-1 perturbations) $L(E) \ni C$ is an injective operator with $CA \subseteq AC$; the convolution like mapping $*$ is given by $f * g(t) := \int_0^t f(t-s)g(s)\,ds$, and the principal branch is always used to take the powers. Given $f \in L^1_{\mathrm{loc}}([0, \infty))$ and $n \in \mathbb{N}$, $f^{*,n}(t)$ denotes the $n$th convolution power of $f(t)$, and $f^{*,0}(t)$ denotes the Dirac $\delta$-distribution. If $s \in \mathbb{R}$ and $\beta \in (0, \pi]$, then $\lceil s \rceil := \inf\{k \in \mathbb{Z} : k \geq s\}$ and $\Sigma_\beta := \{z \in \mathbb{C} : z \neq 0, |\arg(z)| < \beta\}$. The domain, range, and resolvent set of $A$ are denoted by $D(A)$, $R(A)$, and $\rho(A)$, respectively. If $D(A)$ is not dense in $E$, then $\overline{D(A)}$ is a closed subspace of $E$ and therefore a SCLCS itself; the fundamental system of seminorms which defines the topology of $\overline{D(A)}$ is $(p_{|\overline{D(A)}})_{p\in\circledast}$. Recall that the $C$-resolvent set of $A$, in short $\rho_C(A)$, is defined by $\rho_C(A) := \{\lambda \in \mathbb{C} : \lambda - A$ is injective and $(\lambda - A)^{-1}C \in L(E)\}$.

Fairly complete information on the general theory of well-posed abstract Volterra equations in Banach spaces can be obtained by consulting the monograph [10] of Prüss. The following notion is crucially important in the theory of ill-posed Volterra equations (cf. [32–35]).

*Definition 1.* (i) Let $E$ be an SCLCS, let $0 < \tau \leq \infty$, $k \in C([0, \tau))$, $k \neq 0$, and let $a \in L^1_{\mathrm{loc}}([0, \tau))$, $a \neq 0$. A strongly continuous operator family $(R(t))_{t\in[0,\tau)}$ is called a (local, if $\tau < \infty$) $(a, k)$-regularized $C$-resolvent family having $A$ as a subgenerator if and only if the following holds:

(a) $R(t)A \subseteq AR(t)$, $t \in [0, \tau)$, $R(0) = k(0)C$, and $CA \subseteq AC$;

(b) $R(t)C = CR(t)$, $t \in [0, \tau)$;

(c) $R(t)x = k(t)Cx + \int_0^t a(t-s)AR(s)x\,ds$, $t \in [0, \tau)$, $x \in D(A)$.

$(R(t))_{t\in[0,\tau)}$ is said to be nondegenerate if the condition $R(t)x = 0$, $t \in [0, \tau)$ implies $x = 0$, and $(R(t))_{t\in[0,\tau)}$ is said to be locally equicontinuous if, for every $t \in (0, \tau)$, the family $\{R(s) : s \in [0, t]\}$ is equicontinuous. In case $\tau = \infty$, $(R(t))_{t\geq 0}$ is said to be exponentially equicontinuous if there exists $\omega \in \mathbb{R}$ such that the family $\{e^{-\omega t}R(t) : t \geq 0\}$ is equicontinuous.

(ii) Let $\beta \in (0, \pi/2]$ and let $(R(t))_{t\geq 0}$ be an $(a, k)$-regularized $C$-resolvent family. Then it is said that $(R(t))_{t\geq 0}$ is an analytic $(a, k)$-regularized $C$-resolvent family of angle $\beta$, if there exists a function $\mathbf{R} : \Sigma_\beta \to L(E)$ satisfying that, for every $x \in E$, the mapping $z \mapsto \mathbf{R}(z)x$, $z \in \Sigma_\beta$ is analytic as well as that

(a) $\mathbf{R}(t) = R(t)$, $t > 0$ and

(b) $\lim_{z\to 0, z\in\Sigma_\gamma} \mathbf{R}(z)x = k(0)Cx$ for all $\gamma \in (0, \beta)$ and $x \in E$.

It is said that $(R(t))_{t\geq 0}$ is an exponentially equicontinuous, analytic $(a, k)$-regularized $C$-resolvent family of angle $\beta$, if for every $\gamma \in (0, \beta)$, there exists $\omega_\gamma \geq 0$ such that the family $\{e^{-\omega_\gamma \Re z}\mathbf{R}(z) : z \in \Sigma_\gamma\}$ is equicontinuous.

Since there is no risk for confusion, we will identify $R(\cdot)$ and $\mathbf{R}(\cdot)$.

(iii) An $(a, k)$-regularized $C$-resolvent family $(R(t))_{t\geq 0}$ is said to be entire if, for every $x \in E$, the mapping $t \mapsto R(t)x$, $t \geq 0$ can be analytically extended to the whole complex plane.

In the sequel of the paper, we will consider only nondegenerate $(a, k)$-regularized $C$-resolvent families. The set which consists of all subgenerators of $(R(t))_{t\in[0,\tau)}$ need not be finite. In case $k(t) = g_{r+1}(t)$, where $r \geq 0$, it is also said that $(R(t))_{t\in[0,\tau)}$ is an *$r$-times integrated $(a, C)$-regularized resolvent family*; 0-times integrated $(a, C)$-regularized resolvent family is also called an *$(a, C)$-regularized resolvent family*.

Instructive examples of integrated $(a, C)$-regularized resolvent families, providing possible applications of Theorem 14 and Corollary 17, can be constructed following the analysis given in the proof of [36, Proposition 2.4]. If $k(t) = \int_0^t K(s)\,ds, t \in [0,\tau)$, where $K \in L^1_{\text{loc}}([0,\tau))$ and $K \neq 0$, then we obtain the unification concept for (local) $K$-convoluted $C$-semigroups and cosine functions [1]. We refer the reader to [23, 28, 32, 37, 38] for some applications of $(g_\alpha, k)$-regularized $C$-resolvent families in the study of the following abstract time-fractional equation with $\alpha > 0$ :

$$\begin{aligned} &\mathbf{D}_t^\alpha u(t) = Au(t), \quad t > 0, \\ &u^{(k)}(0) = Cx_k, \quad k = 0, 1, \ldots, \lceil\alpha\rceil - 1, \end{aligned} \tag{1}$$

where $x_k \in D(A)$, $k = 0, 1, \ldots, \lceil\alpha\rceil - 1$ and $\mathbf{D}_t^\alpha$ denotes the *Caputo fractional derivative of order* $\alpha$ ([28]). Henceforth, we assume that $k(t)$ and $k_1(t)$ are scalar-valued continuous kernels.

The following conditions will be used frequently:

(P1) : $k(t)$ is Laplace transformable, that is, it is locally integrable on $[0,\infty)$, and there exists $\beta \in \mathbb{R}$ such that $\widetilde{k}(\lambda) := \mathscr{L}(k)(\lambda) := \lim_{b\to\infty}\int_0^b e^{-\lambda t}k(t)\,dt := \int_0^\infty e^{-\lambda t}k(t)\,dt$ exists for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \beta$. Put $\text{abs}(k) := \inf\{\Re\lambda : \widetilde{k}(\lambda) \text{ exists}\}$, $\widetilde{\delta}(\lambda) := 1$ and denote by $\mathscr{L}^{-1}$ the inverse Laplace transform.

(P2) : $k(t)$ satisfies (P1) and $\widetilde{k}(\lambda) \neq 0$, $\Re\lambda > \beta$ for some $\beta \geq \text{abs}(k)$.

For the sake of convenience, we recall the following result from [32, 33].

**Lemma 2.** *Let $k(t)$ and $a(t)$ satisfy* (P1) *and let $(R(t))_{t\geq 0}$ be a strongly continuous operator family such that there exists $\omega \geq 0$ satisfying that the family $\{e^{-\omega t}R(t) : t \geq 0\}$ is equicontinuous. Put $\omega_0 := \max(\omega, \text{abs(a)}, \text{abs(k)})$.*

(i) *Assume $A$ is a subgenerator of the global $(a,k)$-regularized $C$-resolvent family $(R(t))_{t\geq 0}$ and*

$$R(t)x = k(t)Cx + A\int_0^t a(t-s)R(s)x\,ds, \quad t \geq 0, \ x \in E. \tag{2}$$

*Then, for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_0$ and $\widetilde{k}(\lambda) \neq 0$, the operator $I - \widetilde{a}(\lambda)A$ is injective and $R(C) \subseteq R(I - \widetilde{a}(\lambda)A)$. Furthermore,*

$$\begin{aligned} &\widetilde{k}(\lambda)(I - \widetilde{a}(\lambda)A)^{-1}Cx = \int_0^\infty e^{-\lambda t}R(t)x\,dt, \\ &\qquad x \in E, \quad \Re\lambda > \omega_0, \quad \widetilde{k}(\lambda) \neq 0, \\ &\left\{\frac{1}{\widetilde{a}(\lambda)} : \Re\lambda > \omega_0, \widetilde{k}(\lambda)\widetilde{a}(\lambda) \neq 0\right\} \subseteq \rho_C(A). \end{aligned} \tag{3}$$

(ii) *Assume* (3). *Then $A$ is a subgenerator of the global $(a,k)$-regularized $C$-resolvent family $(R(t))_{t\geq 0}$ satisfying* (2).

Let $A$ be a subgenerator of a locally equicontinuous $(a,k)$-regularized $C$-resolvent family $(R(t))_{t\in[0,\tau)}$ satisfying the equality (2) for all $t \in [0,\tau)$ and $x \in E$. Given $s \in [0,\tau)$ and $x \in E$, set $u(t) := R(t)R(s)x - R(s)R(t)x$, $t \in [0,\tau)$. Then it is not difficult to prove that $u \in C([0,\tau) : E)$ and $A\int_0^t a(t-s)u(s)\,ds = u(t)$, $t \in [0,\tau)$. Using the proof of [35, Theorem 2.7] (cf. also [33, Theorem 2.5]), it follows that $\int_0^t k(t-s)Cu(s)\,ds = 0$, $t \in [0,\tau)$. Since $k(t)$ is a kernel and $C$ is injective, we obtain $R(t)R(s) = R(s)R(t)$, $t, s \in [0,\tau)$, which remains true for perturbed resolvent families considered in the paper. Assuming additionally that $(R(t))_{t\geq 0}$ is a global exponentially equicontinuous $(a,k)$-regularized $C$-resolvent family as well as that $a(t)$ and $k(t)$ satisfy (P1), one can define the integral generator $\widehat{A}$ of $(R(t))_{t\geq 0}$ by setting

$$\begin{aligned} \widehat{A} := \Big\{ &(x,y) \in E \times E : R(t)x - k(t)Cx \\ &= \int_0^t a(t-s)R(s)y\,ds, \ t \geq 0 \Big\}. \end{aligned} \tag{4}$$

In case that $a(t)$ is a kernel, the definition of integral generator $\widehat{A}$ of $(R(t))_{t\geq 0}$ coincides with the corresponding one introduced in [33]. Notice that $\widehat{A}$ is the maximal subgenerator of $(R(t))_{t\geq 0}$ with respect to the set inclusion and that Lemma 2 implies $\widehat{A} = C^{-1}AC$.

## 2. Bounded Perturbation Theorems

Assume $\alpha > 0$ and $l \in \mathbb{N}$. Set, for any $E$-valued function $f(t)$ satisfying (P1), $F_{\alpha,f}(z) := \int_0^\infty e^{-z^{1/\alpha}t}f(t)\,dt$, $z > \max(\text{abs}(f), 0)^\alpha$. Using induction and elementary operational properties of vector-valued Laplace transform, one can simply prove that there exist uniquely determined real numbers $(c_{l_0,l,\alpha})_{1\leq l_0\leq l}$, independent of $E$ and $f(t)$, such that

$$\begin{aligned} \frac{d^l}{dz^l}F_{\alpha,f}(z) = \sum_{l_0=1}^{l} c_{l_0,l,\alpha} z^{(l_0/\alpha)-l}\int_0^\infty e^{-z^{(1/\alpha)}t}t^{l_0}f(t)\,dt, \\ z > \max(\text{abs}(f), 0)^\alpha. \end{aligned} \tag{5}$$

Furthermore, $c_{l,l,\alpha} = (-1)^l/\alpha^l$, $l \geq 1$, $c_{1,l,\alpha} = ((-1)/\alpha)((1/\alpha)-1)\cdots((1/\alpha)-(l-1))$, $l \geq 2$ and the following nonlinear recursive formula holds:

$$c_{l_0,l+1,\alpha} = \frac{(-1)}{\alpha}c_{l_0-1,\,l,\alpha} + \left(\frac{l_0}{\alpha} - l\right)c_{l_0,l,\alpha}, \quad l_0 = 2, \ldots, l. \tag{6}$$

The precise computation of coefficients $(c_{l_0,l,\alpha})$ is a nontrivial problem.

**Lemma 3.** *There exists $\zeta \geq 1$ such that*

$$\sum_{l_0=1}^{l} l_0!\left|c_{l_0,l,\alpha}\right| \leq \zeta^l l! \quad \forall l \in \mathbb{N}. \tag{7}$$

*Proof.* Clearly, $L_\alpha := \sup_{n\in\mathbb{N}_0} |\binom{1/\alpha}{n}| < \infty$. Applying (6), one gets

$$\begin{aligned}
&\sum_{l_0=1}^{l+1} l_0! \left|c_{l_0,l+1,\alpha}\right| \\
&\quad \le \left|\frac{1}{\alpha}\left(\frac{1}{\alpha}-1\right)\cdots\left(\frac{1}{\alpha}-l\right)\right| \\
&\qquad + \sum_{l_0=2}^{l}\left[\frac{l_0!}{\alpha}\left|c_{l_0-1,l,\alpha}\right| + l\left(\frac{1}{\alpha}+1\right) l_0! \left|c_{l_0,l,\alpha}\right|\right] + \frac{(l+1)!}{\alpha^{l+1}} \\
&\quad \le L_\alpha\left(\frac{1}{\alpha}+l\right) l! + \frac{l}{\alpha}\sum_{l_0=1}^{l-1} l_0! \left|c_{l_0,l,\alpha}\right| \\
&\qquad + l\left(\frac{1}{\alpha}+1\right)\sum_{l_0=2}^{l} l_0! \left|c_{l_0,l,\alpha}\right| + \frac{(l+1)!}{\alpha^{l+1}}, \quad l \ge 2.
\end{aligned} \tag{8}$$

The preceding inequality implies inductively that (7) holds provided $\zeta \ge 4 + (4/\alpha) + 4L_\alpha(1 + (1/\alpha))$. □

Set $\zeta_\alpha := \inf\{\zeta \ge 1 : \sum_{l_0=1}^{l} l_0!|c_{l_0,l,\alpha}| \le \zeta^l l! \text{ for all } l \in \mathbb{N}\}$ and $((1/\alpha)+1)\cdots((1/\alpha)+(l-1)) := 1$ if $l = 1$. Clearly, $\zeta_1 = 1$, $\zeta_\alpha > 1/\alpha$, $\alpha \in (0,1)$ and $\sum_{l_0=1}^{l} l_0!|c_{l_0,l,\alpha}| \le \zeta_\alpha^l l!$ for all $l \in \mathbb{N}$.

The following lemma will be helpful in the analysis of growth order of perturbed integrated $(g_\alpha, C)$-regularized resolvent families.

**Lemma 4.** *Let $\alpha > 1$. Then $\zeta_\alpha = 1$ and*

$$\sum_{l_0=1}^{l} l_0! \left|c_{l_0,l,\alpha}\right| = \frac{1}{\alpha}\left(\frac{1}{\alpha}+1\right)\cdots\left(\frac{1}{\alpha}+(l-1)\right) \le \frac{l!}{\alpha} \quad \forall l \in \mathbb{N}. \tag{9}$$

*Proof.* Plugging $f(t) \equiv 1$ in (5), we obtain

$$\sum_{l_0=1}^{l} l_0! c_{l_0,l,\alpha} = (-1)^l \frac{1}{\alpha}\left(\frac{1}{\alpha}+1\right)\cdots\left(\frac{1}{\alpha}+(l-1)\right) \quad \forall l \in \mathbb{N}. \tag{10}$$

Since $\alpha > 1$, it follows inductively from (6) that $(-1)^l c_{l_0,l,\alpha} > 0$, provided $l \ge 1$ and $1 \le l_0 \le l$. Combined with (10), the above implies (9) and $\zeta_\alpha = 1$. □

Now we are in a position to state the following important result.

**Theorem 5.** *Suppose $\alpha > 0$, $k(t)$ and $k_1(t)$ satisfy* (P1), *$A$ is a subgenerator of a $(g_\alpha, k)$-regularized $C$-resolvent family $(R(t))_{t\ge0}$ satisfying* (2) *with $a(t) = g_\alpha(t)$, $\omega \ge \max(\mathrm{abs}(k), 0)$, the family $\{e^{-\omega t}R(t) : t \ge 0\}$ is equicontinuous and the following conditions hold*

(i) *$B \in L(E)$, there exists $|B|_\circledast > 0$ such that $p(Bx) \le |B|_\circledast p(x)$, $x \in E$, $p \in \circledast$, $BA \subseteq AB$ and $BC = CB$.*

*There exist $M \ge 1$, $\omega' \ge 0$, $\omega'' \ge 0$ and $\omega''' \ge \max(\omega+\omega', \omega+\omega'', \mathrm{abs}(k_1))$ such that*

$$\begin{aligned}
&\left\{\lambda \in \mathbb{C} : \Re\lambda > \omega''', \widetilde{k_1}(\lambda) \ne 0\right\} \\
&\qquad \subseteq \left\{\lambda \in \mathbb{C} : \Re\lambda > \omega''', \widetilde{k}(\lambda) \ne 0\right\}
\end{aligned} \tag{11}$$

*as well as*

(ii) *For every $i, l_0, l \in \mathbb{N}$ with $1 \le l \le i$ and $1 \le l_0 \le l$, there exists a function $k_{i,l_0,l}(t)$ satisfying* (P1) *and*

$$\mathscr{L}\left(k_{i,l_0,l}(t)\right)(\lambda) = c_{l_0,l,\alpha}\lambda^{l_0-\alpha(l-1)}\widetilde{k_1}(\lambda)\left(\frac{1}{z\widetilde{k}(z^{1/\alpha})}\right)^{(i-l)}_{z=\lambda^\alpha}, \tag{12}$$

*provided $\Re\lambda > \omega'''$ and $\widetilde{k_1}(\lambda) \ne 0$.*

(iii) *For every $i \in \mathbb{N}_0$, there exists a function ${}_ik(t)$ satisfying (P1) and a constant $c_i \in \mathbb{C}$ so that*

$$\begin{aligned}
&c_i + {}_i\widetilde{k}(\lambda) = \lambda^\alpha\widetilde{k_1}(\lambda)\left(\frac{1}{z\widetilde{k}(z^{1/\alpha})}\right)^{(i)}_{z=\lambda^\alpha}, \\
&\Re\lambda > \omega''', \quad \widetilde{k_1}(\lambda) \ne 0.
\end{aligned} \tag{13}$$

(iv)

$$\sum_{i=0}^{\infty}\left|c_i\right|\frac{|B|_\circledast^i}{i!} < \infty, \qquad \sum_{i=0}^{\infty}\frac{|B|_\circledast^i}{i!}\int_0^t \left|{}_ik(s)\right| ds \le Me^{\omega' t}, \quad t \ge 0, \tag{14}$$

(v)

$$\sum_{i=1}^{\infty}\sum_{l=1}^{i}\sum_{l_0=1}^{l}\frac{|B|_\circledast^i}{i!}\binom{i}{l}\int_0^t (t-s)^{l_0}\left|k_{i,l_0,l}(s)\right| ds \le Me^{\omega' t}, \quad t \ge 0, \tag{15}$$

(vi)

$$\sum_{i=2}^{\infty}\sum_{l=2}^{i}\sum_{l_0=1}^{l-1}\frac{|B|_\circledast^i}{i!}l\binom{i}{l}\int_0^t (t-s)^{l_0}\left|k_{i-1,l_0,l-1}(s)\right| ds \le Me^{\omega'' t}, \quad t \ge 0. \tag{16}$$

*Then $A + B$ is a subgenerator of an exponentially equicontinuous $(g_\alpha, k_1)$-regularized $C$-resolvent family $(R_B(t))_{t\ge0}$, which is given by the following formula:*

$$\begin{aligned}
R_B(t)x := &\sum_{i=0}^{\infty}\frac{(-B)^i}{i!}\left[c_iR(t)x + \left({}_ik * R(\cdot)x\right)(t)\right] \\
&+ \sum_{i=1}^{\infty}\sum_{l=1}^{i}\sum_{l_0=1}^{l}\frac{(-B)^i}{i!}\binom{i}{l}\left(k_{i,l_0,l} * {}^{\cdot l_0}R(\cdot)x\right)(t), \\
&t \ge 0, \quad x \in E.
\end{aligned} \tag{17}$$

*Furthermore,*

$$R_B(t)x = k_1(t)Cx + (A+B)\int_0^t g_\alpha(t-s)R_B(s)x\,ds, \quad t \geq 0, \quad x \in E, \tag{18}$$

*and the family* $\{e^{-(\omega+\omega')t}R_B(t) : t \geq 0\}$ *is equicontinuous.*

*Proof.* By (iv)-(v), we obtain that the series in (17) converge uniformly on compact subsets of $[0,\infty)$ as well as that $(R_B(t))_{t\geq 0}$ is strongly continuous and that the family $\{e^{-(\omega+\omega')t}R_B(t) : t \geq 0\}$ is equicontinuous. By (i) and Lemma 2, $(z-A)^{-1}CB = B(z-A)^{-1}C$, $z \in \rho_C(A)$ and, for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $\widetilde{k}(\lambda) \neq 0$ : $\lambda^\alpha\widetilde{k}(\lambda)(\lambda^\alpha - A)^{-1}CBx = \int_0^\infty e^{-\lambda t}R(t)Bx\,dt$, $x \in E$ and $\lambda^\alpha\widetilde{k}(\lambda)B(\lambda^\alpha - A)^{-1}Cx = \int_0^\infty e^{-\lambda t}BR(t)x\,dt$, $x \in E$. By the uniqueness theorem for Laplace transform, one gets $R(t)B = BR(t)$, $t \geq 0$. The closedness of $A$, $R(t)A \subseteq AR(t)$, $t \geq 0$ and (iv)-(v) taken together imply $R_B(t)A \subseteq AR_B(t)$, $t \geq 0$. Hence, $R_B(t)(A+B) \subseteq (A+B)R_B(t)$, $t \geq 0$. By Lemma 2,

$$z\widetilde{k}\left(z^{1/\alpha}\right)(z-A)^{-1}Cx = \int_0^\infty e^{-z^{1/\alpha}t}R(t)x\,dt, \quad x \in E, \quad \Re\left(z^{1/\alpha}\right) > \omega, \quad \widetilde{k}\left(z^{1/\alpha}\right) \neq 0. \tag{19}$$

Exploiting the closedness of $A$ and the product rule, we easily infer from (19) that, for every $x \in E, l \in \mathbb{N}$ and for every $z \in \mathbb{C}$ with $\Re(z^{1/\alpha}) > \omega$ and $\widetilde{k}(z^{1/\alpha}) \neq 0$ :

$$\begin{aligned} A\frac{d^l}{dz^l}\int_0^\infty e^{-z^{1/\alpha}t}R(t)x\,dt &= A\frac{d^l}{dz^l}\left[z\widetilde{k}\left(z^{1/\alpha}\right)(z-A)^{-1}Cx\right] \\ &= \frac{d^l}{dz^l}\left[z\widetilde{k}\left(z^{1/\alpha}\right)A(z-A)^{-1}Cx\right] \\ &= z\frac{d^l}{dz^l}\left[z\widetilde{k}\left(z^{1/\alpha}\right)(z-A)^{-1}Cx\right] \\ &\quad + l\frac{d^{l-1}}{dz^{l-1}}\left[z\widetilde{k}\left(z^{1/\alpha}\right)(z-A)^{-1}Cx\right] \\ &\quad - \frac{d^l}{dz^l}\left[z\widetilde{k}\left(z^{1/\alpha}\right)Cx\right]. \end{aligned} \tag{20}$$

Fix, for the time being, $x \in E$ and $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega'''$ and $\widetilde{k_1}(\lambda) \neq 0$. Then (11) implies $\widetilde{k}(\lambda) \neq 0$. By (iv)-(v) and the dominated convergence theorem, it follows that the Laplace transform of power series appearing in (17) can be computed term by term. Using this fact as well as (5), (19), and (ii)-(iii), we obtain that

$$\begin{aligned} \mathscr{L}\left(R_B(t)x\right)(\lambda) &= \lambda^\alpha\widetilde{k_1}(\lambda)\sum_{i=0}^\infty\frac{(-B)^i}{i!} \\ &\quad \times \sum_{l=0}^i\binom{i}{l}\left(\frac{1}{z\widetilde{k}\left(z^{1/\alpha}\right)}\right)^{(i-l)}_{z=\lambda^\alpha} \\ &\quad \times\left(\frac{d^l}{dz^l}\left[z\widetilde{k}\left(z^{1/\alpha}\right)(z-A)^{-1}Cx\right]\right)_{z=\lambda^\alpha}. \end{aligned} \tag{21}$$

Our goal is to prove that

$$\left(I - \frac{A+B}{\lambda^\alpha}\right)\mathscr{L}\left(R_B(t)x\right)(\lambda) = \widetilde{k_1}(\lambda)Cx. \tag{22}$$

By the product rule, we get

$$\begin{aligned} &\sum_{i=1}^\infty\frac{(-B)^i}{i!}\sum_{l=1}^i\binom{i}{l}\left(\frac{1}{z\widetilde{k}\left(z^{1/\alpha}\right)}\right)^{(i-l)}_{z=\lambda^\alpha}\left(\frac{d^l}{dz^l}\left[z\widetilde{k}\left(z^{1/\alpha}\right)Cx\right]\right)_{z=\lambda^\alpha} \\ &\quad = -\lambda^\alpha\widetilde{k}(\lambda)\sum_{i=1}^\infty\frac{(-B)^i}{i!}\left(\frac{1}{z\widetilde{k}\left(z^{1/\alpha}\right)}\right)^{(i)}_{z=\lambda^\alpha}Cx; \end{aligned} \tag{23}$$

notice that the convergence of last series follows from the conditions (iii)-(iv). Taking into account (5), (ii), and (vi), one yields that

$$\begin{aligned} &\frac{1}{\lambda^\alpha\widetilde{k_1}(\lambda)}\int_0^\infty e^{-\lambda t}\sum_{i=2}^\infty\sum_{l=2}^i\sum_{l_0=1}^{l-1}\frac{(-B)^i}{i!}l\binom{i}{l} \\ &\qquad \times\int_0^t(t-s)^{l_0}R(t-s) \\ &\qquad \times k_{i-1,l_0,l-1}(s)x\,ds\,dt \\ &= \frac{1}{\lambda^\alpha\widetilde{k_1}(\lambda)}\sum_{i=2}^\infty\sum_{l=2}^i\sum_{l_0=1}^{l-1}\frac{(-B)^i}{i!}l\binom{i}{l} \\ &\qquad \times\int_0^\infty e^{-\lambda t}\int_0^t(t-s)^{l_0}R(t-s) \\ &\qquad \times k_{i-1,l_0,l-1}(s)x\,ds\,dt \\ &= \sum_{i=2}^\infty\sum_{l=2}^i\sum_{l_0=1}^{l-1}\frac{(-B)^i}{i!}l\binom{i}{l}c_{l_0,l-1,\alpha}\lambda^{l_0-\alpha(l-1)} \\ &\qquad \times\int_0^\infty e^{-\lambda t}t^{l_0}R(t)x\,dt \\ &= \sum_{i=2}^\infty\frac{(-B)^i}{i!} \\ &\qquad \times\sum_{l=2}^i l\binom{i}{l}\left(\frac{1}{z\widetilde{k}\left(z^{1/\alpha}\right)}\right)^{(i-l)}_{z=\lambda^\alpha} \\ &\qquad \times\left(\frac{d^{l-1}}{dz^{l-1}}\left[z\widetilde{k}\left(z^{1/\alpha}\right)(z-A)^{-1}Cx\right]\right)_{z=\lambda^\alpha}, \end{aligned} \tag{24}$$

which implies that the series

$$\sum_{i=1}^{\infty}\frac{(-B)^i}{i!}\sum_{l=1}^{i}l\binom{i}{l}\left(\frac{1}{z\widetilde{k}\left(z^{1/\alpha}\right)}\right)^{(i-l)}_{z=\lambda^\alpha} \times\left(\frac{d^{l-1}}{dz^{l-1}}\left[z\widetilde{k}\left(z^{1/\alpha}\right)(z-A)^{-1}Cx\right]\right)_{z=\lambda^\alpha} \tag{25}$$

is also convergent. Now we get from (20)-(21) and (23)-(24):

$$\begin{aligned}
&\left(I-\frac{A+B}{\lambda^\alpha}\right)\mathscr{L}\left(R_B(t)x\right)(\lambda)\\
&=\lambda^\alpha\sum_{i=0}^{\infty}\frac{(-B)^i}{i!}\sum_{l=0}^{i}\binom{i}{l}\left(\frac{1}{z\widetilde{k}\left(z^{1/\alpha}\right)}\right)^{(i-l)}_{z=\lambda^\alpha}\\
&\qquad\times\left(\frac{d^l}{dz^l}\left[z\widetilde{k}\left(z^{1/\alpha}\right)(z-A)^{-1}Cx\right]\right)_{z=\lambda^\alpha}\\
&\quad-\sum_{i=1}^{\infty}\frac{(-B)^i}{i!}\sum_{l=1}^{i}\binom{i}{l}\left(\frac{1}{z\widetilde{k}\left(z^{1/\alpha}\right)}\right)^{(i-l)}_{z=\lambda^\alpha}\\
&\qquad\times\left(z\frac{d^l}{dz^l}\left[z\widetilde{k}\left(z^{1/\alpha}\right)(z-A)^{-1}Cx\right]\right.\\
&\qquad\quad+l\frac{d^{l-1}}{dz^{l-1}}\left[z\widetilde{k}\left(z^{1/\alpha}\right)(z-A)^{-1}Cx\right]\\
&\qquad\quad\left.-\frac{d^l}{dz^l}\left[z\widetilde{k}\left(z^{1/\alpha}\right)\right]Cx\right)_{z=\lambda^\alpha}\\
&\quad-\lambda^\alpha\widetilde{k}(\lambda)\sum_{i=0}^{\infty}\frac{(-B)^i}{i!}\left(\frac{1}{z\widetilde{k}\left(z^{1/\alpha}\right)}\right)^{(i)}_{z=\lambda^\alpha}A\left(\lambda^\alpha-A\right)^{-1}Cx\\
&\quad+\lambda^\alpha\sum_{i=0}^{\infty}\frac{(-B)^{i+1}}{i!}\sum_{l=0}^{i}\binom{i}{l}\left(\frac{1}{z\widetilde{k}\left(z^{1/\alpha}\right)}\right)^{(i-l)}_{z=\lambda^\alpha}\\
&\qquad\times\left(\frac{d^l}{dz^l}\left[z\widetilde{k}\left(z^{1/\alpha}\right)(z-A)^{-1}Cx\right]\right)_{z=\lambda^\alpha}\\
&=\lambda^{2\alpha}\widetilde{k}(\lambda)\sum_{i=0}^{\infty}\frac{(-B)^i}{i!}\left(\frac{1}{z\widetilde{k}\left(z^{1/\alpha}\right)}\right)^{(i)}_{z=\lambda^\alpha}\left(\lambda^\alpha-A\right)^{-1}Cx\\
&\quad-\sum_{i=1}^{\infty}\frac{(-B)^i}{i!}\sum_{l=1}^{i}\binom{i}{l}\left(\frac{1}{z\widetilde{k}\left(z^{1/\alpha}\right)}\right)^{(i-l)}_{z=\lambda^\alpha}\\
&\qquad\times\left(l\frac{d^{l-1}}{dz^{l-1}}\left[z\widetilde{k}\left(z^{1/\alpha}\right)(z-A)^{-1}Cx\right]\right)_{z=\lambda^\alpha}\\
&\quad-\lambda^\alpha\widetilde{k}(\lambda)\sum_{i=1}^{\infty}\frac{(-B)^i}{i!}\left(\frac{1}{z\widetilde{k}\left(z^{1/\alpha}\right)}\right)^{(i)}_{z=\lambda^\alpha}Cx\\
&\quad-\lambda^\alpha\widetilde{k}(\lambda)\sum_{i=0}^{\infty}\frac{(-B)^i}{i!}\left(\frac{1}{z\widetilde{k}\left(z^{1/\alpha}\right)}\right)^{(i)}_{z=\lambda^\alpha}\\
&\qquad\times\left(-Cx+\lambda^\alpha\left(\lambda^\alpha-A\right)^{-1}Cx\right)\\
&\quad+\lambda^\alpha\sum_{i=0}^{\infty}\frac{(-B)^{i+1}}{i!}\sum_{l=0}^{i}\binom{i}{l}\left(\frac{1}{z\widetilde{k}\left(z^{1/\alpha}\right)}\right)^{(i-l)}_{z=\lambda^\alpha}\\
&\qquad\times\left(\frac{d^l}{dz^l}\left[z\widetilde{k}\left(z^{1/\alpha}\right)(z-A)^{-1}Cx\right]\right)_{z=\lambda^\alpha}\\
&=Cx-\sum_{i=1}^{\infty}\frac{(-B)^i}{i!}\\
&\qquad\times\sum_{l=1}^{i}\binom{i}{l}\left(\frac{1}{z\widetilde{k}\left(z^{1/\alpha}\right)}\right)^{(i-l)}_{z=\lambda^\alpha}\\
&\qquad\times\left(l\frac{d^{l-1}}{dz^{l-1}}\left[z\widetilde{k}\left(z^{1/\alpha}\right)(z-A)^{-1}Cx\right]\right)_{z=\lambda^\alpha}\\
&\quad+\lambda^\alpha\sum_{i=0}^{\infty}\frac{(-B)^{i+1}}{i!}\sum_{l=0}^{i}\binom{i}{l}\left(\frac{1}{z\widetilde{k}\left(z^{1/\alpha}\right)}\right)^{(i-l)}_{z=\lambda^\alpha}\\
&\qquad\times\left(\frac{d^l}{dz^l}\left[z\widetilde{k}\left(z^{1/\alpha}\right)(z-A)^{-1}Cx\right]\right)_{z=\lambda^\alpha}\\
&=Cx,
\end{aligned}\tag{26}$$

because the sum of coefficients of $(-B)^i$ $(i\geq 1)$ in the last two series equals 0; this follows from an elementary calculus involving only the product rule. Assume now $x\in D(A)$, $\Re\lambda>\omega'''$, $\widetilde{k_1}(\lambda)\neq 0$ and $(I-((A+B)/\lambda^\alpha))x=0$. By (22) and $R_B(t)(A+B)\subseteq(A+B)R_B(t)$, $t\geq 0$, we obtain that

$$\begin{aligned}
\widetilde{k_1}(\lambda)Cx&=\left(I-\frac{A+B}{\lambda^\alpha}\right)\mathscr{L}\left(R_B(t)x\right)(\lambda)\\
&=\mathscr{L}\left(R_B(t)\left(I-\frac{A+B}{\lambda^\alpha}\right)x\right)(\lambda)=0,
\end{aligned}\tag{27}$$

which implies $Cx=x=0$. Thus, $\{\lambda^\alpha:\Re\lambda>\omega''',\ \widetilde{k_1}(\lambda)\neq 0\}\subseteq\rho_C(A+B)$ and

$$\widetilde{k_1}(\lambda)\left(I-\frac{A+B}{\lambda^\alpha}\right)^{-1}Cx=\int_0^\infty e^{-\lambda t}R_B(t)x\,dt,\quad x\in E,\quad \Re\lambda>\omega''',\quad \widetilde{k_1}(\lambda)\neq 0. \tag{28}$$

The proof of theorem completes an application of Lemma 2. □

*Remark 6.* (i) By [33, Proposition 2.4(i)], we get that $(R_B(t))_{t\geq 0}$ is a unique $(g_\alpha,k_1)$-regularized $C$-resolvent family with the properties stated in the formulation of Theorem 5.

(ii) The following comment is also applicable to Theorem 7 given below. Assume $k(t)=k_1(t), t\geq 0, n\in\mathbb{N}$ and the conditions (iv)-(vi) of Theorem 5 hold with $|B|_{\circledast}^i/i!$ replaced by $|B|_{\circledast}^i/n^i i!$ therein. Writing $A+B$ as $A+\sum_{i=1}^n B/n$ and applying Theorem 5 successively $n$ times, we obtain that $A+B$ is a subgenerator of a global $(g_\alpha,k)$-regularized

$C$-resolvent family $(R_B(t))_{t\geq 0}$ satisfying (18). Furthermore, the family $\{e^{-(\omega+n\omega')t}R_B(t) : t \geq 0\}$ is equicontinuous.

(iii) It is not clear whether there exist functions $k(t)$ and $k_1(t)$ such that the conditions (ii)–(vi) of Theorem 5 are fulfilled in the case $\alpha \in (0,1)$.

**Theorem 7.** *Consider the situation of Theorem 5 with $(R(t))_{t\geq 0}$ being an exponentially equicontinuous, analytic $(g_\alpha, k)$-regularized $C$-resolvent family of angle $\beta \in (0,\pi/2]$. Assume that, for every $\gamma \in (0,\beta)$, there exists $\omega_\gamma \geq 0$ such that the set $\{e^{-\omega_\gamma \Re z}R(z) : z \in \Sigma_\gamma\}$ is equicontinuous. Assume, additionally, that there exists $\varepsilon > 0$ such that, for every $\gamma \in (0,\beta)$, there exist $\omega_{\gamma,1} \geq \max(\sup\{\mathrm{abs}({}_ik) : i \geq 1\}, \omega_\gamma)$ and $\omega_{\gamma,2} \geq \max(\sup\{\mathrm{abs}(k_{i,l_0,l}) : 1 \leq l \leq i,\ 1 \leq l_0 \leq l\}, \omega_\gamma + \varepsilon)$ with the following properties.*

(i) *For every $i \in \mathbb{N}_0$, the function $\lambda \mapsto {}_i\widetilde{k}(\lambda)$, $\lambda > \omega_{\gamma,1}$ can be analytically extended to the sector $\omega_{\gamma,1} + \Sigma_{(\pi/2)+\gamma}$ and the following holds:*

$$\sum_{i=0}^{\infty} \frac{|B|_{\circledast}^{i}}{i!} \sup_{\lambda \in \omega_{\gamma,1}+\Sigma_{(\pi/2)+\gamma}} \left|{}_i\widetilde{k}(\lambda)\right| < \infty. \tag{29}$$

(ii) *For every $i, l_0, l \in \mathbb{N}$ with $1 \leq l \leq i$ and $1 \leq l_0 \leq l$, the function $\lambda \mapsto \mathscr{L}(k_{i,l_0,l}(t))(\lambda)$, $\lambda > \omega_{\gamma,2}$ can be analytically extended to the sector $\omega_{\gamma,2} + \Sigma_{(\pi/2)+\gamma}$ and the following holds:*

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{l=1}^{i}\sum_{l_0=1}^{l} &\frac{|B|_{\circledast}^{i}}{i!}\binom{i}{l}\frac{l_0!}{\sqrt{2\pi l_0}(\varepsilon\cos\gamma)^{l_0}} \\ &\times \sup_{\lambda\in\omega_{\gamma,2}+\Sigma_{(\pi/2)+\gamma}} \left|\mathscr{L}\left(k_{i,l_0,l}(t)\right)(\lambda)\right| < \infty.\end{aligned} \tag{30}$$

*Then $(R_B(t))_{t\geq 0}$ is an exponentially equicontinuous, analytic $(g_\alpha, k_1)$-regularized $C$-resolvent family of angle $\beta$.*

*Proof.* Let $p \in \circledast$, $x \in E$, $\gamma \in (0,\beta)$ and $\varepsilon \in (0,(1/3)\min(\gamma,(\pi/2)-\gamma))$. Then Stirling's formula implies that there exists $\kappa \geq 1$ such that

$$\begin{aligned}|z|^{l_0}e^{\omega_\gamma\Re z} &\leq \frac{(\Re z)^{l_0}}{(\cos\gamma)^{l_0}}e^{\omega_\gamma\Re z} \leq \frac{e^{-l_0}}{(\cos\gamma)^{l_0}}\frac{l_0^{l_0}}{\varepsilon^{l_0}}e^{(\omega_\gamma+\varepsilon)\Re z} \\ &\leq \kappa\frac{l_0!}{\sqrt{2\pi l_0}(\varepsilon\cos\gamma)^{l_0}}e^{(\omega_\gamma+\varepsilon)\Re z} \\ &\leq \kappa\frac{l_0!}{\sqrt{2\pi l_0}(\varepsilon\cos\gamma)^{l_0}}e^{\omega_{\gamma,2}\Re z}\end{aligned} \tag{31}$$

for all $z \in \Sigma_\gamma$ and $l_0 \in \mathbb{N}$. By [33, Theorem 3.4(i)] and the proof of implication (i) $\Rightarrow$ (ii) of [39, Theorem 2.6.1], we obtain that the mapping $\lambda \mapsto \mathscr{L}({}_ik * R(\cdot)x)(\lambda)$, $\lambda > \omega_{\gamma,1}$, respectively, $\lambda \mapsto \mathscr{L}(k_{i,l_0,l} * {}^{\cdot l_0}R(\cdot)x)(\lambda)$, $\lambda > \omega_{\gamma,2}$ can be analytically extended to the sector $\omega_{\gamma,1} + \Sigma_{(\pi/2)+\gamma}$, respectively, $\omega_{\gamma,2} + \Sigma_{(\pi/2)+\gamma}$, as well as that there exist $c_p > 0$ and $q_p \in \circledast$, independent of $x$, such that

$$\begin{aligned}&\sup_{\lambda\in\omega_{\gamma,1}+\Sigma_{(\pi/2)+\gamma-\varepsilon}} p\left(\left(\lambda-\omega_{\gamma,1}\right){}_i\widetilde{k}(\lambda)\,\mathscr{L}(R(t)x)(\lambda)\right) \\ &\quad\leq \frac{c_pq_p(x)}{\sin\varepsilon}\sup_{\lambda\in\omega_{\gamma,1}+\Sigma_{(\pi/2)+\gamma-\varepsilon}}\left|{}_i\widetilde{k}(\lambda)\right|\end{aligned} \tag{32}$$

and that, for every $i, l_0, l \in \mathbb{N}$ with $1 \leq l \leq i$ and $1 \leq l_0 \leq l$,

$$\begin{aligned}&\sup_{\lambda\in\omega_{\gamma,2}+\Sigma_{(\pi/2)+\gamma-\varepsilon}} p\left(\left(\lambda-\omega_{\gamma,2}\right)\mathscr{L}\left(k_{i,l_0,l}(t)\right)(\lambda)\,\mathscr{L}\left(t^{l_0}R(t)x\right)(\lambda)\right) \\ &\quad\leq \kappa\frac{c_pq_p(x)}{\sin\varepsilon}\frac{l_0!}{\sqrt{2\pi l_0}(\varepsilon\cos\gamma)^{l_0}} \\ &\qquad\times\sup_{\lambda\in\omega_{\gamma,2}+\Sigma_{(\pi/2)+\gamma-\varepsilon}}\left|\mathscr{L}\left(k_{i,l_0,l}(t)\right)(\lambda)\right|.\end{aligned} \tag{33}$$

Using (32)-(33), [33, Theorem 3.4(i)] and the proof of implication (ii) $\Rightarrow$ (i) of [39, Theorem 2.6.1], it follows that the functions $t \mapsto ({}_ik * R(\cdot)x)(t)$, $t > 0$ and $t \mapsto (k_{i,l_0,l} * {}^{\cdot l_0}R(\cdot)x)(t)$, $t > 0$ can be analytically extended to the sector $\Sigma_\gamma$ and that the following estimates hold:

$$\begin{aligned}&p\left(\left({}_ik * R(\cdot)x\right)(z)\right) \\ &\quad\leq \frac{c_pq_p(x)}{\sin\varepsilon}\sup_{\lambda\in\omega_{\gamma,1}+\Sigma_{(\pi/2)+\gamma-\varepsilon}}\left|{}_i\widetilde{k}(\lambda)\right|\left(e^{1+\omega_{\gamma,1}\Re z}+\frac{e^{\omega_{\gamma,1}\Re z}}{\pi\sin\varepsilon}\right), \\ &\qquad\qquad z\in\Sigma_{\gamma-3\varepsilon}, \\ &p\left(\left(k_{i,l_0,l} * {}^{\cdot l_0}R(\cdot)x\right)(z)\right) \\ &\quad\leq \kappa\frac{c_pq_p(x)}{\sin\varepsilon}\frac{l_0!}{\sqrt{2\pi l_0}(\varepsilon\cos\gamma)^{l_0}} \\ &\qquad\times\sup_{\lambda\in\omega_{\gamma,2}+\Sigma_{(\pi/2)+\gamma-\varepsilon}}\left|\mathscr{L}\left(k_{i,l_0,l}(t)\right)(\lambda)\right| \\ &\qquad\qquad\times\left(e^{1+\omega_{\gamma,2}\Re z}+\frac{e^{\omega_{\gamma,2}\Re z}}{\pi\sin\varepsilon}\right), \quad z\in\Sigma_{\gamma-3\varepsilon}.\end{aligned} \tag{34}$$

Since Vitali's theorem holds in our framework (cf. e.g. [33, Lemma 3.3]), we easily infer from (29)-(30), (34), and the arbitrariness of $\gamma$ and $\varepsilon$ that the mapping $t \mapsto R_B(t)x$, $t > 0$ can be analytically extended to the sector $\Sigma_\beta$ by the formula (17). Thanks to the proof of Theorem 5, the series appearing in (17) converge uniformly on compact subsets of $[0,\infty)$, which implies $\lim_{t\to 0+}\sum_{i=0}^{\infty}((-B)^i/i!)c_iR(t)x = \sum_{i=0}^{\infty}((-B)^i/i!)c_iR(0)$, $\lim_{t\to 0+}\sum_{i=0}^{\infty}((-B)^i/i!)({}_ik * R(\cdot)x)(t) = 0$, and $\lim_{t\to 0+}\sum_{i=1}^{\infty}\sum_{l=1}^{i}\sum_{l_0=1}^{l}((-B)^i/i!)\binom{i}{l}(k_{i,l_0,l} * {}^{\cdot l_0}R(\cdot)x)(t) = 0$. Furthermore, the functions $z \mapsto f_1(z) := \sum_{i=0}^{\infty}((-B)^i/i!)c_iR(z)x$, $z \in \Sigma_\beta$, $z \mapsto f_2(z) := \sum_{i=0}^{\infty}((-B)^i/i!)({}_ik * R(\cdot)x)(z)$, $z \in \Sigma_\beta$, and $z \mapsto f_3(z) := \sum_{i=1}^{\infty}\sum_{l=1}^{i}\sum_{l_0=1}^{l}((-B)^i/i!)\binom{i}{l}(k_{i,l_0,l} * {}^{\cdot l_0}R(\cdot)x)(z)$, $z \in \Sigma_\beta$

are analytic, and the set $\{e^{-(\omega_\gamma+\omega_{\gamma,1}+\omega_{\gamma,2})\Re z} f_j(z) : 1 \le j \le 3,\ z \in \Sigma_{\gamma-3\varepsilon}\}$ is bounded. An application of [33, Theorem 3.4(ii)] gives that the mapping $z \mapsto R_B(z)x,\ z \in \Sigma_\beta \cup \{0\}$ is continuous on any closed subsector of $\Sigma_\beta \cup \{0\}$, which completes the proof of theorem. □

It would take too long to go into details concerning stability of certain differential properties ([40, 41]) under bounded commuting perturbations described in Theorem 5.

Let $\alpha > 0$, let $\beta > 0$, and let the *Mittag-Leffler function* $E_{\alpha,\beta}(z)$ be defined by $E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} z^n/\Gamma(\alpha n+\beta),\ z \in \mathbb{C}$. Set $E_\alpha(z) := E_{\alpha,1}(z),\ z \in \mathbb{C}$. Then it is well known (cf. [28, 42–44]) that $E'_\alpha(z) = E_{\alpha,\alpha}(z)/\alpha,\ z \in \mathbb{C}$ and that, for every $\alpha > 1$, there exist $b_\alpha \ge 1$ and $c_\alpha \ge 1$ such that

$$\begin{aligned} E_{\alpha,\alpha}(t) &\le b_\alpha t^{(1-\alpha)/\alpha} \exp\left(t^{1/\alpha}\right), \quad t > 0, \\ E_\alpha(t) &\le c_\alpha \exp\left(t^{1/\alpha}\right), \quad t \ge 0. \end{aligned} \tag{35}$$

It is noteworthy that the assumptions of Theorems 5 and 7 hold provided $\alpha > 1$ and $k(t) = k_1(t) = g_{r+1}(t)$, where $r \ge 0$. In this case, $c_0 = 1,\ k_0(t) = 0,\ c_i = 0,\ i \ge 1$,

$$ {}_ik(t) = \left(\frac{r+1}{\alpha} - 1\right)\cdots\left(\frac{r+1}{\alpha} - i\right) g_{\alpha i}(t), \quad t \ge 0,\ i \ge 1 \tag{36}$$

and, for every $i, l_0, l \in \mathbb{N}$ with $1 \le l \le i$ and $1 \le l_0 \le l$,

$$k_{i,l_0,l}(t) = c_{l_0,l,\alpha}\left(\frac{r+1}{\alpha} - 1\right)\cdots\left(\frac{r+1}{\alpha} - (i-l)\right) g_{\alpha i - l_0}(t), \quad t > 0. \tag{37}$$

In order to verify (iv)–(vi), notice that there exists a constant $c_{r,\alpha} \ge 1$ such that $|(((r+1)/\alpha) - 1)\cdots(((r+1)/\alpha) - i)| \le c_{r,\alpha} i!$ for all $i \in \mathbb{N}$. Then we obtain from (35) and Lemmas 3–4 that

$$\begin{aligned} &\sum_{i=1}^{\infty} \frac{|B|_\circledast^i}{i!} \left|\left(\frac{r+1}{\alpha} - 1\right)\cdots\left(\frac{r+1}{\alpha} - i\right)\right| \int_0^t g_{\alpha i}(s)\, ds \\ &\quad \le c_{r,\alpha} \sum_{i=1}^{\infty} \frac{\left(t^\alpha |B|_\circledast\right)^i}{\Gamma(\alpha i + 1)} = c_{r,\alpha}\left(E_\alpha\left(t^\alpha |B|_\circledast\right) - 1\right) \\ &\quad \le c_{r,\alpha} c_\alpha e^{t|B|_\circledast^{1/\alpha}}, \quad t \ge 0, \end{aligned} \tag{38}$$

$$\begin{aligned} &\sum_{i=1}^{\infty}\sum_{l=1}^{i}\sum_{l_0=1}^{l} \frac{|B|_\circledast^i}{i!} \binom{i}{l} \int_0^t (t-s)^{l_0} \left|k_{i,l_0,l}(s)\right| ds \\ &\quad \le \sum_{i=1}^{\infty}\sum_{l=1}^{i}\sum_{l_0=1}^{l} \frac{|B|_\circledast^i}{i!} \binom{i}{l} \left|c_{l_0,l,\alpha}\right| \\ &\qquad \times \left|\left(\frac{r+1}{\alpha} - 1\right)\cdots\left(\frac{r+1}{\alpha} - (i-l)\right)\right| l_0! g_{\alpha i+1}(t) \\ &\quad \le \frac{c_{r,\alpha}}{\alpha} \sum_{i=1}^{\infty} i \frac{\left(t^\alpha |B|_\circledast\right)^i}{\Gamma(\alpha i + 1)} = \frac{c_{r,\alpha}}{\alpha} t^\alpha |B|_\circledast \sum_{i=1}^{\infty} i \frac{\left(t^\alpha |B|_\circledast\right)^{i-1}}{\Gamma(\alpha i + 1)} \\ &\quad = \frac{c_{r,\alpha}}{\alpha} t^\alpha |B|_\circledast \left(\sum_{i=1}^{\infty} \frac{z^i}{\Gamma(\alpha i + 1)}\right)'_{z=t^\alpha |B|_\circledast} \\ &\quad = \frac{c_{r,\alpha}}{\alpha^2} t^\alpha |B|_\circledast E_{\alpha,\alpha}\left(t^\alpha |B|_\circledast\right) \\ &\quad \le \frac{c_{r,\alpha} b_\alpha}{\alpha^2} t^\alpha |B|_\circledast \left(t^\alpha |B|_\circledast\right)^{(1-\alpha)/\alpha} e^{t|B|_\circledast^{1/\alpha}} \\ &\quad \le \frac{c_{r,\alpha} b_\alpha}{\alpha^2} t |B|_\circledast^{1/\alpha} e^{t|B|_\circledast^{1/\alpha}}, \quad t \ge 0, \end{aligned} \tag{39}$$

proving the conditions (iv)-(v) and

$$\begin{aligned} &\sum_{i=2}^{\infty}\sum_{l=2}^{i}\sum_{l_0=1}^{l-1} \frac{|B|_\circledast^i}{i!} l \binom{i}{l} \int_0^t (t-s)^{l_0} \left|k_{i-1,l_0,l-1}(s)\right| ds \\ &\quad \le c_{r,\alpha} \sum_{i=2}^{\infty}\sum_{l=2}^{i}\sum_{l_0=1}^{l-1} \frac{|B|_\circledast^i}{(l-1)!} l_0! \left|c_{l_0,l-1,\alpha}\right| g_{\alpha(i-1)+1}(t) \\ &\quad \le \frac{c_{r,\alpha}}{\alpha} |B|_\circledast \sum_{i=2}^{\infty} (i-1) \frac{\left(t^\alpha |B|_\circledast\right)^{i-1}}{\Gamma(\alpha(i-1)+1)} \\ &\quad = \frac{c_{r,\alpha}}{\alpha} |B|_\circledast \sum_{i=1}^{\infty} i \frac{\left(t^\alpha |B|_\circledast\right)^i}{\Gamma(\alpha i + 1)} \\ &\quad \le \frac{c_{r,\alpha} b_\alpha}{\alpha^2} t |B|_\circledast^{(1+\alpha)/\alpha} e^{t|B|_\circledast^{1/\alpha}}, \quad t \ge 0, \end{aligned} \tag{40}$$

proving the condition (vi). Assume now, with the notation used in the formulation of Theorem 7 that $\gamma \in (0, \beta),\ \omega_{\gamma,1} \ge \omega_\gamma,\ \omega_{\gamma,1} > 0, (\omega_{\gamma,1}\cos\gamma)^\alpha > |B|_\circledast,\ \varepsilon = 1/\cos\gamma,\ \omega_{\gamma,2} \ge \omega_\gamma + \varepsilon$, and $(1 + \omega_{\gamma,2}\cos\gamma)^{\alpha-1} > |B|_\circledast$. Then

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{|B|_\circledast^i}{i!} \sup_{\lambda \in \omega_{\gamma,1} + \Sigma_{(\pi/2)+\gamma}} \left|{}_i\widetilde{k}(\lambda)\right| &\le c_{r,\alpha} \sum_{i=0}^{\infty} |B|_\circledast^i \sup_{\lambda \in \omega_{\gamma,1} + \Sigma_{(\pi/2)+\gamma}} \frac{1}{|\lambda|^{\alpha i}} \\ &\le c_{r,\alpha} \sum_{i=0}^{\infty} \frac{|B|_\circledast^i}{\left(\omega_{\gamma,1}\cos\gamma\right)^{\alpha i}} < \infty, \end{aligned} \tag{41}$$

$$\begin{aligned} &\sum_{i=1}^{\infty}\sum_{l=1}^{i}\sum_{l_0=1}^{l} \frac{|B|_\circledast^i}{i!} \binom{i}{l} \frac{l_0!}{\sqrt{2\pi l_0}(\varepsilon\cos\gamma)^{l_0}} \\ &\qquad \times \sup_{\lambda \in \omega_{\gamma,2} + \Sigma_{(\pi/2)+\gamma}} \left|\mathscr{L}\left(k_{i,l_0,l}(t)\right)(\lambda)\right| \\ &\quad \le c_{r,\alpha} \sum_{i=1}^{\infty}\sum_{l=1}^{i}\sum_{l_0=1}^{l} \frac{|B|_\circledast^i}{l!} l_0! \left|c_{l_0,l,\alpha}\right| \sup_{\lambda \in \omega_{\gamma,2} + \Sigma_{(\pi/2)+\gamma}} \frac{1}{|\lambda|^{\alpha i - l_0}} \\ &\quad \le \frac{c_{r,\alpha}}{\alpha} \sum_{i=1}^{\infty}\sum_{l=1}^{i}\sum_{l_0=1}^{l} \frac{|B|_\circledast^i}{\left(1 + \omega_{\gamma,2}\cos\gamma\right)^{\alpha i - l_0}} \\ &\quad \le \frac{c_{r,\alpha}}{\alpha} \sum_{i=1}^{\infty} \frac{i^2 |B|_\circledast^i}{\left(1 + \omega_{\gamma,2}\cos\gamma\right)^{(\alpha-1)i}} < \infty, \end{aligned} \tag{42}$$

proving the conditions (29)-(30).

**Corollary 8.** *Suppose $\alpha > 1$, $\omega \geq 0$, $r \geq 0$ and $A$ is a subgenerator of a global $r$-times integrated $(g_\alpha, C)$-regularized resolvent family $(R(t))_{t\geq 0}$ satisfying (2) with $a(t) = g_\alpha(t)$ and $k(t) = g_{r+1}(t)$. Let the family $\{e^{-\omega t}R(t) : t \geq 0\}$ be equicontinuous and let $B \in L(E)$ satisfy the condition (i) quoted in the formulation of Theorem 5. Then $A + B$ is a subgenerator of a global $r$-times integrated $(g_\alpha, C)$-regularized resolvent family $(R_B(t))_{t\geq 0}$ satisfying (18) with $k_1(t) = k(t)$. Furthermore, the family $\{(1+t)^{-1}\exp(-(\omega + |B|_{\circledast}^{1/\alpha})t)R_B(t) : t \geq 0\}$ is equicontinuous, and $(R_B(t))_{t\geq 0}$ is an exponentially equicontinuous, analytic $r$-times integrated $(g_\alpha, C)$-regularized resolvent family of angle $\beta \in (0, \pi/2]$ provided that $(R(t))_{t\geq 0}$ is.*

*Remark 9.* It is worthwhile to mention (cf. [1, Theorem 2.5.6]) that Corollary 8 remains true, with a different upper bound for the growth order of $(R_B(t))_{t\geq 0}$, in the case $\alpha = 1$. Using [33, Lemma 3.3] and the proof of cited theorem, it follows that $(R_B(t))_{t\geq 0}$ is entire provided that $\alpha \in \mathbb{N}$ and that $(R(t))_{t\geq 0}$ is entire.

*Example 10.* Corollary 8 is a proper extension of [45, Lemma 4.7] provided $\alpha = 2$ and $B = zI$ $(z \in \mathbb{C})$, which can be applied in the analysis of the problem

$$u_{tt} + 2\beta u_t - \Delta u + 2\sum_{i=1}^{n}\alpha_i u_{x_i} + \mu u = 0 \tag{43}$$

in $L^p([0,\pi]^n)$, with Dirichlet boundary conditions; here we assume $n \in \mathbb{N}$, $1 \leq p < \infty$ and $\beta$, $\alpha_i$, $\mu \in \mathbb{C}$ (see e.g., [46, pages 144-145] and [15, Theorem 4.2]). It is clear that Corollary 8 can be applied to ($r$-coercive) differential operators generating integrated cosine functions ([2, 14, 15, 47–50]) or exponentially equicontinuous $(g_\alpha, C)$-regularized resolvent families ([23, 24]); in what follows, we will apply Corollary 8 to abstract differential operators generating $C$-regularized cosine functions. Let $E$ be one of the spaces $L^p(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$, $C_0(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$, $\mathrm{BUC}(\mathbb{R}^n)$, let $0 \leq l \leq n$ and let $\mathscr{F}^{-1}$ denote the inverse Fourier transform. Put $\mathbb{N}_0^l := \{\alpha \in \mathbb{N}_0^n : \alpha_{l+1} = \cdots = \alpha_n = 0\}$ and, for every $l = 0, 1, \ldots, n$, $E_l := \{f \in E : f^{(\alpha)} \in E \quad \text{for all} \quad \alpha \in \mathbb{N}_0^l\}$. Then the family of seminorms $(p_\alpha(f) := \|f^{(\alpha)}\|_E,\ f \in E_l;\ \alpha \in \mathbb{N}_0^l)$ induces a Fréchet topology on $E_l$. Let $\mathbf{T_1}$ possess the same meaning as in [51] and let $m \in \mathbb{N}, a_\alpha \in \mathbb{C}$, $0 \leq |\alpha| \leq m$. Consider the operator $P(D)f := \sum_{|\alpha|\leq m} a_\alpha f^{(\alpha)}$ with its maximal distributional domain. Set $P(x) := \sum_{|\alpha|\leq m} a_\alpha i^{|\alpha|}x^\alpha, x \in \mathbb{R}^n, h_{t,\beta}(x) := (1+|x|^2)^{-\beta/2}\sum_{j=0}^{\infty}(t^{2j}P(x)^j/(2j)!), x \in \mathbb{R}^n, t \geq 0, \beta \geq 0, \Omega(\omega) := \{\lambda^2 : \Re\lambda > \omega\}$, if $\omega > 0$, and $\Omega(\omega) := \mathbb{C} \setminus (-\infty, \omega^2]$, if $\omega \leq 0$. Assume $r \in [0, m], \omega \in \mathbb{R}$, and the following condition:

(♭) $P(x) \notin \Omega(\omega), x \in \mathbb{R}^n$ and, in the case $r \in (0, m]$, there exist $\sigma > 0$ and $\sigma' > 0$ such that $\Re P(x) \leq -\sigma|x|^r + \sigma', x \in \mathbb{R}^n$.

Then, for every $l = 0, 1, \ldots, n$, there exists $M \geq 1$ such that, for every $\beta > (m - (r/2))(n/4)$, $P(D)$ generates an exponentially equicontinuous $\mathbf{T_1}(\langle(1+|x|^2)^{-\beta}\rangle)$-regularized cosine function $(C_\beta(t))_{t\geq 0}$ in $E_l$ satisfying $C_\beta(t)f = \mathscr{F}^{-1}h_{t,\beta} * f, t \geq 0, f \in E_l$ and $p_\alpha(C_\beta(t)f) \leq Mp_\alpha(f)g_{n/2}(t), t \geq 0, f \in E_l, \alpha \in \mathbb{N}_0^l$, with $g_{n/2}(t)$ being the function defined on [52, page 40]; cf. [33, 51, 52] for full details. If $1 < p < \infty$ and $E = L^p(\mathbb{R}^n)$, then the previous result can be slightly refined by allowing that $\beta$ takes the value $(1/2)(m - r/2)n|1/p - 1/2|$. Given $\varphi \in L^1(\mathbb{R}^n)$, define the bounded linear operator $B$ on $E_l$ by $(Bf)(x) := \int_{\mathbb{R}^n}\varphi(x-t)f(t)dt, f \in E_l, x \in \mathbb{R}^n$. Then $BP(D) \subseteq P(D)B, \mathbf{T_1}(\langle(1+|x|^2)^{-\beta}\rangle)B = B\,\mathbf{T_1}(\langle(1+|x|^2)^{-\beta}\rangle)$ and $p_\alpha(Bf) \leq \|\varphi\|_{L^1(\mathbb{R}^n)}p_\alpha(f), f \in E, \alpha \in \mathbb{N}_0^l$. Applying Corollary 8, we get that $P(D) + B$ generates an exponentially equicontinuous $\mathbf{T_1}(\langle(1+|x|^2)^{-\beta}\rangle)$-regularized cosine function $(C_{\beta,B}(t))_{t\geq 0}$ in $E_l$.

Assume now $\alpha > 1$, $\varrho > 0, \sigma \in (0,1)$ and $k(t) = k_1(t) = \mathscr{L}^{-1}(\lambda^{-\alpha}e^{-\varrho\lambda^\sigma})(t)$, $t \geq 0$. By the consideration given in [1, Remark 2.5.4(iii)], it follows that, for every $l \in \mathbb{N}$, there exist real numbers $(p_{m,l,\alpha,\varrho,\sigma})_{1\leq m\leq l}$ such that $p_{1,l,\alpha,\varrho,\sigma} = \varrho(\sigma/\alpha)((\sigma/\alpha)-1)\cdots((\sigma/\alpha)-(l-1))$, $p_{l,l,\alpha,\varrho,\sigma} = (\varrho(\sigma/\alpha))^l$, and that the following holds:

$$\begin{aligned}
&\frac{d^l}{dz^l}\left(\frac{1}{z\tilde{k}(z^{1/\alpha})}\right) = \frac{d^l}{dz^l}e^{\varrho z^{\sigma/\alpha}} = e^{\varrho z^{\sigma/\alpha}}\sum_{m=1}^{l}p_{m,l,\alpha,\varrho,\sigma}z^{m(\sigma/\alpha)-l} \\
&\qquad z > 0, \\
&p_{m,l+1,\alpha,\varrho,\sigma} = \varrho\frac{\sigma}{\alpha}p_{m-1,l,\alpha,\varrho,\sigma} + \left(m\frac{\sigma}{\alpha} - l\right)p_{m,l,\alpha,\varrho,\sigma}, \\
&\qquad 2 \leq m \leq l.
\end{aligned} \tag{44}$$

This implies $c_0 = 1, k_0(t) = 0, c_i = 0, i \geq 1$,

$$\begin{aligned}
{}_ik(t) &= \sum_{m=1}^{i}p_{m,i,\alpha,\varrho,\sigma}g_{\alpha i-m\sigma}(t), \quad t > 0,\ i \geq 1, \\
k_{i,l_0,l}(t) &= c_{l_0,l,\alpha}\sum_{m=1}^{i-l}p_{m,i-l,\alpha,\varrho,\sigma}g_{\alpha i-l_0-m\sigma}(t), \quad t > 0, \\
&\qquad 1 \leq l < i, \quad 1 \leq l_0 \leq l,
\end{aligned} \tag{45}$$

$$k_{i,l_0,i}(t) = c_{l_0,i,\alpha}g_{\alpha i-l_0}(t), \quad t > 0,\ 1 \leq l_0 \leq i. \tag{46}$$

By means of (44) and the proof of Lemma 3, we obtain the existence of a constant $\zeta_{\alpha,\varrho,\sigma} \geq 1$ such that

$$\sum_{m=1}^{l}m!\left|p_{m,l,\alpha,\varrho,\sigma}\right| \leq \zeta_{\alpha,\varrho,\sigma}^l l! \quad \forall l \in \mathbb{N}. \tag{47}$$

In what follows, we assume that $\zeta_{\alpha,\varrho,\sigma} \geq 1$ is minimal with respect to (47); notice that $\zeta_{\alpha,\varrho,\sigma} > \varrho\sigma/\alpha$ and that it is not clear

whether Lemma 4 can be reconsidered in the newly arisen situation. Then

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{l_0=1}^{i}\frac{|B|_{\circledast}^{i}}{i!}\int_0^t (t-s)^{l_0}\left|k_{i,l_0,i}(s)\right| ds \\
&\quad=\sum_{i=1}^{\infty}\sum_{l_0=1}^{i}\frac{|B|_{\circledast}^{i}}{i!}\left|c_{l_0,i,\alpha}\right| l_0!\int_0^t \frac{(t-s)^{l_0}}{l_0!}\frac{s^{\alpha i-l_0-1}}{\Gamma(\alpha i-l_0)}ds \\
&\quad\le\sum_{i=1}^{\infty}\frac{\left(t^{\alpha}|B|_{\circledast}\right)^{i}}{\Gamma(\alpha i+1)}\le E_{\alpha}\left(t^{\alpha}|B|_{\circledast}\right)\le c_{\alpha}e^{t|B|_{\circledast}^{1/\alpha}},\qquad t\ge 0.
\end{aligned}\tag{48}$$

Since $\Gamma(\cdot)$ is increasing in $(\xi,\infty)$, where $\xi\sim 1.4616\ldots$, we obtain that

$$\begin{aligned}
\frac{\Gamma(\alpha i+1)}{\Gamma(\alpha i-m\sigma+1)\,m!}&\le\frac{\Gamma(\alpha i+1)}{\Gamma(\alpha i-m+1)\,m!} \\
&=\frac{\alpha i(\alpha i-1)\cdots(\alpha i-m+1)}{m!}\le\binom{\lceil\alpha i\rceil}{m},
\end{aligned}\tag{49}$$

provided $i\ge 2$ and $1\le m\le i-1$. Combining this with (35), Lemmas 3 and 4 and (47), we get

$$\begin{aligned}
&\sum_{i=2}^{\infty}\sum_{l=1}^{i-1}\sum_{l_0=1}^{l}\frac{|B|_{\circledast}^{i}}{i!}\binom{i}{l}\int_0^t (t-s)^{l_0}\left|k_{i,l_0,l}(s)\right| ds \\
&\quad\le\sum_{i=2}^{\infty}\sum_{l=1}^{i-1}\sum_{l_0=1}^{l}\frac{|B|_{\circledast}^{i}}{i!}\binom{i}{l}l_0!\left|c_{l_0,l,\alpha}\right| \\
&\qquad\times\sum_{m=1}^{i-l}m!\left|p_{m,i-l,\alpha,\varrho,\sigma}\right|\frac{t^{\alpha i-m\sigma}}{\Gamma(\alpha i-m\sigma+1)\,m!} \\
&\quad\le\frac{1}{\alpha}\sum_{i=2}^{\infty}\sum_{l=1}^{i-1}\frac{|B|_{\circledast}^{i}}{\Gamma(\alpha i+1)}\zeta_{\alpha,\varrho,\sigma}^{i-l}\sum_{m=1}^{i-l}\frac{t^{\alpha i-m\sigma}\Gamma(\alpha i+1)}{\Gamma(\alpha i-m\sigma+1)\,m!} \\
&\quad\le\frac{1}{\alpha}\sum_{i=2}^{\infty}\sum_{l=1}^{i-1}\frac{|B|_{\circledast}^{i}}{\Gamma(\alpha i+1)}\zeta_{\alpha,\varrho,\sigma}^{i-l}\sum_{m=1}^{i-l}t^{\alpha i-m\sigma}\binom{\lceil\alpha i\rceil}{m} \\
&\quad\le\frac{1}{\alpha}\left(1+t^{-\sigma}\right)\sum_{i=2}^{\infty}\sum_{l=1}^{i-1}\frac{|B|_{\circledast}^{i}}{\Gamma(\alpha i+1)}\zeta_{\alpha,\varrho,\sigma}^{i-l}\left(t+t^{1-\sigma}\right)^{\alpha i} \\
&\quad\le\frac{1}{\alpha\zeta_{\alpha,\varrho,\sigma}}\left(1+t^{-\sigma}\right)\sum_{i=1}^{\infty}i\frac{\left(|B|_{\circledast}\left(t+t^{1-\sigma}\right)^{\alpha}\zeta_{\alpha,\varrho,\sigma}\right)^{i+1}}{\Gamma(\alpha i+1)} \\
&\quad\le\frac{1}{\alpha}\left(1+t^{-\sigma}\right)|B|_{\circledast}\left(t+t^{1-\sigma}\right)^{\alpha} \\
&\qquad\times\left(\sum_{i=1}^{\infty}i\frac{z^{i}}{\Gamma(\alpha i+1)}\right)_{z=|B|_{\circledast}(t+t^{1-\sigma})^{\alpha}\zeta_{\alpha,\varrho,\sigma}} \\
&\quad\le\frac{b_{\alpha}}{\alpha^{2}}\left(1+t^{-\sigma}\right)|B|_{\circledast}^{(1+\alpha)/\alpha} \\
&\qquad\times\left(t+t^{1-\sigma}\right)^{1+\alpha}\zeta_{\alpha,\varrho,\sigma}^{1/\alpha}e^{|B|_{\circledast}^{1/\alpha}(t+t^{1-\sigma})\zeta_{\alpha,\varrho,\sigma}^{1/\alpha}},\qquad t\ge 1.
\end{aligned}\tag{50, 51}$$

Noticing that $t^{\alpha i-m\sigma}\le 1, t\in[0,1), i\ge 2, 1\le m\le i-1$, we obtain from (50) that there exists $\xi_{\alpha,\varrho,\sigma}\ge 1$ such that

$$\begin{aligned}
&\sum_{i=2}^{\infty}\sum_{l=1}^{i-1}\sum_{l_0=1}^{l}\frac{|B|_{\circledast}^{i}}{i!}\binom{i}{l}\int_0^t (t-s)^{l_0}\left|k_{i,l_0,l}(s)\right| ds \\
&\qquad\le\xi_{\alpha,\varrho,\sigma},\qquad t\in[0,1).
\end{aligned}\tag{52}$$

By (48)–(52), (v) holds for any $\omega'>(|B|_{\circledast}\zeta_{\alpha,\varrho,\sigma})^{1/\alpha}$. In almost the same way, one can prove that (iv) and (vi) hold for any $\omega'>(|B|_{\circledast}\zeta_{\alpha,\varrho,\sigma})^{1/\alpha}$. Assume now that $(R(t))_{t\ge 0}$ is an exponentially equicontinuous, analytic $(g_\alpha,k)$-regularized $C$-resolvent family of angle $\beta\in(0,\pi/2]$, $\gamma\in(0,\beta)$, $\omega_{\gamma,1}\ge\omega_\gamma$, $\omega_{\gamma,1}>0$, $(\omega_{\gamma,1}\cos\gamma)^{\alpha-\sigma}>|B|_{\circledast}\zeta_{\alpha,\varrho,\sigma}$, $\varepsilon=1/\cos\gamma$, $\omega_{\gamma,2}\ge\omega_\gamma+\varepsilon$ and $(1+\omega_{\gamma,2}\cos\gamma)^{\alpha-1}>|B|_{\circledast}\zeta_{\alpha,\varrho,\sigma}$. Then

$$\begin{aligned}
\sum_{i=0}^{\infty}\frac{|B|_{\circledast}^{i}}{i!}\sup_{\lambda\in\omega_{\gamma,1}+\Sigma_{(\pi/2)+\gamma}}\left|{}_{i}\widetilde{k}(\lambda)\right|&\le\sum_{i=0}^{\infty}\frac{|B|_{\circledast}^{i}}{i!}\sum_{m=1}^{i}\frac{m!\left|p_{m,i,\alpha,\varrho,\sigma}\right|}{\left(\omega_{\gamma,1}\cos\gamma\right)^{\alpha i-m\sigma}} \\
&\le\mu_{\sigma,\gamma}\sum_{i=0}^{\infty}\frac{|B|_{\circledast}^{i}\zeta_{\alpha,\varrho,\sigma}^{i}}{\left(\omega_{\gamma,1}\cos\gamma\right)^{(\alpha-\sigma)i}}<\infty
\end{aligned}\tag{53}$$

for an appropriate constant $\mu_{\sigma,\gamma}\ge 1$,

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{l_0=1}^{i}\frac{|B|_{\circledast}^{i}}{i!}\frac{l_0!}{\sqrt{2\pi l_0}(\varepsilon\cos\gamma)^{l_0}}\sup_{\lambda\in\omega_{\gamma,2}+\Sigma_{(\pi/2)+\gamma}}\left|\mathscr{L}\left(k_{i,l_0,i}(t)\right)(\lambda)\right| \\
&\quad\le\sum_{i=1}^{\infty}i\frac{|B|_{\circledast}^{i}}{\left(1+\omega_{\gamma,2}\right)^{(\alpha-1)i}}<\infty,
\end{aligned}\tag{54}$$

and, for every $\varepsilon>0$ with $(1+\omega_{\gamma,2}\cos\gamma)^{\alpha-1}>|B|_{\circledast}(\zeta_{\alpha,\varrho,\sigma}+\varepsilon)$, there exists $\mu_\varepsilon>0$ such that

$$\begin{aligned}
&\sum_{i=2}^{\infty}\sum_{l=1}^{i-1}\sum_{l_0=1}^{l}\frac{|B|_{\circledast}^{i}}{i!}\binom{i}{l}\frac{l_0!}{\sqrt{2\pi l_0}(\varepsilon\cos\gamma)^{l_0}} \\
&\qquad\times\sup_{\lambda\in\omega_{\gamma,2}+\Sigma_{(\pi/2)+\gamma}}\left|\mathscr{L}\left(k_{i,l_0,l}(t)\right)(\lambda)\right| \\
&\quad\le\sum_{i=2}^{\infty}\sum_{l=1}^{i-1}\sum_{l_0=1}^{l}\frac{|B|_{\circledast}^{i}}{(i-l)!l!}l_0!\left|c_{l_0,l,\alpha}\right|\sum_{m=1}^{i-l}\frac{\left|p_{m,i-l,\alpha,\varrho,\sigma}\right|}{\left(1+\omega_{\gamma,2}\cos\gamma\right)^{\alpha i-l_0-m\sigma}} \\
&\quad\le\sum_{i=2}^{\infty}\sum_{l=1}^{i-1}\sum_{l_0=1}^{l}\frac{|B|_{\circledast}^{i}}{(i-l)!l!}l_0!\left|c_{l_0,l,\alpha}\right|\sum_{m=1}^{i-l}\frac{\left|p_{m,i-l,\alpha,\varrho,\sigma}\right|}{\left(1+\omega_{\gamma,2}\cos\gamma\right)^{(\alpha-1)i}} \\
&\quad\le\frac{1}{\alpha}\sum_{i=2}^{\infty}\sum_{l=1}^{i-1}\frac{|B|_{\circledast}^{i}\zeta_{\alpha,\varrho,\sigma}^{i-l}}{\left(1+\omega_{\gamma,2}\cos\gamma\right)^{(\alpha-1)i}} \\
&\quad\le\frac{1}{\alpha}\sum_{i=2}^{\infty}(i-1)\frac{|B|_{\circledast}^{i}\left(\zeta_{\alpha,\varrho,\sigma}+\varepsilon\right)^{i}}{\left(1+\omega_{\gamma,2}\cos\gamma\right)^{(\alpha-1)i}}<\infty,
\end{aligned}\tag{55}$$

proving the conditions (29)-(30).

**Corollary 11.** *Let $\alpha > 1, \omega \geq 0, \varrho > 0, \sigma \in (0,1), k(t) = \mathscr{L}^{-1}(\lambda^{-\alpha}e^{-\varrho\lambda^{\sigma}})(t), t \geq 0$, and let $A$ be a subgenerator of a global $(g_\alpha, k)$-regularized $C$-resolvent family $(R(t))_{t\geq 0}$ satisfying* (2) *with $a(t) = g_\alpha(t)$. Let $B \in L(E)$ satisfy the condition* (i) *quoted in the formulation of Theorem 5. Then $A + B$ is a subgenerator of a global $(g_\alpha, k)$-regularized $C$-resolvent family $(R_B(t))_{t\geq 0}$ satisfying* (18) *with $k_1(t) = k(t)$. Furthermore, for every $\varepsilon > 0$, the family $\{\exp(-(\omega + (|B|_{\circledast}\zeta_{\alpha,\varrho,\sigma})^{1/\alpha} + \varepsilon)t)R_B(t) : t \geq 0\}$ is equicontinuous, and $(R_B(t))_{t\geq 0}$ is an exponentially equicontinuous, analytic $(g_\alpha, k)$-regularized $C$-resolvent family of angle $\beta \in (0, \pi/2]$ provided that $(R(t))_{t\geq 0}$ is.*

*Example 12.* Let $s > 1$,

$$E := \left\{ f \in C^{\infty}\ [0,1]\ ;\ \|f\| = \sup_{p\geq 0} \frac{\|f^{(p)}\|_{\infty}}{p!^{s}} < \infty \right\}, \quad (56)$$

$$A := \frac{-d}{ds}, \qquad D(A) := \left\{ f \in E;\ f' \in E,\ f(0) = 0 \right\}. \quad (57)$$

Then $\rho(A) = \mathbb{C}$, $A$ generates a tempered ultradistribution semigroup of $(p!^{s})$-class, and $A$ cannot be the generator of a distribution semigroup since $A$ is not stationary dense (see e.g., [53, Example 1.6] and [41]). If $f \in E, t \in [0,1]$ and $\lambda \in \mathbb{C}$, set $f_\lambda^1(t) := \int_0^t e^{-\lambda(t-s)} f(s)\,ds$ and $f_\lambda^2(t) := \int_0^t e^{\lambda(t-s)} f(s)\,ds$. Then $f_\lambda^1(\cdot),\ f_\lambda^2(\cdot) \in E, \lambda \in \mathbb{C}$, and there exist $b > 0$ and $M \geq 1$, independent of $f(\cdot)$, such that

$$\left\| f_\lambda^1(\cdot) \right\| \leq M \|f\| e^{b|\lambda|^{1/s}}, \quad \Re\lambda \geq 0,\ f \in E. \quad (58)$$

It is clear that $\|f_\lambda^2(\cdot)\|_{L^\infty[0,1]} \leq e^{|\lambda|}\|f\|, \Re\lambda \geq 0$ and $\|(d/dt)f_\lambda^2(\cdot)\|_{L^\infty[0,1]} \leq (|\lambda|e^{|\lambda|} + 1)\|f\|, \Re\lambda \geq 0$. Proceeding by induction, we obtain that, for every $n \geq 2, t \in [0,1]$ and $\lambda \in \mathbb{C}$ with $\Re\lambda \geq 0$ :

$$\frac{d^n}{dt^n} f_\lambda^2(t) = \frac{d^{n-1}}{dt^{n-1}} f(t) + \sum_{k=1}^{n-1} \lambda^k \frac{d^{n-1-k}}{dt^{n-1-k}} f(t) + \lambda^n f_\lambda^2(t). \quad (59)$$

On the other hand, [54, Proposition 4.5] implies that there exists $c > 0$ such that $\sum_{p=0}^{\infty} t^p/p!^{s} = O(\exp(ct^{1/s}))$, $t \geq 0$. Combined with (59) and the logarithmic convexity, the last estimate yields

$$\begin{aligned} \frac{1}{n!^{s}} \left\| \frac{d^n}{dt^n} f_\lambda^2(\cdot) \right\|_{L^\infty[0,1]} &\leq \|f\| + \|f\| e^{c|\lambda|^{1/s}} + \frac{|\lambda|^n}{n!^{s}} e^{|\lambda|} \|f\| \\ &\leq \left(1 + e^{c|\lambda|^{1/s}} + e^{c|\lambda|^{1/s}} e^{|\lambda|}\right) \|f\|, \\ &\qquad \Re\lambda \geq 0,\ \lambda \neq 0. \end{aligned} \quad (60)$$

In view of (60) we get that, for every $\eta > 1$, there exists $M_\eta \geq 1$, independent of $f(\cdot)$, such that

$$\left\| f_\lambda^2(\cdot) \right\| \leq M_\eta \|f\| e^{\eta|\lambda|}, \quad \Re\lambda \geq 0,\ f \in E. \quad (61)$$

Consider now the complex polynomial $P(z) = \sum_{j=0}^{n} a_j z^j, z \in \mathbb{C}, a_n \neq 0, n \geq 2$. Set, for every $\lambda \in \mathbb{C}, P_\lambda(\cdot) := P(\cdot) - \lambda$ and consider the operator $P(A)$ defined by

$$\begin{aligned} D(P(A)) &:= D(A^n), \\ P(A) f &:= \sum_{j=0}^{n} a_j A^j f, \quad f \in D(P(A)). \end{aligned} \quad (62)$$

Clearly, $P(A)$ is not stationary dense. Let $r > 0$ and $d > 0$ be such that $P(z) \neq 0, |z| \geq r$ and $P'(z) \neq 0, |z| \geq d$. Let $z_{1,\lambda}, \ldots, z_{n,\lambda}$ denote the zeros of the polynomial $z \mapsto P_\lambda(z), z \in \mathbb{C}$ and let $0 < m := \min_{|z|\geq d+1}|P'(z)|$. Then an old result of Walsh [55] says that $|z_{j,\lambda}| \leq r + |a_n|^{-1/n}|\lambda|^{1/n}, 1 \leq j \leq n, \lambda \in \mathbb{C}$. Furthermore, it is checked at once that there exists a sufficiently large $\lambda_0 > 0$ such that $z_{j,\lambda}$ is a simple zero of $P_\lambda(z)$ and that $|z_{j,\lambda}| \geq d + 1$, provided $|\lambda| \geq \lambda_0$ and $1 \leq j \leq n$. Therefore, for every $\lambda \in \mathbb{C}$ with $|\lambda| \geq \lambda_0$ and for every $i,\ j \in \{1, \ldots, n\}$ with $i \neq j$, the following holds:

$$\begin{aligned} d + 1 &\leq \left|z_{j,\lambda}\right| \leq r + \left|a_n\right|^{-1/n}|\lambda|^{1/n}, \\ \left|P'\left(z_{j,\lambda}\right)\right| &\geq m, \quad z_{i,\lambda} \neq z_{j,\lambda}. \end{aligned} \quad (63)$$

It is straightforward to verify that

$$\begin{aligned} \rho(p(A)) &= \mathbb{C}, \\ R(\lambda : p(A)) &= (-1)^{n+1} a_n^{-1} R(z_{1,\lambda} : A) \cdots R(z_{n,\lambda} : A), \\ &\qquad \lambda \in \mathbb{C}. \end{aligned} \quad (64)$$

Assume now $|\lambda| \geq \lambda_0$. Then de L'Hospital's rule implies

$$a_n \prod_{\substack{1\leq i\leq n \\ i\neq j}} \left(z_{i,\lambda} - z_{j,\lambda}\right) = (-1)^{n+1} P'\left(z_{j,\lambda}\right), \quad 1 \leq j \leq n. \quad (65)$$

Using the resolvent equation, (58), (61)–(63), and (65), one can rewrite and evaluate the right-hand side of equality appearing in (64) as follows:

$$\begin{aligned} &\left\| (-1)^{n+1} a_n^{-1} R(z_{1,\lambda} : A) \cdots R(z_{n,\lambda} : A) \right\| \\ &\quad = \left\| (-1)^{n+1} a_n^{-1} \sum_{j=1}^{n} \frac{R(z_{j,\lambda} : A)}{\prod_{\substack{1\leq i\leq n \\ i\neq j}} \left(z_{i,\lambda} - z_{j,\lambda}\right)} \right\| \\ &\quad = \left\| \sum_{j=1}^{n} \frac{R(z_{j,\lambda} : A)}{P'\left(z_{j,\lambda}\right)} \right\| \leq \frac{1}{m} \sum_{j=1}^{n} \left\| R(z_{j,\lambda} : A) \right\|. \end{aligned} \quad (66)$$

By (64) and (66) we finally get that, for every $\eta > 1$,

$$\left\| R(\lambda : p(A)) \right\| = O\left(e^{b|a_n|^{-1/n}|\lambda|^{1/ns}} + e^{\eta|a_n|^{-1/n}|\lambda|^{1/n}}\right), \quad \lambda \in \mathbb{C}. \quad (67)$$

Since the preceding estimate holds for any $\lambda \in \mathbb{C}$, it is quite complicated to inscribe here all of its consequences

(cf. [1, 16], [32, (2.35)–(2.37)], [56, 57]); for example, $P(A)$ generates a tempered ultradistribution sine of $(p!^s)$-class provided $n \geq 2s$, and $P(A)$ generates an exponentially bounded, $\mathscr{L}^{-1}(e^{-\varrho\lambda^{1/n}})$-convoluted group provided $\varrho > |a_n|^{-1/n}/\cos(\pi/2n)$. Let us also mention that the consideration given in example following [41, Corollary 3.8] enables one to construct important examples of (pseudo-)differential operators generating ultradistribution sines, and that the estimate (67) can be derived, with insubstantial technical modifications, in the case of a general sequence $(M_p)$ of positive numbers satisfying $M_0 = 1$ and $M_{p+q} \geq M_p M_q (p, q \geq 0)$. In what follows, we will illustrate an application of Corollary 11. Suppose $n > \alpha \geq 1, \delta \in (0, \pi/2], (\pi/2 + \delta)\alpha/n < \pi/2, \varrho \geq 1/\cos((\pi/2 + \delta)\alpha/n)$ and $k(t) = \mathscr{L}^{-1}(\lambda^{-\alpha} e^{-\varrho\lambda^{\alpha/n}})(t), t \geq 0$. By [32, Theorem 2.17] and (67), $P(A)$ is the integral generator of an exponentially bounded, analytic $(g_\alpha, k)$-regularized resolvent family of angle $\delta$ (cf. also [41, Proposition 3.12]). Let $\varphi \in E$ and $Bf(t) := (\varphi * f)(t), t \in [0, 1], f \in E$. Then $B \in L(E), BP(A) \subseteq P(A)B$ and, by Corollary 11, $P(A) + B$ is the integral generator of an exponentially bounded, analytic $(g_\alpha, k)$-regularized resolvent family of angle $\delta$.

The following extension of [9, Theorem 3.1], [32, Theorem 2.12] has been recently established in [33]; cf. also [39, Theorem 3.15.6], [4, Theorem 1.1].

**Theorem 13.** *Suppose $M > 0$, $\omega_1 \geq \omega \geq 0$, $A$ is a subgenerator of an $(a, k)$-regularized $C$-resolvent family $(R(t))_{t\geq 0}$ such that $p(R(t)x) \leq Me^{\omega t} p(x)$, $x \in E$, $t \geq 0$, $p \in \circledast$ and $z \in \mathbb{C}$. Let $B : \overline{D(A)} \to E$ be a linear operator such that $BCx = CBx$, $x \in D(A)$ and that, for every $p \in \circledast$, there exist $c_p > 0$ and $q \in \circledast$ satisfying $p(C^{-1}Bx) \leq c_p q(x)$, $x \in \overline{D(A)}$. Let (P1) hold for $a(t)$, $k(t), b(t)$ and let $\tilde{a}(\lambda)/\tilde{k}(\lambda) = \tilde{b}(\lambda) + z$, $\Re\lambda > \omega_1$, $\tilde{k}(\lambda) \neq 0$. Suppose $\mu > \omega_1$, $\gamma \in [0, 1)$ and*

$$\overline{D(A)} = E, \tag{68}$$

$$\int_0^\infty e^{-\mu t} p\left(C^{-1}B\int_0^t b(t-s)R(s)x\,ds + zC^{-1}BR(t)x\right)dt \leq \gamma p(x), \quad x \in D(A), \ p \in \circledast \tag{69}$$

*or*

$$(R(t))_{t\geq 0} \text{ satisfies (2)}, \ \overline{D(A)} \neq E \text{ and (69) holds for any } x \in E, \ p \in \circledast. \tag{70}$$

*Then the operator $A + B$ is a subgenerator of an $(a, k)$-regularized $C$-resolvent family $(R_B(t))_{t\geq 0}$ satisfying (18) with $k_1(t)$ and $g_\alpha(t)$ replaced by $k(t)$ and $a(t)$ therein. Furthermore,*

$$p(R_B(t)x) \leq \frac{M}{1-\gamma} e^{\mu t} p(x), \quad x \in E, \ t \geq 0, \ p \in \circledast, \tag{71}$$

$$R_B(t)x = R(t)x + \int_0^t R_B(t-r)\left(C^{-1}B\int_0^r b(r-s)R(s)x\,ds + zC^{-1}BR(t)x\right)dr, \quad t \geq 0, \quad x \in D(A), \tag{72}$$

*and (72) holds for any $t \geq 0$ and $x \in E$ provided (70).*

In many cases, we do not have the existence of a function $b(t)$ and a complex number $z$ such that $\tilde{a}(\lambda)/\tilde{k}(\lambda) = \tilde{b}(\lambda) + z, \Re\lambda > \omega_1, \tilde{k}(\lambda) \neq 0$. The following theorem is an attempt to fill this gap.

**Theorem 14.** *Suppose $M$, $M_1 > 0, \omega \geq 0, l \in \mathbb{N}$ and $A$ is a subgenerator of an $(a, k)$-regularized $C$-resolvent family $(R(t))_{t\geq 0}$ such that $p(R(t)x) \leq Me^{\omega t} p(x)$, $x \in E$, $t \geq 0$, $p \in \circledast$ and that (2) holds. Let $a(t)$ and $k(t)$ satisfy (P1) and let the following conditions hold:*

(i) *$BCx = CBx$, $x \in D(A)$,*

$$\begin{gathered} p\left(CA^jC^{-1}Bx\right) \leq M_1 p(x), \\ x \in \overline{D(A)}, \quad p \in \circledast, \quad 0 \leq j \leq l-1, \\ p\left(A^lC^{-1}Bx\right) \leq M_1 p(x), \quad x \in \overline{D(A)}, \ p \in \circledast. \end{gathered} \tag{73}$$

(ii) *There exist a function $b(t)$ satisfying (P1) and a complex number $z$ such that*

$$\frac{\tilde{a}(\lambda)^{l+1}}{\tilde{k}(\lambda)} = \tilde{b}(\lambda) + z, \quad \Re\lambda > \max(\omega, \operatorname{abs}(a), \operatorname{abs}(k)), \quad \tilde{k}(\lambda) \neq 0. \tag{74}$$

(iii) *$\lim_{\lambda\to+\infty} \int_0^\infty e^{-\lambda t}|a(t)|\,dt = 0$ and $\lim_{\lambda\to+\infty} \int_0^\infty e^{-\lambda t} \times |b(t)|\,dt = 0$.*

*Then, for every $x \in E$, there exists a unique solution of the integral equation*

$$R_B(t)x = R(t)x + (S * R_B)(t)x, \quad t \geq 0; \tag{75}$$

*furthermore, $(R_B(t))_{t\geq 0}$ is an $(a, k)$-regularized $C$-resolvent family with a subgenerator $A + B$, there exist $\mu \geq \max(\omega, \operatorname{abs}(a), \operatorname{abs}(k))$ and $\gamma \in [0, 1)$ such that (71) holds and that (18) holds with $k_1(t)$ and $g_\alpha(t)$ replaced by $k(t)$ and $a(t)$ therein.*

*Proof.* It is clear that $\int_0^\infty e^{-\lambda t}|a^{*,j}(t)|\,dt \leq (\int_0^\infty e^{-\lambda t}|a(t)|\,dt)^j$, $j \in \mathbb{N}, \lambda > \operatorname{abs}(a)$. Define, for every $x \in \overline{D(A)}$ and $t \geq 0$,

$$S(t)x := \sum_{j=0}^{l-1} a^{*,j+1}(t)CA^jC^{-1}Bx + \int_0^t b(t-s)R(s)A^lC^{-1}Bx\,ds + zR(t)A^lC^{-1}Bx. \tag{76}$$

By [58, Theorem 1.7, page 3] it follows that $R(t)x \in \overline{D(A)}$, $t \geq 0$, $x \in E$. Using this fact and (i), we get that $S(t) \in L(\overline{D(A)})$, $t \geq 0$. Keeping in mind the condition (ii), it is not difficult to prove that, for every $x \in \overline{D(A)}$,

$$\mathscr{L}(S(t)x)(\lambda) = \tilde{a}(\lambda)(I - \tilde{a}(\lambda)A)^{-1}CC^{-1}Bx, \quad x \in \overline{D(A)}, \quad \Re\lambda > \omega_1, \quad \tilde{k}(\lambda) \neq 0. \tag{77}$$

Using the conditions (i) and (iii), we obtain the existence of numbers $\mu > \max(\omega, \operatorname{abs}(a), \operatorname{abs}(k))$ and $\gamma \in [0,1)$ such that

$$\int_0^\infty e^{-\mu t} p(S(t)x)\,dt \leq \gamma p(x), \quad x \in \overline{D(A)}, \ p \in \circledast \tag{78}$$

and that (H1) holds, where

(H1) : For every strongly continuous function $f : [0,\infty) \to L(E, \overline{D(A)})$ such that $p(f(t)x) \leq Mp(x)$, $x \in E$, $t \geq 0$, $p \in \circledast$, the following inequality holds:

$$\int_0^t e^{-\mu(t-s)} p(S(t-s)f(s)x)\,ds \leq \gamma M p(x), \quad x \in E, \quad t \geq 0, \quad p \in \circledast. \tag{79}$$

Now one can define inductively, for every $t \geq 0$, the sequence $(T_n(t))_{n\in\mathbb{N}_0}$ in $L(E, \overline{D(A)})$ by $T_0(t) := R(t)$ and $T_{n+1}(t)x := \int_0^t S(t-s)T_n(s)x\,ds$, $x \in E$, $n \in \mathbb{N}_0$; observe that, for every $n \in \mathbb{N}_0$, $(T_n(t))_{t\geq 0}$ is strongly continuous and that the family $\{T_n(t) : t \geq 0\}$ is locally equicontinuous (with clear meaning). By (78), (H1), and the proof of [9, Theorem 3.1], it follows inductively that

$$p(T_n(t)x) \leq M\gamma^n e^{\mu t} p(x), \quad x \in E, \ t \geq 0, \ p \in \circledast \tag{80}$$

and that, for every $x \in E$ and $t \geq 0$, the sequence $(R_B^n(t)x := \sum_{i=0}^n T_i(t)x)_n$ is Cauchy in $E$ and therefore convergent. Set $R_B(t)x := \lim_{n\to\infty} R_B^n(t)x$, $x \in E$, $t \geq 0$. It is obvious that the mapping $t \mapsto R_B(t)x$, $t \geq 0$ is continuous for every fixed $x \in E$ as well as that (71) and (75) hold. Therefore, it suffices to show that

$$\tilde{k}(\lambda)(I - \tilde{a}(\lambda)(A+B))^{-1}Cx = \int_0^\infty e^{-\lambda t} R_B(t)x\,dt, \quad x \in E, \quad \Re\lambda > \mu, \quad \tilde{k}(\lambda) \neq 0. \tag{81}$$

Towards this end, notice that (78) and (H1) together imply that $I - \tilde{S}(\lambda)$ is invertible for $\Re\lambda > \mu$ and $(I - \tilde{S}(\lambda))^{-1} = \sum_{n=0}^\infty [\tilde{S}(\lambda)]^n$, $\Re\lambda > \mu$. Now we obtain from (75)

$$\widetilde{R_B}(\lambda)x = \left(I - \tilde{S}(\lambda)\right)^{-1}\tilde{k}(\lambda)(I - \tilde{a}(\lambda)A)^{-1}Cx, \quad \Re\lambda > \mu, \quad x \in E, \tag{82}$$

which immediately implies with (77) the validity of (81) in case $\tilde{a}(\lambda) = 0$, $\Re\lambda > \mu$ and $\tilde{k}(\lambda) \neq 0$. Assume now $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$ and $\Re\lambda > \mu$. Then a straightforward computation involving the equality $BCx = CBx$, $x \in D(A)$ as well as (77) and (82) shows that the operator $I - \tilde{a}(\lambda)(A+B)$ is injective and

$$\left(I - \tilde{S}(\lambda)\right)^{-1}\tilde{k}(\lambda)(I - \tilde{a}(\lambda)A)^{-1}C(I - \tilde{a}(\lambda)(A+B))x = \tilde{k}(\lambda)Cx, \quad x \in D(A). \tag{83}$$

The representation $(I - \tilde{S}(\lambda))^{-1} = \sum_{n=0}^\infty [((1/\tilde{a}(\lambda)) - A)^{-1} \times CC^{-1}B]^n$ implies

$$\left(\frac{1}{\tilde{a}(\lambda)} - (A+B)\right)\left(I - \tilde{S}(\lambda)\right)^{-1}\tilde{k}(\lambda)\left(\frac{1}{\tilde{a}(\lambda)} - A\right)^{-1}Cx = \tilde{k}(\lambda)Cx, \quad x \in E, \tag{84}$$

$R(C) \subseteq R(I - \tilde{a}(\lambda)(A+B))$ and (81), finishing the proof of theorem. □

*Remark 15.* Now we will explain how one can reformulate Theorem 13 in case in which $B$ is not necessarily bounded operator from $\overline{D(A)}$ into $E$ (cf. also [3, 7, 8] and the next section). Consider the situation of Theorem 13 with $E$ being complete. Assume (69) and, instead of condition $C^{-1}B \in L(E)$ :

($\natural$) $C^{-1}B : D(A) \to E$ and, for every $p \in \circledast$, there exist $c_p > 0$ and $q \in \circledast$ such that $p(C^{-1}Bx) \leq c_p(q(x) + q(Ax))$, $x \in D(A)$.

Denote, with a little abuse of notation, $T_0(t) = R(t)$, $t \geq 0$, $S(t)x = C^{-1}B(\int_0^t b(t-s)R(s)x\,ds + zR(t)x)$, $t \geq 0$, $x \in D(A)$ and $T_1(t)x = \int_0^t T_0(t-s)S(s)x\,ds$, $t \geq 0$, $x \in D(A)$. Then ($\natural$) implies that the mapping $t \mapsto S(t)x$, $t \geq 0$ is continuous for every $x \in D(A)$ and $p(T_1(t)x) \leq \gamma M p(x)$, $x \in D(A)$, $t \geq 0$, $p \in \circledast$. By [59, Lemma 22.19] and the completeness of $E$, one can extend the operator $T_1(t)$ to the whole space $E$ ($t \geq 0$). Proceeding inductively, one can define for each $t \geq 0$ the sequence $(T_n(t) = \int_0^t T_{n-1}(t-s)S(s)\,ds)_{n\in\mathbb{N}_0}$ in $L(E)$ such that $p(T_n(t)x) \leq \gamma^n M p(x)$, $x \in E$, $t \geq 0$, $p \in \circledast$. The preceding inequality implies that, for every $x \in E$, the sequence $(R_B^n(t)x \equiv \sum_{i=0}^n T_i(t)x)_n$ is Cauchy in $E$ and therefore convergent. Put $R_B(t)x = \lim_{n\to\infty} R_B^n(t)x$, $x \in E$, $t \geq 0$. As in the proof of Theorem 14, the mapping $t \mapsto R_B(t)x$, $t \geq 0$ is continuous for every fixed $x \in E$ and (71)-(72) hold. Using the closedness of $A$ and the condition ($\natural$), we get $\int_0^\infty e^{-\lambda t}S(t)x\,dt = C^{-1}B\tilde{a}(\lambda)(I - \tilde{a}(\lambda)A)^{-1}Cx$, $x \in D(A)$, $\Re\lambda > \mu$, $\tilde{k}(\lambda) \neq 0$ and $C^{-1}B\tilde{a}(\lambda)(I - \tilde{a}(\lambda)A)^{-1}C \in L(E)$, $\Re\lambda > \max(\omega_1, \mu)$, $\tilde{k}(\lambda) \neq 0$. In view of (69), $p(\tilde{S}(\lambda)x) \leq \gamma p(x)$, $x \in D(A)$, $\Re\lambda > \mu$, $p \in \circledast$; by the denseness of $D(A)$ in $E$, the last estimate holds for all $x \in E$. Hence, the operator $I - \tilde{S}(\lambda)$ is invertible and $(I - \tilde{S}(\lambda))^{-1}x = \sum_{n=0}^\infty [C^{-1}B\tilde{a}(\lambda)(I - \tilde{a}(\lambda)A)^{-1}C]^n x$, $x \in E$, $\Re\lambda > \mu$, $\tilde{k}(\lambda) \neq 0$. Suppose, for the time being, $\Re\lambda > \mu$ and $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$. The closedness of the operator $A + B$ can be proved as follows. Let a net $(x_\tau)_{\tau\in T}$ in $E$ satisfy $x_\tau \to x$, $\tau \to \infty$ and $(I -$

$\widetilde{a}(\lambda)(A+B))x_\tau \to y$, $\tau \to \infty$. Then a simple computation shows that $(I-\widetilde{a}(\lambda)(A+B))x_\tau = (I-\widetilde{S}(\lambda))(I-\widetilde{a}(\lambda)A)\ x_\tau$, which implies $(I-\widetilde{a}(\lambda)A)x_\tau = (I-\widetilde{S}(\lambda))^{-1}(I-\widetilde{a}(\lambda)(A+B))x_\tau \to (I-\widetilde{S}(\lambda))^{-1}y, \tau \to \infty$. Since $I-\widetilde{a}(\lambda)A$ is closed, we infer that $x \in D(A)$, $(I-\widetilde{a}(\lambda)A)x = (I-\widetilde{S}(\lambda))^{-1}y$ and $(I-\widetilde{a}(\lambda)(A+B))x = y$. Therefore, the closedness of $A+B$ follows from that of $I-\widetilde{a}(\lambda)(A+B)$. Suppose now $\Re\lambda > \mu$ and $\widetilde{k}(\lambda) \neq 0$. Similarly as in the proof of Theorem 14, we get $\widetilde{R}_B(\lambda)x = \widetilde{R}(\lambda)(I-\widetilde{S}(\lambda))^{-1}x$, $x \in E$, the injectiveness of $I-\widetilde{a}(\lambda)(A+B)$, $R(C) \subseteq R(I-\widetilde{a}(\lambda)(A+B))$ and $\widetilde{k}(\lambda)(I-\widetilde{a}(\lambda)(A+B))^{-1}Cx = \widetilde{R}_B(\lambda)x$, $x \in E$, which implies that the conclusions of Theorem 13 continue to hold. We left to the interested reader details concerning the possibilities of the extension of [8, Theorems 3.1 and 3.2] and results of [3, 7, 11, 12] to abstract Volterra equations in SCLCSs.

*Remark 16.* The local Hölder continuity with exponent $\sigma \in (0,1]$ is an example of the property which is stable under perturbations described in Theorems 13–14 and Remark 15, as indicated below. Consider the situation of Theorem 13 in which $D(A)$ is dense in $E$. Using the same notation as in Remark 15, one has $p(S(t)x) \leq c_p M q(x)[\int_0^t |b(t-r)|e^{\omega r}dr + |z|e^{\omega t}]$, $p \in \circledast$, $t \geq 0, x \in D(A)$. Suppose now that, for every $T > 0$ and $p \in \circledast$, there exist $c_{T,p} > 0$ and $h_{T,p} \in \circledast$ such that

$$p(R(t)x - R(s)x) \leq c_{T,p}(t-s)^\sigma h_{T,p}(x), \quad x \in E, \quad 0 \leq s < t \leq T. \tag{85}$$

Let $T > 0$ and $p \in \circledast$ be fixed. Then, for every $x \in D(A)$ and $0 \leq s < t \leq T$,

$$\begin{aligned} &p(S(t)x - S(s)x) \\ &\quad \leq c_p q\left(\int_0^s b(r)(R(t-r)x - R(s-r)x)\ dr + \int_t^s b(r)R(t-r)x\,dr\right) \\ &\quad + c_p|z|q(R(t)x - R(s)x) \\ &\quad \leq c_p\left[c_{T,q}(t-s)^\sigma h_{T,q}(x)\left(\int_0^T |b(r)|\,dr + |z|\right) + Me^{\omega T}q(x)\int_t^s |b(r)|\ dr\right], \end{aligned} \tag{86}$$

which implies by (72) that

$$\begin{aligned} &p((R_B * S)(t)x - (R_B * S)(s)x) \\ &\quad \leq \frac{Me^{\mu T}}{1-\gamma}\left(\int_0^s p(S(t-r)x - S(s-r)x)\ dr + \int_s^t p(S(t-r)x)\ dr\right) \\ &\quad \leq \frac{c_p M e^{\mu T}}{1-\gamma}\left(c_{T,q}(t-s)^\sigma h_{T,q}(x)\int_0^s\int_0^T |b(v)|\ dv\,dr + Me^{\omega T}q(x)\int_0^s\int_{s-r}^{t-r}|b(v)|\,dv\,dr + |z|c_{T,q}T(t-s)^\sigma h_{T,q}(x)\right) \\ &\quad + \frac{c_p M^2 e^{2\mu T}q(x)}{1-\gamma} \times\left(\int_s^t\int_0^{t-r}|b(t-r-v)|\,dv\,dr + |z|(t-s)\right). \end{aligned} \tag{87}$$

One can simply prove that there exists $c_T > 0$ such that, for $0 \leq s < t \leq T$,

$$\int_0^s\int_{s-r}^{t-r}|b(v)|\ dv\,dr + \int_s^t\int_0^{t-r}|b(t-r-v)|\ dv\,dr \leq c_T(t-s)^\sigma, \tag{88}$$

which implies with (72), the previous computation and the denseness of $A$ that there exists $b_{T,p} > 0$ such that, for every $x \in E$ and $0 \leq s < t \leq T$:

$$p(R_B(t)x - R_B(s)x) \leq b_{T,p}(t-s)^\sigma \max\left(p(x), h_{T,p}(x), q(x)\right). \tag{89}$$

The same estimate holds provided (70), while in the case of Remark 15 we obtain that, for every $x \in D(A)$ and $0 \leq s < t \leq T$,

$$\frac{p(R_B(t)x - R_B(s)x)}{(t-s)^\sigma} \leq b_{T,p}\max\left(p(x) + p(Ax), h_{T,p}(x) + h_{T,p}(Ax), q(x) + q(Ax)\right). \tag{90}$$

Assuming additionally

$$\sup_{0\leq s<t\leq T}\frac{1}{(t-s)^\sigma}\sum_{j=0}^{l-1}\left(\int_0^{t-s}\left|a^{*,j+1}(r)\right|\ dr + \int_0^s\left|a^{*,j+1}(t-r) - a^{*,j+1}(s-r)\right|\ dr\right) < \infty, \tag{91}$$

then an estimate of the form (89) holds in the case of Theorem 14.

The following corollary is an immediate consequence of Theorems 13–14 and Remark 15.

**Corollary 17.** *Suppose $M$, $M_1 > 0, \omega \geq 0, \alpha > 0, \beta \geq 0$, $A$ is a subgenerator of a $(g_\alpha, g_{\alpha\beta+1})$-regularized $C$-resolvent family $(R(t))_{t\geq 0}$ satisfying $p(R(t)x) \leq Me^{\omega t}p(x), x \in E, t \geq 0, p \in \circledast$ and (2) with $a(t) = g_\alpha(t)$ and $k(t) = g_{\alpha\beta+1}(t)$. Assume exactly one of the following conditions:*

(i) $\alpha - 1 - \alpha\beta \geq 0, BCx = CBx, x \in D(A)$, *and* (a) $\vee$ (b), *where*

(a) $p(C^{-1}Bx) \leq M_1 p(x), x \in \overline{D(A)}, p \in \circledast$.

(b) *E is complete,* (69) *and* ($\natural$).

(ii) $\alpha - 1 - \alpha\beta < 0, BCx = CBx, x \in D(A), l = \lceil(\alpha\beta + 1 - \alpha)/\alpha\rceil$ *and* (73) *holds.*

*Then there exist* $\mu > \omega$ *and* $\gamma \in [0,1)$ *such that* $A + B$ *is a subgenerator of a* $(g_\alpha, g_{\alpha\beta+1})$*-regularized* $C$*-resolvent family* $(R_B(t))_{t\geq 0}$ *satisfying* (71), *and* (18) *with* $k_1(t)$ *replaced by* $g_{\alpha\beta+1}(t)$ *therein.*

*Remark 18.* Let $0 < \alpha < 2$ and let $(R(t))_{t\geq 0}$ be an exponentially equicontinuous, analytic $(g_\alpha, g_{\alpha\beta+1})$-regularized $C$-resolvent family of angle $\delta \in (0, \pi/2]$. Suppose additionally that, for every $\zeta \in (0,\delta)$, there exist $M_\zeta \geq 1$ and $\omega_\zeta \geq 0$ such that $p(R(z)x) \leq M_\zeta e^{\omega_\zeta \Re z} p(x), x \in E, z \in \Sigma_\zeta, p \in \circledast$. If (ii) or (i)(a) holds, then we obtain from Corollary 17 and the proofs of Kato's analyticity criteria [60, Theorems 4.3 and 4.6] that $(R_B(t))_{t\geq 0}$ is also an exponentially equicontinuous, analytic $(g_\alpha, g_{\alpha\beta+1})$-regularized $C$-resolvent family of angle $\delta$; furthermore, for every $\zeta \in (0,\delta)$, there exist $M'_\zeta \geq 1$ and $\omega'_\zeta \geq 0$ such that $p(R_B(z)x) \leq M'_\zeta e^{\omega'_\zeta \Re z} p(x), x \in E, z \in \Sigma_\zeta, p \in \circledast$. If (i)(b) holds, then one has to assume additionally that there exist $\eta > \omega$ and $\gamma \in [0,1)$ such that, for every $\zeta \in (-\delta, \delta), x \in D(A)$ and $p \in \circledast$, the following holds:

$$\int_0^\infty e^{-\mu t} p\left(C^{-1}B\int_0^t b(t-s)\, R\left(se^{i\zeta}\right)x\, ds + zC^{-1}BR\left(te^{i\zeta}\right)x\right) dt \leq \gamma p(x). \tag{92}$$

The question whether perturbations considered in Theorems 13–14 retain analytical properties requires further analysis and will not be discussed in the context of this paper.

*Example 19* (cf. [28, Example 2.24]). Let $E := l^1, 0 < \alpha < 1$ and $l := \lceil(1-\alpha)/\alpha\rceil$. Define a closed densely defined linear operator $A_\alpha$ on $E$ by $D(A_\alpha) := \{\langle x_n\rangle \in l^1 : \sum_{n=1}^\infty n|x_n| < \infty\}$ and $A_\alpha\langle x_n\rangle := \langle e^{i\alpha(\pi/2)} n x_n\rangle, \langle x_n\rangle \in D(A_\alpha)$. Then $A_\alpha$ is the integral generator of a bounded $(g_\alpha, 1)$-regularized resolvent family, $A_\alpha + I$ is not the integral generator of an exponentially bounded $(g_\alpha, 1)$-regularized resolvent family, and $\sigma(A_\alpha) = \{e^{i\alpha(\pi/2)} n : n \in \mathbb{N}\}$. Suppose

$$B \in L(E), \quad R(B) \subseteq D\left(A^l\right) = \left\{\langle x_n\rangle \in l^1 : \sum_{n=1}^\infty n^l |x_n| < \infty\right\}. \tag{93}$$

Then it follows from Corollary 17 that $A + B$ is the integral generator of an exponentially bounded $(g_\alpha, 1)$-regularized resolvent family.

## 3. Unbounded Perturbation Theorems

In the subsequent theorems, we transfer the assertions of [19, Theorems 3.1 and 3.3] and [1, Theorem 2.5.9, Corollary 2.5.10] to abstract Volterra equations.

**Theorem 20.** *Suppose* $E$ *is a Banach space,* $k(t)$ *and* $a(t)$ *satisfy* (P1)-(P2) *and* $A$ *is the integral generator of an exponentially bounded* $(a,k)$*-regularized resolvent family* $(R(t))_{t\geq 0}$ *satisfying* (2) *with* $C = I$. *Let* $M > 0$ *and* $\omega \geq 0$ *be such that* $\|R(t)\| \leq Me^{\omega t}, t \geq 0$ *and let* $\lambda_0 > \max(\omega, \mathrm{abs}(a), \mathrm{abs}(k))$ *satisfy* $\widetilde{k}(\lambda)\widetilde{a}(\lambda) \neq 0, \Re\lambda \geq \lambda_0$. *Suppose that, for every* $\varepsilon > 0$, *there exists* $C_\varepsilon > 0$ *such that*

$$\frac{1}{\left|\widetilde{k}(\lambda)\right|} \leq C_\varepsilon e^{\varepsilon|\lambda|}, \quad \Re\lambda \geq \lambda_0,$$
$$\frac{|\widetilde{a}(\lambda)|}{\left|\widetilde{a}(\lambda_0 + i\Im\lambda)\right|} \leq C_\varepsilon e^{\varepsilon|\lambda|}, \quad \Re\lambda \geq \lambda_0. \tag{94}$$

(i) *Let* $B$ *be a linear operator, let* $D(A) \subseteq D(B)$ *and let*

$$\left\|BR\left(\frac{1}{\widetilde{a}(\lambda)} : A\right)\right\| \leq M_\varrho |\lambda|^{-\varrho}, \quad \Re\lambda = \lambda_0 \tag{95}$$

*for some* $\varrho > 0$ *and* $M_\varrho > 0$ *(for* $\varrho = 0$ *and some* $M_0 \in (0,1)$*). Then, for every* $\zeta > 1$, $A + B$ *is the integral generator of an exponentially bounded,* $(a, k*_0 g_\zeta)$*-regularized resolvent family* $(R_B(t))_{t\geq 0}$ *satisfying* (18) *with* $k_1(t) = (k*_0 g_\zeta)(t), C = I$, *and* $g_\alpha(t)$ *replaced by* $a(t)$ *therein.*

(ii) *Let* $B$ *be a densely defined linear operator and let*

$$\left\|R\left(\frac{1}{\widetilde{a}(\lambda)} : A\right)Bx\right\| \leq M_\varrho |\lambda|^{-\varrho}\|x\|, \quad x \in D(B), \ \Re\lambda = \lambda_0 \tag{96}$$

*for some* $\varrho > 0$ *and* $M_\varrho > 0$*(for* $\varrho = 0$ *and some* $M_0 \in (0,1)$*). Then there exists a closed extension* $D$ *of the operator* $A + B$ *such that, for every* $\zeta > 1$, $D$ *is the integral generator of an exponentially bounded,* $(a, k*_0 g_\zeta)$*-regularized resolvent family* $(R_B(t))_{t\geq 0}$ *satisfying* (18) *with* $k_1(t) = (k*_0 g_\zeta)(t), C = I$, *and* $g_\alpha(t)$ *replaced by* $a(t)$ *therein. Furthermore, if* $A$ *and* $A^*$ *are densely defined, then* $D$ *is the part of the operator* $(A^* + B^*)^*$ *in* $E$.

*Proof.* By Lemma 2, $\{1/\widetilde{a}(\lambda) : \Re\lambda > \lambda_0\} \subseteq \rho(A)$ and

$$\left\|R\left(\frac{1}{\widetilde{a}(\lambda)} : A\right)\right\| \leq \frac{M|\widetilde{a}(\lambda)|}{\left|\widetilde{k}(\lambda)\right|(\Re\lambda - \omega)}, \quad \Re\lambda > \lambda_0. \tag{97}$$

Given $z \in \mathbb{C}$ with $\Re z > \lambda_0$, put $\lambda_z := \lambda_0 + i\Im z$. Then the prescribed assumptions combined with (97) imply

$$\begin{aligned}
&\left\| BR\left(\frac{1}{\widetilde{a}(z)} : A\right)\right\| \\
&= \left\| BR\left(\frac{1}{\widetilde{a}(\lambda_z)} : A\right)\left(I + \left(\frac{1}{\widetilde{a}(\lambda_z)} - \frac{1}{\widetilde{a}(z)}\right) R\left(\frac{1}{\widetilde{a}(z)} : A\right)\right)\right\| \\
&\leq \left\| BR\left(\frac{1}{\widetilde{a}(\lambda_z)} : A\right)\right\| \left[1 + \left|\frac{1}{\widetilde{a}(\lambda_z)} - \frac{1}{\widetilde{a}(z)}\right| \left\| R\left(\frac{1}{\widetilde{a}(z)} : A\right)\right\|\right] \\
&\leq \frac{M_\varrho}{\lambda_0^\varrho}\left[1 + \left|\frac{1}{\widetilde{a}(\lambda_z)} - \frac{1}{\widetilde{a}(z)}\right| \frac{M\,|\widetilde{a}(z)|}{\left|\widetilde{k}(z)\right|(\Re z - \omega)}\right] \\
&\leq \frac{M_\varrho}{\lambda_0^\varrho} + \frac{M_\varrho M}{\lambda_0^\varrho(\lambda_0 - \omega)\left|\widetilde{k}(z)\right|}\left|\frac{\widetilde{a}(z)}{\widetilde{a}(\lambda_z)} - 1\right|.
\end{aligned} \tag{98}$$

Consider now the function $h : \{z \in \mathbb{C} : \Re z \geq 0\} \to L(E)$ defined by $h(z) := z^\varrho BR((1/\widetilde{a}(\lambda_0 + z)) : A), \Re z \geq 0, z \neq 0, h(0) := 0$ if $\varrho > 0$, and $h(0) := BR((1/\widetilde{a}(\lambda_0)) : A)$ if $\varrho = 0$. Then the function $z \mapsto h(z)$ is continuous for $\Re z \geq 0$ and analytic for $\Re z > 0$. Furthermore, $\|h(it)\| \leq M_\varrho, t \in \mathbb{R}$ and, by (94)–(98), one has that, for every $\varepsilon > 0$, there exists $C'_\varepsilon > 0$ such that $\|h(z)\| \leq C'_\varepsilon e^{\varepsilon|z|}$ for all $z \in \mathbb{C}$ with $\Re z \geq 0$. By the Phragmén-Lindelöf type theorems (cf. for instance [39, Theorem 3.9.8]), we get that $\|h(z)\| \leq M_\varrho$ for all $z \in \mathbb{C}$ with $\Re z \geq 0$. This, in turn, implies that there exists $a > \lambda_0$ such that $\|BR(1/\widetilde{a}(\lambda) : A)\| < 1/2, \Re\lambda \geq a$ if $\varrho > 0$, and that $\|BR(1/\widetilde{a}(\lambda) : A)\| < M_0, \Re\lambda \geq a$ if $\varrho = 0$. Therefore, $1/\widetilde{a}(\lambda) \in \rho(A + B), \Re\lambda \geq a$ and there exists $c_\varrho > 0$ such that, for $\Re\lambda \geq a$ :

$$\begin{aligned}
&\left\| \frac{1}{\widetilde{a}(\lambda)} R\left(\frac{1}{\widetilde{a}(\lambda)} : A + B\right)\right\| \\
&\quad = \left\| \frac{1}{\widetilde{a}(\lambda)} R\left(\frac{1}{\widetilde{a}(\lambda)} : A\right)\left(I - BR\left(\frac{1}{\widetilde{a}(\lambda)} : A\right)\right)^{-1}\right\| \leq \frac{c_\rho}{\left|\widetilde{k}(\lambda)\right|}.
\end{aligned} \tag{99}$$

The proof of (i) follows from [32, Theorem 2.7(i), Remark 2.3(v)]. Using [19, Lemma 3.2] and a similar argumentation, we obtain the validity of (ii). □

Recall that a Banach space $E$ has Fourier type $p \in [1, 2]$ if and only if the Fourier transform extends to a bounded linear operator from $L^p(\mathbb{R} : E)$ to $L^q(\mathbb{R} : E)$, where $1/p + 1/q = 1$. Each Banach space $E$ has Fourier type 1, and $E^*$ has the same Fourier type as $E$. A space of the form $L^p(\Omega, \mu)$ has Fourier type $\min(p, p/p-1)$, and there exist examples of nonreflexive Banach spaces which do have nontrivial Fourier type.

**Theorem 21.** *Let $E$ be a Banach space of Fourier type $p \in (1, 2]$.*

(i) *Let the assumptions of Theorem 20(i) hold and let $\zeta > 1/p$. Assume that at least one of the following conditions holds:*

(a) *$A$ and $A^*$ are densely defined, there exist $\lambda'_0 > \lambda_0$ and $\eta > 1$ such that*

$$\begin{aligned}
&\left|\widetilde{k}(\lambda)\right| = O\left(|\lambda|^{\zeta-\eta}\right), \quad \Re\lambda > \lambda'_0, \\
&|\widetilde{a}(\lambda)| = O\left(|\lambda|^{\zeta-\eta}\right), \quad \Re\lambda > \lambda'_0.
\end{aligned} \tag{100}$$

(b) *$A$ is densely defined and $E$ is reflexive.*

(c) *$B(D(A^2)) \subseteq D(A)$ and $BAx = ABx, x \in D(A^2)$.*

*Then $A + B$ is the integral generator of an exponentially bounded, $(a, k*_0 g_\zeta)$-regularized resolvent family $(R_B(t))_{t\geq 0}$ satisfying (18) with $k_1(t) = (k*_0 g_\zeta)(t), C = I$, and $g_\alpha(t)$ replaced by $a(t)$ therein.*

(ii) *Let the assumptions of Theorem 20(ii) hold and let $\zeta > 1/p$. Then there exists a closed extension $D$ of the operator $A + B$ such that $D$ is the integral generator of an exponentially bounded, $(a, k*_0 g_\zeta)$-regularized resolvent family $(R_B(t))_{t\geq 0}$ satisfying (18) with $k_1(t) = (k*_0 g_\zeta)(t), C = I$, and $g_\alpha(t)$ replaced by $a(t)$ therein. Furthermore, if $A$ and $A^*$ are densely defined, then $D$ is the part of the operator $(A^* + B^*)^*$ in $E$.*

*Proof.* Assume that (c) holds. According to (100), $R(1/\widetilde{a}(\lambda) : A)(I - BR(1/\widetilde{a}(\lambda) : A))^{-1} = (I - BR(1/\widetilde{a}(\lambda) : A))^{-1}R(1/\widetilde{a}(\lambda) : A), \Re\lambda \geq a$ and $(1/\widetilde{a}(\lambda))R(1/\widetilde{a}(\lambda) : A + B) = (1/\widetilde{a}(\lambda))(I - BR(1/\widetilde{a}(\lambda) : A))^{-1}R(1/\widetilde{a}(\lambda) : A), \Re\lambda \geq a$. Define

$$\begin{aligned}
R_B(t)\,x &:= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\lambda t}\lambda^{-\zeta}\left[\frac{\widetilde{k}(\lambda)}{\widetilde{a}(\lambda)} R\left(\frac{1}{\widetilde{a}(\lambda)} : A+B\right)x\right] d\lambda, \\
&\qquad x \in E,\ t \geq 0.
\end{aligned} \tag{101}$$

By the first part of the proof of [19, Theorem 3.3], $A + B$ is the integral generator of an exponentially bounded, $(a, k*_0 g_\zeta)$-regularized resolvent family $(R_B(t))_{t\geq 0}$ satisfying (18) with $k_1(t) = (k*_0 g_\zeta)(t), C = I$, and $g_\alpha(t)$ replaced by $a(t)$ therein. The property (18) holds in any particular case considered below and the assertion (ii) is also an immediate consequence of the proof of [19, Theorem 3.3]. Assume now that (b) holds. Then $A^*$ is densely defined and, by [33, Theorem 2.14(ii)], $(R^*(t))_{t\geq 0}$ is an exponentially bounded, $(a, k)$-regularized resolvent family with the integral generator $A^*$. Let $q$ be such that $1/p + 1/q = 1$ and let $J : E \to E^{**}$ denote the canonical embedding of $E$ in its bidual $E^{**}$. Since $E^*$ has Fourier type $p$ and $(1/\widetilde{a}(\lambda))R(1/\widetilde{a}(\lambda) : A+B)^* = (1/\widetilde{a}(\lambda))((I - BR((1/\widetilde{a}(\lambda)) : A))^{-1})^* R((1/\widetilde{a}(\lambda)) : A)^*, \Re\lambda \geq a$, it follows that there exists $c_1 > 0$ such that, for every $x^* \in E^*$ and $r \geq a$,

$$\int_{-\infty}^{\infty}\left\|\frac{\widetilde{k}(r+is)}{\widetilde{a}(r+is)} R\left(\frac{1}{\widetilde{a}(r+is)} : A + B\right)^* x^*\right\|^q ds \leq c_1\|x^*\|^q. \tag{102}$$

Set, for every $x^* \in E^*$ and $t \geq 0$ :

$$R_{B,*}(t)x^* := \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\lambda t}\lambda^{-\zeta}\left[\frac{\widetilde{k}(\lambda)}{\widetilde{a}(\lambda)}R\left(\frac{1}{\widetilde{a}(\lambda)} : A+B\right)^* x^*\right] d\lambda. \tag{103}$$

Then $(R_{B,*}(t))_{t\geq 0} \subseteq L(E^*)$ is strongly continuous, exponentially bounded and

$$\frac{\widetilde{k}(\lambda)}{\widetilde{a}(\lambda)}R\left(\frac{1}{\widetilde{a}(\lambda)} : A+B\right)^* x^* = \lambda^{\zeta}\int_0^{\infty} e^{-\lambda t}R_{B,*}(t)x^*\,dt, \quad \Re\lambda > a, \quad x^* \in E^*. \tag{104}$$

By Lemma 2, $(R_{B,*}(t))_{t\geq 0}$ is an $(a, k*_0 g_\zeta)$-regularized resolvent family with the integral generator $(A+B)^*$. By [33, Theorem 2.14(iii)], it follows that $(R_B(t) \equiv J^{-1}R_{B,*}(t)^*J)_{t\geq 0}$ is an $(a, k*_0 g_\zeta)$-regularized resolvent family with the integral generator $A+B = J^{-1}((A+B)^*)^*J$. We continue the proof by assuming that (a) holds. Using (99)-(100), we easily infer that the improper integral in (101) converges absolutely for $x \in D(A)$ and that

$$\frac{\widetilde{k}(\lambda)}{\widetilde{a}(\lambda)}R\left(\frac{1}{\widetilde{a}(\lambda)} : A+B\right)x = \lambda^{\zeta}\int_0^{\infty} e^{-\lambda t}R_B(t)x\,dt, \quad \Re\lambda > a, \quad x \in D(A). \tag{105}$$

By (104)-(105) and the uniqueness theorem for Laplace transform, we get

$$\langle R_{B,*}(t)x^*, x\rangle = \langle x^*, R_B(t)x\rangle, \quad t \geq 0, \quad x^* \in E^*, \quad x \in D(A) \tag{106}$$

and $R_{B,*}(t)^*J_x = J_{R_B(t)x}$, $t \geq 0$, $x \in D(A)$. Now one can simply prove that $((R_{B,*}(t)^*)_{|E})_{t\geq 0}$ is an exponentially bounded, $(a, k*_0 g_\zeta)$-regularized resolvent family with the integral generator $A+B$. □

*Remark 22.* (i) It is noteworthy that Kaiser and Weis analyzed in [61, Theorem 3.1] an analogue of Theorem 21 for operator semigroups in Hilbert spaces. The question whether the perturbed semigroup $(R_B(t))_{t\geq 0}$ is strongly continuous at $t = 0$ was answered in the affirmative by Batty [62]; here we would like to note that it is not clear in which way one can transfer the assertion of [62, Theorem 1] to abstract Volterra equations.

(ii) To the author's knowledge, the denseness of $D(A^*)$ in $E^*$ cannot be so simply dropped from the formulation of (a). The main problem is that we do not know whether the mapping $t \mapsto R(t)^*x^*$, $t \geq 0$ is measurable provided $x^* \in E^* \setminus \overline{D(A^*)}$ (cf. [19, (5)-(6), page 221; l. 7-8, page 222] and [63, Section 3]). Notice also that the assertion (c), although practically irrelevant, may help one to better understand the proof of [19, Theorem 3.3].

(iii) Let $\alpha > 0$ and $a(t) = g_\alpha(t)$. Then the assumptions of Theorems 20 and 21[(i)(b)-(c), (ii)] hold while the assumptions of Theorem 21(i)(a) hold provided $\zeta + \alpha > 1$.

In the following nontrivial example, we will transfer the assertion of [19, Proposition 8.1] to abstract time-fractional equations.

*Example 23.* Let $1 < p < \infty$, $1/p + 1/q = 1$, $k \in \mathbb{N}_0$, $0 < \beta \leq 2$ and $E := L^p(\mathbb{R})$. Define a closed linear operator $A_{\beta,k}$ on $E$ by $D(A_{\beta,k}) := W^{4k+2,p}(\mathbb{R})$ and $A_{\beta,k}f := e^{i(2-\beta)(\pi/2)}f^{(4k+2)}$, $f \in D(A_{\beta,k})$. Put $Bf(x) := V(x)f^{(l)}(x)$, $x \in \mathbb{R}$ with maximal domain $D(B) := \{f \in E : V\cdot f^{(l)} \in E\}$; here $V(x)$ is a potential and $l \in \mathbb{N}_0$. Assume first that

$$V \in L^p(\mathbb{R}), \qquad l \leq \frac{1}{p}\left((4k+2)p - 1 - \frac{(4k+2)(p-1)}{\beta}\right). \tag{107}$$

Given $\Re\lambda > 0$, denote by $\mu_{j,\lambda}$ $(1 \leq j \leq 2k+1)(2k+1)$ solutions of the equation $\mu_{j,\lambda}^{4k+2} = \lambda^{\beta}e^{i(\beta(\pi/2)-\pi)}$ with $\Re\mu_{j,\lambda} > 0$. Then $D(A) \subseteq D(B)$,

$$\left(R\left(\lambda^{\beta} : A_{\beta,k}\right)f\right)(x) = \frac{e^{i\beta(\pi/2)}}{4k+2}\int_{-\infty}^{\infty}\sum_{j=1}^{2k+1}\frac{e^{-\mu_{j,\lambda}|x-s|}}{\left(-\mu_{j,\lambda}\right)^{4k+1}}f(s)\,ds, \tag{108}$$

provided $f \in E$, $x \in \mathbb{R}$, $\Re\lambda > 0$,

$$\left(BR\left(\lambda^{\beta} : A_{\beta,k}\right)f\right)(x) = \frac{e^{i\beta(\pi/2)}}{4k+2}V(x) \times \sum_{j=1}^{2k+1}\int_{-\infty}^{x}\left(\frac{e^{-\mu_{j,\lambda}(x-s)}}{\left(-\mu_{j,\lambda}\right)^{4k-l+1}}f(s)\,ds - \int_x^{\infty}\frac{e^{\mu_{j,\lambda}(x-s)}}{\mu_{j,\lambda}^{4k-l+1}}f(s)\,ds\right), \quad f \in E, \quad x \in \mathbb{R}, \quad \Re\lambda > 0,$$

$$\left\|R\left(\lambda^{\beta} : A_{\beta,k}\right)\right\| \leq \left(|\lambda|^{\beta(1-(1/(4k+2)))}\min\left(\Re\mu_{1,\lambda}, \ldots, \Re\mu_{2k+1,\lambda}\right)\right)^{-1}, \quad \Re\lambda > 0,$$

$$\left\|BR\left(\lambda^{\beta} : A_{\beta,k}\right)\right\| \leq \|V\|_p\left(|\lambda|^{\beta(1-((l+1)/(4k+2)))} \times \min\left(\left(\Re\mu_{1,\lambda}\right)^{1/q}, \ldots, \left(\Re\mu_{2k+1,\lambda}\right)^{1/q}\right)\right)^{-1}, \tag{109}$$

provided $\Re\lambda > 0$. Furthermore, $\Re\mu_{j,\lambda} = |\lambda|^{\beta/(4k+2)}\cos(\arg(\mu_{j,\lambda}))$, $\Re\lambda > 0$, $1 \le j \le 2k+1$, and

$$\begin{aligned}\min\left\{\Re\mu_{j,\lambda} : 1 \le j \le 2k+1\right\} &= |\lambda|^{\beta/(4k+2)} \\ &\times \min\left(\cos\left(\frac{\arg(\lambda)\beta + (\beta\pi)/(2)}{4k+2} + \frac{(2k-1)\pi}{4k+2}\right),\right. \\ &\left.\quad -\cos\left(\frac{\arg(\lambda)\beta + (\beta\pi)/2}{4k+2} + \frac{\pi}{2}\right)\right),\end{aligned} \tag{110}$$

provided $\Re\lambda > 0$. The above implies that there exists a constant $c_{\beta,k} > 0$ such that

$$|\lambda|^{\beta/(4k+2)}\cos(\arg(\lambda)) / \min(\Re\mu_{1,\lambda}, \ldots, \Re\mu_{2k+1,\lambda}) \le c_{\beta,k}, \quad \Re\lambda > 0. \tag{111}$$

Keeping in mind (107)-(111), we obtain that

$$\left\|R\left(\lambda^\beta : A_{\beta,k}\right)\right\| = O\left(|\lambda|^{1-\beta}(\Re\lambda)^{-1}\right), \quad \Re\lambda > 0 \tag{112}$$

$$\begin{aligned}&\left\|BR\left(\lambda^\beta : A_{\beta,k}\right)\right\| \\ &\quad = O\left(\|V\|_p(\Re\lambda)^{-\beta(1-((l+1)/(4k+2))+(1/(4k+2)q))}\right) \\ &\quad = O\left(\|V\|_p(\Re\lambda)^{(-1)/q}\right),\end{aligned} \tag{113}$$

provided $\Re\lambda > 0$. Denote by $\beta_k$ the infimum of all nonnegative real numbers $r \ge 0$ such that the operator $A_{\beta,k}$ generates an exponentially bounded $(g_\beta, g_{r+1})$-regularized resolvent family. The precise computation of integration rate $\beta_k$ falls out from the framework of this paper (cf. also the representation formula [28, Example 3.7, (3.15)] and notice that it is not clear whether Theorem 13 or Remark 15 can be applied in case $\beta \in (1,2]$). Clearly, (112) yields the imprecise estimate $\beta_k \le 1$; furthermore, $\beta_k = 0$ provided $p = 2$ ([24]), and $\beta_k \le |(1/2) - (1/p)|$ provided $\beta \in \{1,2\}$ [14, 50]. Set $\kappa_p := \min(1/p, (p-1)/p)$. By Theorem 21, $A_{\beta,k} + B$ generates an exponentially bounded $(g_\beta, g_{\sigma_{\beta,k,p}+1})$-regularized resolvent family for any $\sigma_{\beta,k,p} > \beta_k + \kappa_p$. By (112)-(113) and the proof of [19, Proposition 8.1], the above remains true provided $(4k+2)p - 1 - ((4k+2)(p-1)/\beta) \ge 0, l = 0$ and $V \in L^p(\mathbb{R}) + L^\infty(\mathbb{R})$; similarly, one can consider the operators $A^1_{\beta,k}(k \in \mathbb{N}, 0 < \beta \le 2)$ and $A^2_{\beta,k}(k \in \mathbb{N}, 0 < \beta \le 1)$ given by $A^1_{\beta,k}f := e^{-i\beta(\pi/2)}f^{(4k)}, f \in W^{4k,p}(\mathbb{R}) := D(A^1_{\beta,k})$ and $A^2_{\beta,k}f := e^{\pm i(\pi/2)(1-\beta)}f^{(2k+1)}, f \in W^{2k+1,p}(\mathbb{R}) := D(A^2_{\beta,k})$.

Notice that Lizama and Prado have recently analyzed in [21] the qualitative properties of the abstract relaxation equation:

$$\begin{aligned}&u'(t) - A\mathbf{D}_t^\alpha u(t) + u(t) = f(t), \\ &\alpha \in (0,1), \quad t \ge 0, \quad u(0) = 0,\end{aligned} \tag{114}$$

where $E$ is a Banach space and $f \in L^1_{\mathrm{loc}}([0,\infty) : E)$. By a (strong) solution of (114) we mean any function $u \in C^1([0,\infty) : E)$ such that (114) holds for a.e. $t \ge 0$. The following extension of [28, Theorem 2.25] (cf. also [10, page 65]) will be helpful in the study of perturbation properties of (114).

**Theorem 24.** *Let $k(t)$ and $a(t)$ satisfy* (P1). *Suppose $\delta \in (0, \pi/2]$, $\omega \ge \max(0, \mathrm{abs}(a), \mathrm{abs}(k))$, there exist analytic functions $\widehat{k} : \omega + \Sigma_{(\pi/2)+\delta} \to \mathbb{C}$ and $\widehat{a} : \omega + \Sigma_{(\pi/2)+\delta} \to \mathbb{C}$ such that $\widehat{k}(\lambda) = \widetilde{k}(\lambda), \Re\lambda > \omega, \widehat{a}(\lambda) = \widetilde{a}(\lambda), \Re\lambda > \omega$ and $\widehat{k}(\lambda)\widehat{a}(\lambda) \ne 0, \lambda \in \omega + \Sigma_{(\pi/2)+\delta}$. Let $A$ be a subgenerator of an analytic $(a,k)$-regularized $C$-resolvent family $(R(t))_{t\ge0}$ of angle $\delta$ and let* (2) *hold. Suppose that, for every $\eta \in (0,\delta)$, there exists $c_\eta > 0$ such that*

$$p\left(e^{-\omega\Re z}R(z)x\right) \le c_\eta p(x), \quad x \in E, \ p \in \circledast, \ z \in \Sigma_\eta \tag{115}$$

*as well as $b, c \ge 0, B$ is a linear operator satisfying $D(C^{-1}AC) \subseteq D(B), BCx = CBx, x \in D(C^{-1}AC)$ and*

$$\begin{aligned}&p\left(C^{-1}Bx\right) \le bp\left(C^{-1}ACx\right) + cp(x), \\ &x \in D\left(C^{-1}AC\right), \quad p \in \circledast.\end{aligned} \tag{116}$$

*Assume that at least one of the following conditions holds:*

(i) *$A$ is densely defined, the numbers $b$ and $c$ are sufficiently small, there exists $|C|_\circledast > 0$ such that $p(Cx) \le |C|_\circledast p(x), x \in E, \ p \in \circledast$ and, for every $\eta \in (0,\delta)$, there exists $\omega_\eta \ge \omega$ such that $|\widehat{k}(\lambda)^{-1}| = O(|\lambda|), \lambda \in \omega_\eta + \Sigma_{(\pi/2)+\eta}$ and $|\widehat{a}(\lambda)/\widehat{k}(\lambda)| = O(|\lambda|), \lambda \in \omega_\eta + \Sigma_{(\pi/2)+\eta}$.*

(ii) *$A$ is densely defined, the number $b$ is sufficiently small, there exists $|C|_\circledast > 0$ such that $p(Cx) \le |C|_\circledast p(x), x \in E, p \in \circledast$ and, for every $\eta \in (0,\delta)$, there exists $\omega_\eta \ge \omega$ such that $|\widehat{k}(\lambda)^{-1}| = O(|\lambda|), \lambda \in \omega_\eta + \Sigma_{(\pi/2)+\eta}$ and $\widehat{a}(\lambda)/(\lambda\widehat{k}(\lambda)) \to 0, |\lambda| \to \infty, \lambda \in \omega_\eta + \Sigma_{(\pi/2)+\eta}$.*

(iii) *$A$ is densely defined, the number $c$ is sufficiently small, $b = 0$ and, for every $\eta \in (0,\delta)$, there exists $\omega_\eta \ge \omega$ such that $|\widehat{a}(\lambda)/\widehat{k}(\lambda)| = O(|\lambda|), \lambda \in \omega_\eta + \Sigma_{(\pi/2)+\eta}$.*

(iv) *$b = 0$ and, for every $\eta \in (0,\delta)$, there exists $\omega_\eta \ge \omega$ such that $\widehat{a}(\lambda)/(\lambda\widehat{k}(\lambda)) \to 0, |\lambda| \to \infty, \lambda \in \omega_\eta + \Sigma_{(\pi/2)+\eta}$.*

*Then $C^{-1}(C^{-1}AC + B)C$ is a subgenerator of an exponentially equicontinuous, analytic $(a,k)$-regularized $C$-resolvent family $(R_B(t))_{t\ge0}$ of angle $\delta$, which satisfies $R_B(z)[C^{-1}(C^{-1}AC + B)C] \subseteq [C^{-1}(C^{-1}AC + B)C]R_B(z), z \in \Sigma_\delta$ and the following condition:*

$$\begin{aligned}&\forall\eta \in (0,\delta) \quad \exists\omega'_\eta > 0 \ \exists M_\eta > 0 \quad \forall p \in \circledast : p\left(R_B(z)x\right) \\ &\quad \le M_\eta e^{\omega'_\eta\Re z}p(x), \quad x \in E, \ z \in \Sigma_\eta.\end{aligned} \tag{117}$$

*Furthermore, in cases* (iii) *and* (iv), *the above remains true with the operator $C^{-1}(C^{-1}AC + B)C$ replaced by $C^{-1}AC + B$.*

*Proof.* First of all, notice that the closedness of the operator $C^{-1}AC+B$ in cases (iii) or (iv) trivially follows and that it is not clear how one can prove that the operator $C^{-1}AC+B$ is closed in cases (i) or (ii). We will only prove the assertion provided that (i) holds and remark the minor modifications in case that (iv) holds. Let $\eta \in (0,\delta)$ and $\sigma \in (0,1)$. Clearly, $A \subseteq C^{-1}AC$, $C[C^{-1}AC] \subseteq [C^{-1}AC]C$, $C[C^{-1}AC+B] \subseteq [C^{-1}AC+B]C$, $C[C^{-1}(C^{-1}AC+B)C] \subseteq [C^{-1}(C^{-1}AC+B)C]C$ and $C^{-1}AC+B \subseteq C^{-1}(C^{-1}AC+B)C$. Invoking (115), [33, Theorem 3.6] and the proof of [39, Theorem 2.6.1], we obtain that

$$\lim_{\lambda\to+\infty} \lambda \frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}\left(\frac{1}{\widehat{a}(\lambda)} - C^{-1}AC\right)^{-1} Cx = k(0)\,Cx, \quad x \in E \tag{118}$$

and that there exists $N_\eta > 0$ such that $\{(1/\widehat{a}(\lambda)) : \lambda \in \omega + \Sigma_{(\pi/2)+\eta}\} \subseteq \rho_C(C^{-1}AC)$ and

$$\begin{aligned}\sup_{\lambda\in\omega+\Sigma_{(\pi/2)+\eta}} & p\left((\lambda-\omega)\frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}\left(\frac{1}{\widehat{a}(\lambda)} - C^{-1}AC\right)^{-1} Cx\right) \\ & \le N_\eta p(x), \quad x \in E.\end{aligned} \tag{119}$$

By (116) and (119), we infer that, for every $\lambda \in \omega_\eta + \Sigma_{(\pi/2)+\eta}$, $x \in E$ and $p \in \circledast$:

$$\begin{aligned} & p\left(C^{-1}B\left(\frac{1}{\widehat{a}(\lambda)} - C^{-1}AC\right)^{-1} Cx\right) \\ & \quad \le b|C|_\circledast p(x) + b\frac{N_\eta c_\eta p(x)}{|\lambda-\omega|\left|\widetilde{k}(\lambda)\right|} + cc_\eta \frac{N_\eta p(x)}{|\lambda-\omega|}\frac{|\widehat{a}(\lambda)|}{\left|\widehat{k}(\lambda)\right|},\end{aligned} \tag{120}$$

which implies by the given assumption the existence of a number $\omega'_\eta > \omega_\eta$ such that $p(C^{-1}B((1/\widehat{a}(\lambda))-C^{-1}AC)^{-1}Cx) \le \sigma p(x)$, $x \in E$, $\lambda \in \omega'_\eta + \Sigma_{(\pi/2)+\eta}$, $p \in \circledast$, provided that the numbers $b$ and $c$ are sufficiently small; if (iv) holds, then

$$\lim_{\lambda\to+\infty} C^{-1}B\left(\frac{1}{\widehat{a}(\lambda)} - C^{-1}AC\right)^{-1} Cx = 0, \quad x \in E. \tag{121}$$

Using the same argument as in the proof of Theorem 14, it follows that, for every $\lambda \in \omega'_\eta + \Sigma_{(\pi/2)+\eta}$, $R(C) \subseteq R((1/\widehat{a}(\lambda)) - (C^{-1}AC+B)) \subseteq R((1/\widehat{a}(\lambda)) - C^{-1}(C^{-1}AC+B)C)$ as well as that the operators $(1/\widehat{a}(\lambda)) - (C^{-1}AC+B)$ and $(1/\widehat{a}(\lambda)) - C^{-1}(C^{-1}AC+B)C$ are injective. Moreover, for any $\lambda \in \omega'_\eta + \Sigma_{(\pi/2)+\eta}$:

$$\begin{aligned} & \left(\frac{1}{\widehat{a}(\lambda)} - \left(C^{-1}AC+B\right)\right)^{-1} C \\ & \quad = \left(\frac{1}{\widehat{a}(\lambda)} - C^{-1}\left(C^{-1}AC+B\right)C\right)^{-1} C \\ & \quad = \left(\frac{1}{\widehat{a}(\lambda)} - C^{-1}AC\right)^{-1} \\ & \qquad \times C\left(I - C^{-1}B\left(\frac{1}{\widehat{a}(\lambda)} - C^{-1}AC\right)^{-1} C\right)^{-1}.\end{aligned} \tag{122}$$

Now we will prove that the operator $C^{-1}(C^{-1}AC+B)C$ is closed. Let $(x_\tau)_{\tau\in T}$ be a net in $E$ satisfying $x_\tau \to x$, $\tau \to \infty$ and $C^{-1}(C^{-1}AC+B)Cx_\tau \to y$, $\tau \to \infty$. Then $((1/\widetilde{a}(\lambda)) - (C^{-1}AC+B))^{-1}CC^{-1}(C^{-1}AC+B)Cx_\tau \to ((1/\widetilde{a}(\lambda)) - (C^{-1}AC+B))^{-1}Cy$, $\tau \to \infty$, that is, $-Cx_\tau + (1/\widetilde{a}(\lambda))((1/\widetilde{a}(\lambda)) - (C^{-1}AC+B))^{-1}Cx_\tau \to ((1/\widetilde{a}(\lambda)) - (C^{-1}AC+B))^{-1}Cy$, $\tau \to \infty$, which simply implies $Cx \in D(C^{-1}AC+B)$ and $(C^{-1}AC+B)Cx = Cy$. Therefore, $x \in D(C^{-1}(C^{-1}AC+B)C)$, $C^{-1}(C^{-1}AC+B)Cx = y$ and $C^{-1}(C^{-1}AC+B)C$ is closed, as required. Notice that, for every $x \in E$, the analyticity of mapping

$$\begin{aligned} \lambda \longmapsto & \left(\frac{1}{\widehat{a}(\lambda)} - C^{-1}\left(C^{-1}AC+B\right)C\right)^{-1} Cx \\ & = \left(\frac{1}{\widehat{a}(\lambda)} - C^{-1}AC\right)^{-1} C \\ & \quad \times \sum_{n=0}^{\infty}\left[C^{-1}B\left(\frac{1}{\widehat{a}(\lambda)} - C^{-1}AC\right)^{-1} C\right]^n x, \\ & \qquad \lambda \in \omega'_\eta + \Sigma_{(\pi/2)+\eta}\end{aligned} \tag{123}$$

follows from [33, Lemma 3.3] and the fact that an $E$-valued mapping is analytic if and only if it is weakly analytic. By [33, Theorem 3.7], $C^{-1}(C^{-1}AC+B)C$ is a subgenerator of an exponentially equicontinuous, analytic $(a,k)$-regularized $C$-resolvent family $(R_B(t))_{t\ge0}$ of angle $\eta$ and (117) holds; assuming (iv), we get from (119)

$$\begin{aligned} & p\left(\left(\lambda-\omega'_\eta\right)\frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}\left(\frac{1}{\widehat{a}(\lambda)} - \left(C^{-1}AC+B\right)\right)^{-1} Cx\right) \\ & \quad = p\left(\frac{\left(\lambda-\omega'_\eta\right)}{(\lambda-\omega)}(\lambda-\omega)\frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}\left(\frac{1}{\widehat{a}(\lambda)} - C^{-1}AC\right)^{-1} C\right. \\ & \qquad \left.\times\left(I - C^{-1}B\left(\frac{1}{\widehat{a}(\lambda)} - C^{-1}AC\right)^{-1} C\right)^{-1} x\right) \\ & \quad \le \left(1+\frac{1}{\cos\eta}\right)\frac{N_\eta}{1-\sigma}p(x), \quad x \in E, \quad \lambda \in \omega'_\eta + \Sigma_{(\pi/2)+\eta}.\end{aligned} \tag{124}$$

In combination with (118) and (121), the above implies

$$\begin{aligned} & \lim_{\lambda\to+\infty} \lambda\frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}\left(\frac{1}{\widehat{a}(\lambda)} - \left(C^{-1}AC+B\right)\right)^{-1} Cx \\ & \quad = \lim_{\lambda\to+\infty} \lambda\frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}\left(\frac{1}{\widehat{a}(\lambda)} - C^{-1}AC\right)^{-1} Cx \\ & \qquad + \lim_{\lambda\to+\infty} \lambda\frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}\left(\frac{1}{\widehat{a}(\lambda)} - C^{-1}AC\right)^{-1} C \\ & \qquad \times C^{-1}B\left(\frac{1}{\widehat{a}(\lambda)} - C^{-1}AC\right)^{-1} \\ & \qquad \times C\left(I - C^{-1}B\left(\frac{1}{\widehat{a}(\lambda)} - C^{-1}AC\right)^{-1} C\right)^{-1} x \\ & \quad = k(0)\,Cx + 0 = k(0)\,Cx, \quad x \in E,\end{aligned} \tag{125}$$

and the proof follows again from an application of [33, Theorem 3.7]. □

*Remark 25.* Using the proof of [33, Theorem 3.7], we get that there exists $\omega_0 > 0$ such that, for every $x \in E$ and for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_0$ :

$$\begin{aligned}\widehat{k}(\lambda)\left(I-\widehat{a}(\lambda)\left(C^{-1}AC+B\right)\right)^{-1}Cx &= \widehat{k}(\lambda)\left(I-\widehat{a}(\lambda)C^{-1}\left(C^{-1}AC+B\right)C\right)^{-1}Cx \\ &= \int_0^\infty e^{-\lambda t}R_B(t)\,x\,dt.\end{aligned} \tag{126}$$

By Lemma 2, we obtain that (18) holds with $A+B$, $k_1(t)$ and $g_\alpha(t)$ replaced, respectively, by $C^{-1}(C^{-1}AC+B)C$, $k(t)$ and $a(t)$ therein; clearly, the above assertion remains true with the operator $C^{-1}(C^{-1}AC+B)C$ replaced by $C^{-1}AC+B$, provided that (iii) or (iv) holds. Taking the Laplace transform, (126) simply implies that $C^{-1}(C^{-1}AC+B)C$ is, in fact, the integral generator of $(R_B(t))_{t\geq 0}$.

*Example 26.* Let $u(t)$ be a solution of (114). Set $a_\alpha(t) := \mathscr{L}^{-1}(\lambda^\alpha/(\lambda+1))(t)$, $t \geq 0$, $k_\alpha(t) := e^{-t}$, $t \geq 0$, and $v(t) = u(t) + (1 * u)(t)$, $t \geq 0$. Then $u(t) = v(t) - (e^{-t} * v)(t)$, $t \geq 0$ and $v(t) = A(a_\alpha * v)(t) + (1 * f)(t)$, $t \geq 0$, which implies that the notion of an $(a_\alpha, k)$-regularized $C$-resolvent family is important in the study of (114). In [21], the authors mainly use the following conditions: $k(t) = k_\alpha(t)$, $C = I$ and $A$ is the generator of a bounded analytic $C_0$-semigroup. Set $\delta := \min(\pi/2, \pi\alpha/(2(1-\alpha)))$ and assume, more generally, that for every $\eta \in (0, ((\pi/2)+\delta)(1-\alpha))$, there exists $\omega_\eta > 0$ such that the family

$$\left\{(1+|\lambda|)^{1-r}(\lambda-A)^{-1}C : \lambda \in \omega_\eta + \Sigma_\eta\right\} \tag{127}$$

is equicontinuous ($r \geq 0$) and that the mapping

$$\begin{gathered}\lambda \longmapsto (\lambda-A)^{-1}Cx, \qquad \lambda \in \omega_\eta + \Sigma_\eta \\ \text{is continuous for every fixed } x \in E.\end{gathered} \tag{128}$$

Notice that (127)-(128) hold provided that $A$ is a subgenerator of an exponentially equicontinuous $r$-times integrated $C$-semigroup $(R_r(t))_{t\geq 0}$; furthermore, if

$$\begin{gathered}\exists M \geq 1 \quad \exists \omega \geq 0 : p\left(R_r(t)x\right) \leq Me^{\omega t}p(x), \\ x \in E, \quad p \in \circledast,\end{gathered} \tag{129}$$

then, for every $\eta \in (0, \pi/2)$ and $\omega_\eta > \omega$, there exists $M_\eta > 0$ such that

$$\begin{gathered}p\left((\lambda-A)^{-1}Cx\right) \leq M_\eta(1+|\lambda|)^{r-1}p(x), \\ x \in E, \quad \lambda \in \omega_\eta + \Sigma_\eta, \quad p \in \circledast.\end{gathered} \tag{130}$$

We refer the reader to [58, Chapter 1] for examples of differential operators generating exponentially equicontinuous, $r$-times integrated $C$-semigroups satisfying (129). Assume, further, that there exist $\omega > \max(0, \operatorname{abs}(k))$ and an analytic function $\widehat{k} : \omega + \Sigma_{(\pi/2)+\delta} \to \mathbb{C}$ such that $\widehat{k}(\lambda) = \widetilde{k}(\lambda)$, $\Re\lambda > \omega$, $\widehat{k}(\lambda) \neq 0$, $\lambda \in \omega + \Sigma_{(\pi/2)+\delta}$ and $|\widehat{k}(\lambda)| = O(|\lambda|^{-1})$, $\lambda \in \omega + \Sigma_{(\pi/2)+\delta}$. Let $\gamma \in (0,\delta)$ and let $((\pi/2)+\gamma)(1-\alpha) < \eta < \pi/2$. Then there exists a sufficiently large $\omega'_\gamma > \omega$ such that $(\lambda+1)/\lambda^\alpha = \lambda^{1-\alpha} + \lambda^{-\alpha} \in \omega_\eta + \Sigma_\eta$ for all $\lambda \in \omega'_\gamma + \Sigma_{(\pi/2)+\gamma}$, which implies with (127)-(128) and [33, Proposition 2.16(iii)] that the mapping $\lambda \mapsto (\widehat{k}(\lambda)/\widehat{a}(\lambda))((1/\widehat{a}(\lambda)) - A)^{-1}Cx, \lambda \in \omega'_\gamma + \Sigma_{(\pi/2)+\gamma}$ is analytic ($x \in E$) and that, for every $\sigma \geq r(1-\alpha)$, the family $\{(\lambda - \omega'_\gamma)(\widehat{k}(\lambda)/|\lambda|^\sigma)(1/\widehat{a}(\lambda))((1/\widehat{a}(\lambda)) - A)^{-1}C : \lambda \in \omega'_\gamma + \Sigma_{(\pi/2)+\gamma}\}$ is equicontinuous (if (129) holds, then there exists $N_\gamma > 0$ such that $p((\lambda-\omega'_\gamma)(\widehat{k}(\lambda)/|\lambda|^\sigma)(1/\widehat{a}(\lambda))((1/\widehat{a}(\lambda))-A)^{-1}Cx) \leq N_\gamma p(x), x \in E, p \in \circledast, \lambda \in \Sigma_\eta$). Using [33, Theorem 3.7] and the arbitrariness of $\eta$, we get that $A$ is a subgenerator of an exponentially equicontinuous, analytic $(a_\alpha, k * g_\zeta)$-regularized $C$-resolvent family $(R(t))_{t\geq 0}$ of angle $\delta$, where

$$\zeta = \begin{cases} r(1-\alpha), & \text{if } \overline{D(A)} = E \\ > r(1-\alpha), & \text{if } \overline{D(A)} \neq E \end{cases} \tag{131}$$

and $g_0(t)$ stands for the Dirac distribution (if (129) holds, then for every $\eta \in (0,\delta)$ there exist $\omega_\eta > 0$ and $L_\eta > 0$ such that $p(R(z)x) \leq L_\eta e^{\omega_\eta \Re z}p(x), x \in E, p \in \circledast$). This is a significant improvement of [21, Theorem 3.1]. In what follows, we will provide the basic information on the $C$-well-posedness of (114). Given $\beta \in (0,1)$ and $T > 0$, set

$$\begin{aligned}&C^0_\beta([0,T]:E) \\ &\quad := \left\{f \in C([0,T]:E) : f(0) = 0, |f|_{\beta,T,p} < \infty, \ \forall p \in \circledast\right\},\end{aligned} \tag{132}$$

where

$$|f|_{\beta,T,p} := \sup_{0\leq s<t\leq T} \frac{p\left(f(t)-f(s)\right)}{(t-s)^\beta}. \tag{133}$$

Let $A$ be densely defined, let $r = 0$ and let $\beta \in (0,1)$ be such that $C^{-1}(1 * f_{|[0,T]}) \in C^0_\beta([0,T] : E)$ for all $T > 0$. Then $\zeta = 0$, and the proof of [10, Theorem 2.4] combined with the Cauchy integral formula (cf. also [33, Section 1, Theorem 3.4(i)]) indicates that the function

$$\begin{aligned}v(t) &= R(t)C^{-1}(1*f)(t) \\ &\quad + \int_0^t R'(t-s)\left(C^{-1}(1*f)(s) - C^{-1}(1*f)(t)\right)ds, \\ &\qquad t \geq 0\end{aligned} \tag{134}$$

satisfies $A(a_\alpha * v)(t) = v(t) - (1*f)(t)$, $t \geq 0$ and that, for every $T > 0$, one has $v_{|[0,T]} \in C^0_\beta([0,T] : E)$; in the above formula, we assume that $(R(t))_{t\geq 0}$ is the exponentially equicontinuous, analytic $(a_\alpha, C)$-regularized resolvent family of angle $\delta$. It is obvious that the function $t \mapsto u(t) = v(t) - (e^{-t} * v)(t)$, $t \geq 0$ is a unique function satisfying (114) in integrated form

$$\begin{gathered}u(t) - A\left(g_{1-\alpha} * u\right)(t) + (1*u)(t) = (1*f)(t), \\ t \geq 0, \quad u(0) = 0\end{gathered} \tag{135}$$

and that $u_{|[0,T]} \in C^0_\beta([0,T] : E)$ for all $T > 0$. If $x \in E$, $x \neq 0$ and $C^{-1}(1 * (f(\cdot) - x)_{|[0,T]}) \in C^0_\beta([0,T] : E)$ for all $T > 0$, then we obtain similarly the unique solution $u(t)$ of the problem

$$\begin{aligned} u(t) - x - A\left(g_{1-\alpha} * (u(\cdot) - x)\right)(t) + (1 * u)(t) \\ = \left(1 * f\right)(t), \quad t \geq 0, \ u(0) = x; \end{aligned} \tag{136}$$

furthermore, $u_{|[0,T]} \in C^0_\beta([0,T] : E)$ for all $T > 0$. Since $a_\alpha \notin BV_{\mathrm{loc}}([0,\infty))$, the above-described method does not work in the case $r > 0$ (cf. [35, Corollary 2.11] and [33, Theorem 2.6(i)]).

We are turning back to the case in which $A$ is not necessarily densely defined. Let $C^{-1}f, AC^{-1}f \in L^1_{\mathrm{loc}}([0,\infty) : E)$ and let $(R_\alpha(t))_{t\geq 0}$ denote the $(a_\alpha, k_\alpha * g_\zeta)$-regularized $C$-resolvent family with a subgenerator $A$. By the proofs of [21, Theorem 3.5, Corollary 3.6], it follows that, for every $x \in R(C)$, there exists a unique solution of the problem

$$\begin{aligned} u(t) - A\left(g_{(1-\alpha)} * u\right)(t) + (1 * u)(t) \\ = \left(1 * f * g_\zeta\right)(t) + g_{\zeta+1}(t)\, x, \quad t \geq 0, \end{aligned} \tag{137}$$

given by $t \mapsto u(t) = R_\alpha(t)C^{-1}x + \int_0^t R_\alpha(t-s)C^{-1}f(s)\, ds, t \geq 0$. Only after assuming some additional conditions, one can differentiate the formulae (135)–(137), obtaining in such a way (114) or its slight modification. Now we are interested in the perturbation properties of (114). Assume $r \in [0,1]$ and $A$ is a subgenerator of an exponentially equicontinuous, $r$-times integrated $C$-semigroup satisfying (129). Let $B$ be a linear operator such that $D(A) \subseteq D(B), BCx = CBx, x \in D(A)$ and let $b$, $c \geq 0$ satisfy $p(C^{-1}Bx) \leq bp(Ax) + cp(x), x \in D(A), p \in \circledast$. By Remark 25 and the proof of Theorem 24, we have the following

(i) If $r = \zeta = 0, b$ is sufficiently small and $|C|_\circledast > 0$ satisfies $p(Cx) \leq |C|_\circledast p(x), x \in E, p \in \circledast$, then $C^{-1}(A+B)C$ is the integral generator of an exponentially equicontinuous, analytic $(a_\alpha, k)$-regularized $C$-resolvent family $(R_B(t))_{t\geq 0}$ of angle $\delta$ (cf. [64, Chapter III] and [65, Chapter 7] for corresponding examples).

(ii) If $b = 0$, $c$ is sufficiently small, $r = 1$, and $\zeta = 1 - \alpha$, then $A+B$, respectively, $C^{-1}(A+B)C$, is a subgenerator, respectively the integral generator, of an exponentially equicontinuous, analytic $(a_\alpha, k * g_\zeta)$-regularized $C$-resolvent family $(R_B(t))_{t\geq 0}$ of angle $\delta$.

(iii) If $b = 0, 0 \leq r < 1$, and $\zeta \geq r(1-\alpha)$, then $A + B$, respectively $C^{-1}(A+B)C$, is a subgenerator, respectively the integral generator, of an exponentially equicontinuous, analytic $(a_\alpha, k * g_\zeta)$-regularized $C$-resolvent family $(R_B(t))_{t\geq 0}$ of angle $\delta$.

We continue this example by observing that Karczewska and Lizama [20] have recently analyzed the following stochastic fractional oscillation equation:

$$u(t) + \int_0^t (t-s)\left[A\mathbf{D}_s^\alpha u(s) + u(s)\right] ds = W(t), \quad t > 0, \tag{138}$$

where $1 < \alpha < 2$, $A$ is the generator of a bounded analytic $C_0$-semigroup on a Hilbert space $H$ and $W(t)$ denotes an $H$-valued Wiener process defined on a stochastic basis $(\Omega, \mathcal{F}, P)$. The theory of $(a,k)$-regularized resolvent families (cf. [20, Theorems 3.1 and 3.2]) is essentially applied in the study of deterministic counterpart of (138) in integrated form

$$\begin{aligned} u(t) + \int_0^t g_{2-\alpha}(t-s)\, Au(s)\, ds + \int_0^t (t-s)\, u(s)\, ds \\ = \int_0^t (t-s)\, f(s)\, ds, \quad t > 0, \end{aligned} \tag{139}$$

where $f \in L^1_{\mathrm{loc}}([0,\infty) : E)$. Equation (139) models an oscillation process with fractional damping term and after differentiation becomes, in some sense,

$$u''(t) + A\mathbf{D}_t^\alpha u(t) + u(t) = f(t), \quad t \geq 0. \tag{140}$$

Without any essential changes, one can consider the $C$-well-posedness and perturbation properties of (139).

*Example 27.* (See [22, 66, 67]). Let $\alpha \in (0,1)$, $m \in \mathbb{N}$, let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with boundary of class $C^{4m}$, and let $E := C^\alpha(\overline{\Omega})$. Consider the operator $A : D(A) \subseteq C^\alpha(\overline{\Omega}) \to C^\alpha(\overline{\Omega})$ given by

$$Au(x) := \sum_{|\beta|\leq 2m} a_\beta(x)\, D^\beta u(x) \ \ \forall x \in \overline{\Omega} \tag{141}$$

with domain $D(A) := \{u \in C^{2m+\alpha}(\overline{\Omega}) : D^\beta u_{|\partial\Omega} = 0$ for all $|\beta| \leq m-1\}$. Here $\beta \in \mathbb{N}_0^n, |\beta| = \sum_{i=1}^n \beta_j, D^\beta = \prod_{i=1}^n ((1/i)(\partial/\partial x_i))^{\beta_i}$, and $a_\beta : \overline{\Omega} \to \mathbb{C}$ satisfy the following conditions:

(i) $a_\beta(x) \in \mathbb{R}$ for all $x \in \overline{\Omega}$ and $|\beta| = 2m$,

(ii) $a_\beta \in C^\alpha(\overline{\Omega})$ for all $|\beta| \leq 2m$, and

(iii) there exists $M > 0$ such that

$$\begin{aligned} M^{-1}|\xi|^{2m} \leq \sum_{|\beta|=2m} a_\beta(x)\, \xi^\beta \leq M|\xi|^{2m} \\ \forall \xi \in \mathbb{R}^n, \quad x \in \overline{\Omega}. \end{aligned} \tag{142}$$

Then there exists a sufficiently large $\sigma > 0$ such that the operator $-A_\sigma \equiv -(A+\sigma)$ satisfies $\Sigma_\omega \cup \{0\} \subseteq \rho(-A_\sigma)$ with some $\omega \in ((\pi/2), \pi)$ and

$$\left\|R(\lambda : -A_\sigma)\right\| = O\left(|\lambda|^{(\alpha/2m)-1}\right), \quad \lambda \in \Sigma_\omega. \tag{143}$$

Notice that $A$ is not densely defined since $D(A) \subseteq \{u \in C^\alpha(\overline{\Omega}) : u_{|\partial\Omega} = 0\}$. Let $\varsigma \in [1, (2\omega/\pi))$ and $\tau \in ((\alpha/2m), 1)$. By (143) and [32, Theorem 2.17], we get that $-A_\sigma$ is the integral generator of an exponentially bounded, analytic $(g_\varsigma, g_{\varsigma\tau+1})$-regularized resolvent family of angle $\delta = (\omega/\varsigma) - (\pi/2) \in (0, (\pi/2))$. Assume now that $B : D(B) \subseteq C^\alpha(\overline{\Omega}) \to C^\alpha(\overline{\Omega})$ is a linear operator satisfying $D(A) \subseteq D(B)$ and $||Bu|| \leq c||u||, u \in D(A)$ for some $c > 0$. Applying Theorem 24(iv),

we obtain that the operator $-(A+B)$ is the integral generator of an exponentially bounded, analytic $(g_\varsigma, g_{\varsigma\tau+1})$-regularized resolvent family of angle $\delta$. Suppose, for example, $m = n = 1$ and $\Omega = (0,1)$. Let $\varphi, \psi \in L^1[0,1]$ and let the operator $B : C^\alpha[0,1] \to C^\alpha[0,1]$ be defined by

$$Bu(x) := \int_0^x (u(x-s) - u(0))\varphi(s)\, ds + \int_0^x (u(1-x+s) - u(1))\psi(s)\, ds, \quad x \in [0,1]. \tag{144}$$

Then $B$ satisfies the conditions stated above since $B \in L(C^\alpha[0,1])$ and $\|Bu\| \le (\|\varphi\|_{L^1[0,1]} + \|\psi\|_{L^1[0,1]})\|u\|, u \in C^\alpha[0,1]$. Finally, it could be interesting to construct an example in which there does not exist $\widehat{B} \in L(C^\alpha[0,1])$ such that $\widehat{B}x = Bx$ for all $x \in D(A)$.

In the remaining part, which is mainly motivated by reading of the paper [25] by Arendt and Batty, we assume that $E$ is a Banach space. We consider rank-1 perturbations of ultradistribution semigroups and sines whose generators possess polynomially bounded resolvent; our intention is also to prove generalizations of [25, Theorem 4.3] and [26, Theorem 1.3] for abstract time-fractional equations.

Given $a \in E, b^* \in E^*$ and $C \in L([D(A)], E)$, we consider the rank-1 perturbation $B \in L([D(A)], E)$ of $A$ given by

$$Bx := b^*(Cx)a, \quad x \in D(A). \tag{145}$$

We also denote this operator $B$ by $ab^*C$. Denote $B_\delta(a,b^*) := \{(x,y^*) \in E \times E^* : \|x-a\| \le \delta,\ \|x^*-b^*\| \le \delta\}(a \in E, b^* \in E^*, \delta > 0)$.

For the sake of convenience to the reader, we will repeat the assertion of [25, Theorem 1.3].

**Lemma 28.** *Let $A$ be a closed linear operator on $E$, let $C \in L([D(A)], E)$ and let $\varepsilon > 0$. Assume that $\Omega_n \subseteq \rho(A)$ and $\sup_{\lambda\in\Omega_n}\|CR(\lambda : A)x\| < \infty$ for all $x$ in a dense subset $E_n$ of $E$ and all $n \in \mathbb{N}$. Let $g_n : \Omega_n \to (0,\infty)$ $(n \in \mathbb{N})$. Assume that for each $(a,b^*) \in B_\varepsilon(0,0)$ there exists $n \in \mathbb{N}$ such that $\Omega_n \subseteq \rho(A+ab^*C)$ and $\|R(\lambda : A+ab^*C)\| \le g_n(\lambda), \lambda \in \Omega_n$. Then there exists $m \in \mathbb{N}$ such that $\sup_{\lambda\in\Omega_m}\|CR(\lambda : A)\| < \infty$.*

Henceforth, we assume that $(M_p)$ is a sequence of positive real numbers such that $M_0 = 1$ and that the following conditions are fulfilled:

$$M_p^2 \le M_{p+1}M_{p-1}, \quad p \in \mathbb{N}, \tag{M.1}$$

$$M_p \le AH^p \sup_{0\le i\le p} M_i M_{p-i}, \quad p \in \mathbb{N}, \text{ for some } A, \quad H > 1, \tag{M.2}$$

$$\sum_{p=1}^\infty \frac{M_{p-1}}{M_p} < \infty. \tag{M.3$'$}$$

Let $s > 1$. Then the Gevrey sequences $(p!^s)$, $(p^{ps})$ and $(\Gamma(1+ps))$ satisfy the above conditions. The associated function of $(M_p)$ is defined by $M(t) := \sup_{p\in\mathbb{N}} \ln(t^p/M_p), t > 0$ and $M(0) := 0$. Recall [54], the function $t \mapsto M(t)$, $t \ge 0$ is increasing, $\lim_{t\to\infty} M(t) = \infty$ and $\lim_{t\to\infty}(M(t)/t) = 0$.

Following [1, 16], a closed linear operator $A$ is said to be the generator of an ultradistribution sine of $(M_p)$-class if and only if the operator $\mathscr{A} := \left(\begin{smallmatrix} 0 & I \\ A & 0 \end{smallmatrix}\right)$ generates an ultradistribution semigroup of $(M_p)$-class (cf. [16, 18, 68, 69] for the notion). The following well-known lemma (cf. [69, Theorem 1.5], [16, Theorem 9] and [1, Chapter 3]) will be helpful in our further work.

**Lemma 29.** *(i) Let $A$ be a closed densely defined operator on $E$. Then $A$ generates an ultradistribution semigroup of $(M_p)$-class if and only if there exist $l \ge 1$, $\alpha > 0$ and $\beta \in \mathbb{R}$ such that*

$$\Lambda_{l,\alpha,\beta} := \{\lambda \in \mathbb{C} : \Re\lambda \ge \alpha M(l|\Im\lambda|) + \beta\} \subseteq \rho(A), \tag{146}$$

$$\|R(\lambda : A)\| = O(\exp(M(l|\lambda|))), \quad \lambda \in \Lambda_{l,\alpha,\beta}. \tag{147}$$

*(ii) Let $A$ be a closed densely defined operator on $E$. Then $A$ generates an ultradistribution sine of $(M_p)$-class if and only if there exist $l \ge 1, \alpha > 0$ and $\beta \in \mathbb{R}$ such that*

$$\{\lambda^2 : \lambda \in \Lambda_{l,\alpha,\beta}\} \subseteq \rho(A), \tag{148}$$

$$\|R(\lambda^2 : A)\| = O(\exp(M(l|\lambda|))), \quad \lambda \in \Lambda_{l,\alpha,\beta}. \tag{149}$$

**Theorem 30.** *Let $l \ge 1, \alpha > 0, \beta \in \mathbb{R}, k \in \mathbb{N}$ and $C > 0$. Let $A$ be a closed densely defined operator on $E$.*

(i) *Assume* (148) *and*

$$\|R(\lambda^2 : A)\| \le C(1+|\lambda|)^k, \quad \lambda \in \Lambda_{l,\alpha,\beta}. \tag{150}$$

*Let $\varepsilon > 0$ and $z \in \mathbb{C}$ be such that for each $(a,b^*) \in B_\varepsilon(0,0)$ the operator $A + ab^*(z-A)$ generates an ultradistribution sine of $(M_p)$-class. Then $A$ must be bounded.*

(ii) *Assume* (146) *and*

$$\|R(\lambda : A)\| \le C(1+|\lambda|)^k, \quad \lambda \in \Lambda_{l,\alpha,\beta}. \tag{151}$$

*Let $\varepsilon > 0$ and $z \in \mathbb{C}$ be such that for each $(a,b^*) \in B_\varepsilon(0,0)$ the operator $A + ab^*(z-A)$ generates an ultradistribution semigroup of $(M_p)$-class. Then $A$ generates an analytic $C_0$-semigroup.*

*Proof.* We will only prove the first part of the theorem. Put $\Omega_n := \{\lambda \in \mathbb{C} : \Re\lambda \ge nM(n|\Im\lambda|) + n\}$. Then $\Omega_n \subseteq \Lambda_{l,\alpha,\beta}$ for all $n \ge \max(l,\alpha,|\beta|)$. By the generalized resolvent equation, it follows that for each $x \in Y_n \equiv D(A^{\lceil k/2\rceil+2})$, the set $\{\|R(\lambda : A)x\| : \lambda \in \Omega_n\}$ is bounded. The prescribed assumption combined with Lemma 29(ii) implies that for each $(a,b^*) \in B_\varepsilon(0,0)$ there exist $n \in \mathbb{N}$ and a function $g_n : \Omega_n \to (0,\infty)$ such that $\Omega_n^2 := \{\lambda^2 : \lambda \in \Omega_n\} \subseteq \rho(A + ab^*(z-A))$ and $\|R(\lambda^2 : A + ab^*(z-A))\| \le g_n(\lambda)$, $\lambda \in \Omega_n$. By Lemma 28, we obtain $m \in \mathbb{N}$ such that $\sup_{\lambda\in\Omega_m^2}\|\lambda R(\lambda : A)\| < \infty$. Let

$\xi + i\eta = \lambda \in \partial(\Omega_m^2)$. Assume $|\eta_1| \leq |\eta|$ and $\mu = \xi + \eta_1$. Then $\xi = (mM(m|t|) + m)^2 - t^2$ and $\eta = 2t(mM(m|t|) + m)$ for some $t \in \mathbb{R}$. Since $\lim_{t\to\infty}(M(t)/t) = 0$, we easily infer that there exist $t_0 > 0$ and $L \geq 1$ such that, for any $|t| \geq t_0$ :

$$\begin{aligned}
&\left\|(\lambda-\mu)\,R(\lambda:A)\right\| \\
&\quad\leq 2\,|\eta|\,\|R(\lambda:A)\| \leq \frac{2L\,|\eta|}{|\lambda|} \\
&\quad\leq \frac{4L\,|t|\,(mM\,(m\,|t|)+m)}{\left(\left((mM\,(m\,|t|)+m)^2-t^2\right)^2+4t^2(mM\,(m\,|t|)+m)^2\right)^{1/2}} \\
&\quad\leq \frac{1}{2},
\end{aligned} \tag{152}$$

which implies that $R(\mu : A)$ exists and $\|R(\mu : A)\| \leq 2\|R(\lambda : A)\| \leq 2L/|\lambda| \leq 2L$. Therefore, there exists $\omega_0 > 0$ such that $\|R(\cdot : A)\|$ is polynomially bounded on $\{\lambda \in \mathbb{C} : \Re\lambda \leq -\omega_0\} \setminus \{\lambda^2 : \lambda \in \Lambda_{l,\alpha,\beta}\}$. The set $\{\lambda \in \mathbb{C} : \Re\lambda \geq -\omega_0\} \setminus \{\lambda^2 : \lambda \in \Lambda_{l,\alpha,\beta}\}$ is compact, which completes the proof by [25, Lemma 2.3]. □

*Remark 31.* (i) It is worth noting that Theorem 30(ii) is an extension of [25, Theorem3.1], and that Theorem 30(i) is an extension of [25, Theorem 2.2] provided $k > 0$ in the formulation of this result. Consider now the situation of [25, Theorem 2.2] with $A$ being the generator of an exponentially bounded $\alpha$-times integrated cosine function ($\alpha \geq 0$). Then there exists $\omega_A > 0$ such that $\sup_{\lambda>\omega_A}\|\lambda^{(2-\alpha)/2}R(\lambda : A)\| < \infty$. Let $\omega > \omega_A$. Then, for every $\gamma \in \mathbb{R}$, one can define the fractional power $A_\gamma := (\omega - A)^\gamma$ (cf. [1, Section 1.4]). Assuming $0 \leq \alpha < 2$ and $0 < \gamma \leq (2-\alpha)/2$, we obtain from [1, Theorem 1.4.10(iii), (x)] that $D(A) \subseteq D(A_\gamma)$ and $A_\gamma \in L([D(A)], E)$, which implies that one can define the rank-1 perturbation $B_\gamma := A + ab^* A_\gamma$ of $A$; notice that the case $\gamma = 1$ has been already considered in Theorem 30. Obviously, $A_\gamma R(\lambda : A)x = A_{\gamma-1}R(\lambda : A)(\omega - A)x$ for all $x \in D(A)$ and $\lambda \in \rho(A)$. By the proof of [25, Theorem 2.2], one gets that there exists $n \in \mathbb{N}$ such that $\{\lambda^2 : \Re\lambda \geq n\} \subseteq \rho(A)$ and $\sup_{\Re\lambda\geq n}\|A_\gamma R(\lambda^2 : A)\| < \infty$. Unfortunately, it is not clear whether the above conclusions together with [25, Lemma 2.4] (cf. also [70, Lemma 2.3]) imply that $\sup_{\Re\lambda\geq n}\|\lambda^{2\gamma}R(\lambda^2 : A)\| < \infty$, unless $\alpha = 0$. Notice also that the assumption $\gamma = 1$ must be imposed in the case $\alpha \geq 2$.

(ii) In the formulation of Theorem 30(ii) and Theorem 30(i), respectively, we do not assume that the operator $A + ab^*(z - A)$ has polynomially bounded resolvent on the square of $\Lambda_{l,\alpha,\beta}$, respectively, on $\Lambda_{l,\alpha,\beta}$. Furthermore, we may assume that the operator $A + ab^*(z - A)$ has a slightly different spectral properties (cf. [25, Remark 2.5] and the formulation of Theorem 32 below).

(iii) Given $\varepsilon \in (0, 1)$ and $C_\varepsilon > 0$, set

$$\Omega_\varepsilon := \{\lambda \in \mathbb{C} : \Re\lambda \geq \varepsilon\,|\lambda| + C_\varepsilon\}. \tag{153}$$

The proof of Theorem 30(i) and Theorem 30(ii) respectively, does not work any longer if, for every $\varepsilon > 0$, the estimate (150), respectively (151), holds with $\Lambda_{l,\alpha,\beta}$ replaced by $\Omega_\varepsilon$. Therefore, it is not clear whether Theorem 3.11 can be reformulated in case of certain classes of hyperfunction semigroups and sines [1, 71].

Recall [32], a (local) $(a, k)$-regularized $C$-resolvent family $(R(t))_{t\in[0,\tau)}$ having $A$ as a subgenerator is of *class* $C^L$ if and only if the following holds

(i) the mapping $t \mapsto R(t)$, $t \in (0, \tau)$ is infinitely differentiable (in the uniform operator topology), and

(ii) for every compact set $K \subseteq (0, \tau)$ there exists $h_K > 0$ such that

$$\sup_{t\in K, p\in\mathbb{N}_0}\left\|\frac{h_K^p\,(d^p/dt^p)\,R(t)}{M_p}\right\| < \infty; \tag{154}$$

$(R(t))_{t\in[0,\tau)}$ is said to be *$\rho$-hypoanalytic*, $1 \leq \rho < \infty$, if $(R(t))_{t\in[0,\tau)}$ is of class $C^L$ with $M_p = p!^\rho$.

By [72, Theorem 5.5] and [32, Theorem 2.23], a $C_0$-semigroup $(T(t))_{t\geq0}$ is $\rho$-hypoanalytic for some $\rho \geq 1$ if $(T(t))_{t\geq0}$ is in the *Crandall-Pazy class* of semigroups. Recall that $(T(t))_{t\geq0}$ is in the Crandall-Pazy class [72] if and only if there exist $\gamma \in (0, 1]$, $b > 0$, $k > 0$ and $c \in \mathbb{R}$ such that

$$\begin{gathered}
E_{\gamma,b,c} := \{\lambda \in \mathbb{C} : \Re\lambda \geq c - b|\Im\lambda|^\gamma\} \subseteq \rho(A), \\
\|R(\lambda : A)\| \leq c, \quad \lambda \in E_{\gamma,b,c}.
\end{gathered} \tag{155}$$

Keeping in mind (155), the subsequent theorem can be viewed as a generalization of [25, Theorem 4.3]. Observe that the operator $(\omega - A)^\gamma (\gamma \in \mathbb{R})$ is defined for a sufficiently large $\omega > 0$, provided that $A$ generates an exponentially bounded $(g_\alpha, g_\beta)$-regularized resolvent family.

**Theorem 32.** *Suppose $0 < \alpha < 2, (\alpha - 1)/\alpha < \gamma < 1, z \in \mathbb{C}, \beta \geq 0$ and a densely defined operator $A$ generates an exponentially bounded $(g_\alpha, g_\beta)$-regularized resolvent family $(R(t))_{t\geq0}$.*

(i) *Assume that $b = 0$ and for each $(a, b) \in B_\varepsilon(0, 0)$ there exists a kernel $k_{a,b^*}(t)$ satisfying* (P1)-(P2) *so that the operator $A + ab^*(\omega - A)^\gamma$ generates an exponentially bounded $(g_\alpha, k_{a,b^*})$-regularized resolvent family. Then $(R(t))_{t\geq0}$ is $(1/(\alpha\gamma + 1 - \alpha))$-hypoanalytic.*

(ii) *Assume that $b > 0$ and for each $(a, b) \in B_\varepsilon(0, 0)$ there exists a kernel $k_{a,b^*}(t)$ satisfying* (P1)-(P2) *so that the operator $A + ab^*(z - A)$ generates an exponentially bounded $(g_\alpha, k_{a,b^*})$-regularized resolvent family. Then $A$ generates an exponentially bounded, analytic $(g_\alpha, 1)$-regularized resolvent family.*

*Proof.* Given $\omega \geq 0$, set $\Phi_{\alpha,\omega} := \{\lambda^\alpha : \Re\lambda \geq \omega\}$. Making use of [25, Lemma 2.4], [32, Theorem 2.7], and Lemma 28, we get that there exist $n \in \mathbb{N}$ and $m > 0$ such that $\Psi_{n,\alpha} := \Phi_{\alpha,n} \cap \{z \in \mathbb{C} : |z - \lambda| \leq m|\lambda|^\gamma$ for some $\lambda \in \partial(\Phi_{\alpha,n})\} \subseteq \rho(A)$ and that $\|R(\lambda : A)\| = O(|\lambda|^{-\gamma})$, $\lambda \in \Psi_{n,\alpha}$. Let $\varepsilon \in (0, 1)$ and let $K_\varepsilon > 0$ be such that

$$\alpha K_\varepsilon(1 + K_\varepsilon)^{\alpha-1} \leq m. \tag{156}$$

Put $\rho := 1/(\alpha\gamma + 1 - \alpha)$. Notice that $\rho \geq 1$ since $\alpha > 0$ and $(\alpha - 1)/\alpha < \gamma < 1$. With the help of (156) and the Darboux inequality, we obtain that for each $\eta \in \mathbb{R}$ :

$$\begin{aligned}
&\left|\left(n - K_\varepsilon|\eta|^{1/\rho} + i\eta\right)^\alpha - (n + i\eta)^\alpha\right| \\
&\quad \leq \alpha K_\varepsilon|\eta|^{1/\rho} \sup_{\nu\in\left[n+i\eta, n+i\eta-K_\varepsilon|\eta|^{1/\rho}\right]} |\nu|^{\alpha-1} \\
&\quad \leq \alpha K_\varepsilon|\eta|^{1/\rho}\left(|n + i\eta| + K_\varepsilon|\eta|^{1/\rho}\right)^{\alpha-1} \\
&\quad \leq \alpha K_\varepsilon(1 + K_\varepsilon)^{\alpha-1}|\eta|^{1/\rho}|n + i\eta|^{\alpha-1} \\
&\quad \leq \alpha K_\varepsilon(1 + K_\varepsilon)^{\alpha-1}|n + i\eta|^{\alpha-1+(1/\rho)} \leq m|n + i\eta|^{\alpha\gamma},
\end{aligned} \tag{157}$$

which implies that $\{\lambda \in \mathbb{C} : \Re\lambda \geq -K_\varepsilon|\Im\lambda|^{1/\rho}+n\} \subseteq \Psi_{n,\alpha}$. The proof of (i) is completed by an application of [32, Theorem 2.23]. Suppose now that the assumptions of (ii) hold. Then $\omega - A$ need not be sectorial, in general. We obtain similarly the existence of an integer $n \in \mathbb{N}$ and a number $c \in (0, 1)$ such that $\Upsilon_{n,\alpha} := \Phi_{\alpha,n} \cap \{z \in \mathbb{C} : |z - \lambda| \leq m|\lambda| \text{ for some } \lambda \in \partial(\Phi_{\alpha,n})\} \subseteq \rho(A)$ and that $||R(\lambda : A)|| \leq c|\lambda|^{-1}, \lambda \in \Upsilon_{n,\alpha}$. Then it readily follows from [32, Theorem 2.17] that $A$ generates an exponentially bounded, analytic $(g_\alpha, 1)$-regularized resolvent family of angle $(\arcsin c)/\alpha$. □

Now we will transfer the assertion of [26, Theorem 1.3] to abstract time-fractional equations. For $a \in E$ and $b^* \in E^*$, define $A_{a,b^*}$ by $D(A_{a,b^*}) := \{x \in E : x + \langle b^*, x\rangle a \in D(A)\}$ and $A_{a,b^*}x := A(x + \langle b^*, x\rangle a), x \in D(A_{a,b^*})$.

We need the following auxiliary lemma (cf. the proofs and formulations of [26, Lemmas 2.1 and 2.2]).

**Lemma 33.** *Let $\omega \geq 0, \alpha \in (0, 2), a \in E, b^* \in E^*$ and $z \in \mathbb{C}_\omega$, where $\mathbb{C}_\omega := \{z \in \mathbb{C} : \Re z > \omega\}$.*

(i) *Then $z^\alpha$ is an eigenvalue of both, $A_{a,b^*}$ and $A + ab^*A$, with $A_{a,b^*}(AR(z^\alpha : A)a) = z^\alpha AR(z^\alpha : A)a$ and $(A + ab^*A)(R(z^\alpha : A)a) = z^\alpha R(z^\alpha : A)a$.*

(ii) *Let $\langle b^*, AR(z^\alpha : A)a\rangle = 1$ and $\langle b^*, AR(z^\alpha : A)a\rangle \neq 0$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $(a_1, b_1^*) \in B_\delta(a, b^*)$ there exists some $z_1 \in B(z, \varepsilon)$ such that $\langle b_1^*, AR(z_1^\alpha : A)a_1\rangle = 1$ and $\langle b_1^*, AR(z_1^\alpha : A)a_1\rangle \neq 0$.*

The following fractional analogue of [26, Lemma 2.3] will be essentially utilized in the proof of Theorem 35 stated below.

**Lemma 34.** *Suppose $\alpha \in (0, 2)$, $n \in \mathbb{N}$, $\omega \geq 0$, $\varepsilon > 0$ and $A$ is the generator of an exponentially bounded, nonanalytic $(g_\alpha, 1)$-regularized resolvent family $(S_\alpha(t))_{t\geq 0}$ satisfying $||S_\alpha(t)|| \leq Me^{\omega t}, t \geq 0$ for some $M \geq 1$. Let $r > \omega, k = \lceil 1/\alpha\rceil, a \in D(A^k), b^* \in D((A^*)^k)$ and $z_j \in \mathbb{C}_\omega$ be such that $\langle b^*, AR(z_j^\alpha : A)a\rangle = 1$ and $\langle b^*, AR(z_j^\alpha : A)a\rangle \neq 0$ $(1 \leq j \leq n)$. Then there exist $(a_1, b_1^*) \in B_\varepsilon(a, b^*) \cap (D(A^k) \times D((A^*)^k))$ and ${}_1z, \ldots, {}_{n+1}z \in \mathbb{C}_\omega$ such that $\Re_{n+1}z = r, |{}_jz - z_j| < \varepsilon$ $(1 \leq j \leq n)$, $\langle b_1^*, AR({}_{n+1}z^\alpha : A)a_1\rangle = 1$ and $\langle b_1^*, AR({}_{n+1}z^\alpha : A)a_1\rangle \neq 0$.*

*Proof.* We will only outline the main details of the proof. First of all, notice that $||R(z^\alpha : A^*)|| = ||R(z^\alpha : A)|| \leq M/((\Re z - \omega)|z|^{\alpha-1}), \Re z > \omega$. By [1, Lemma 1.4.10], we get that $A^*$ is stationary dense with $n(A^*) \leq 1$, which implies that $\overline{D((A^*)^k)} = \overline{D(A^*)}$ in the strong topology of $E^*$. Let the numbers $\delta_1 > 0, \ldots, \delta_n > 0$ be given by Lemma 33(i) and let $\delta := \min(1, \varepsilon, \delta_1, \ldots, \delta_n)$. By the generalized resolvent equation (see e.g., the proof of [25, Theorem 2.2]) and the fact that $^* : L(E) \to L(E^*)$ is an isometrically isomorphism, we obtain that for each $(a, b^*) \in D(A^k) \times D((A^*)^k)$, the following supremum

$$\begin{aligned}
&\sup_{\Re z=r} \max\left(1, \left|\left|A^*R\left(z^\alpha : A^*\right)b^*\right|\right|, \left|\left|AR\left(z^\alpha : A\right)a\right|\right|,\right. \\
&\qquad \left.\left|\left\langle b^*, AR\left(z^\alpha : A\right)a\right\rangle\right|\right) := K
\end{aligned} \tag{158}$$

is finite. The nonanalyticity of $(S_\alpha(t))_{t\geq 0}$ yields that $\sup_{\Re z=r}||AR(z^\alpha : A)|| = \infty$. By the denseness of $D(A^k)$ in $E$, we get the existence of an element $u \in D(A^k)$ and a complex number ${}_{n+1}z := z$ such that $\Re z = r$, $||AR(z^\alpha : A)u|| > (10K/\delta^2)$, $AR(z^\alpha : A)^2u \neq 0$ and $||u|| < 1$. Now one can proceed as in the proof of [26, Lemma 2.3] so as to obtain $v^* \in D(A^*)$ such that $|\langle v^*, AR(z^\alpha : A)u\rangle| > (10K/\delta^2)$, $||v^*|| < 1$ and $\langle b^* + (\delta/2)v^*, AR(z^\alpha : A)^2u\rangle \neq 0$. Since $D((A^*)^k)$ is dense in $D(A^*)$ with respect to the strong topology of $E^*$, we may assume that $v^* \in D((A^*)^k)$. Copying the final part of the proof of the aforementioned lemma, with $AR(z : A)$ and $AR(z : A)^2$ replaced by $AR(z^\alpha : A)$ and $AR(z^\alpha : A)^2$ there, we obtain that there exists $(a_1, b_1^*) \in B_\varepsilon(a, b^*) \cap (D(A^k) \times D(A^*)^k)$ with required properties (cf. [26, page 474, $l$.1 – 4]). □

If $\alpha \in (0, 2)$ and $A$ is the generator of an exponentially bounded $(g_\alpha, 1)$-regularized resolvent family $(S_\alpha(t))_{t\geq 0}$ satisfying the properties stated above, then one can simply prove that for each $r > \omega$ there exist $z \in \mathbb{C}_\omega$ with $\Re z = r$ and $(a, b^*) \in D(A^k) \times D((A^*)^k)$ such that $\langle b^*, AR(z^\alpha : A)a\rangle = 1$ and $\langle b^*, AR(z^\alpha : A)^2a\rangle \neq 0$. Using induction, Lemma 34 and the proof of [26, Theorem 1.3], we obtain the following theorem.

**Theorem 35.** *Suppose $\alpha \in (0, 2), n \in \mathbb{N}, \omega \geq 0, \varepsilon > 0$ and $A$ is the generator of an exponentially bounded, nonanalytic $(g_\alpha, 1)$-regularized resolvent family $(S_\alpha(t))_{t\geq 0}$ satisfying $||S_\alpha(t)|| \leq Me^{\omega t}$, $t \geq 0$ for some $M \geq 1$. Let $I_j \subseteq (\omega^\alpha, \infty)$ be an open interval $(1 \leq j < \infty)$. Then there exist $a \in E$ and $b^* \in E^*$ such that the operators $A_{a,b^*}$ and $A + ab^*A$ have a sequence of eigenvalues $z_j$ with $\Re z_j \in I_j$ for all $j = 1, 2, \ldots$.*

We close the paper with the observation that perturbation theorems for q-exponentially equicontinuous $(a, k)$-regularized $C$-resolvent families have been recently analyzed in [73].

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