Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 354287, 11 pages
http://dx.doi.org/10.1155/2013/354287

Research Article

Stability Analysis of a Multigroup Epidemic Model with General Exposed Distribution and Nonlinear Incidence Rates

Ling Zhang,[1] Jingmei Pang,[2] and Jinliang Wang[2]

[1] *School of Science, Department of Fundamental Mathematics, Jiamusi University, Jiamusi 154007, China*
[2] *School of Mathematical Science, Heilongjiang University, Harbin 150080, China*

Correspondence should be addressed to Jinliang Wang; jinliangwang@hit.edu.cn

Received 26 January 2013; Revised 16 June 2013; Accepted 22 July 2013

Academic Editor: Pagavathi Balasubramaniam

We investigate a class of multigroup epidemic models with general exposed distribution and nonlinear incidence rates. For a simpler case that assumes an identical natural death rate for all groups, and with a gamma distribution for exposed distribution is considered. Some sufficient conditions are obtained to ensure that the global dynamics are completely determined by the basic production number R_0. The proofs of the main results exploit the method of constructing Lyapunov functionals and a graph-theoretical technique in estimating the derivatives of Lyapunov functionals.

1. Introduction

Multigroup epidemic models have been used in the literature to describe the transmission dynamics of many different infectious diseases such as mumps, measles, gonorrhea, HIV/AIDS and vector borne diseases such as Malaria [1]. In the models, heterogeneous host population can be divided into several homogeneous groups according to modes of transmission, contact patterns, or geographic distributions, so that within-group and intergroup interactions can be modeled separately. It is well known that global dynamics of multigroup models with higher dimensions, especially the global stability of the endemic equilibrium, are a very challenging problem. Guo et al. [2] proposed a graph-theoretic approach to the method of global Lyapunov functions and used it to resolve the open problem on the uniqueness and global stability of the endemic equilibrium of a multigroup SIR model with varying subpopulation sizes. Subsequently, a series of studies on the global stability of multigroup epidemic models were produced in the literature (see e.g., [2–5]).

In the present paper, a more general multigroup epidemic model is proposed and studied to describe the disease spread in a heterogeneous host population with general exposed distribution and nonlinear incidence rate. The host population is divided into m distinct groups ($m \geq 1$). For $1 \leq i \leq m$, the ith group is further partitioned into four disjoint classes: the susceptible individuals, exposed individuals, infectious individuals, and recovered individuals, whose numbers of individuals at time t are denoted by $S_i(t)$, $E_i(t)$, $I_i(t)$, and $R_i(t)$, respectively. Susceptible individuals infected with the disease but not yet infective are in the exposed (latent) class.

It is pointed in [6] that a fixed latent period can be considered as an approximation of the mean latent period, and this would be appropriate for those diseases whose latent periods vary only relatively slightly. For example, poliomyelitis has a latent period of 1–3 days (comparing to its much longer infectious period of 14–20 days). However disease such as tuberculosis, including bovine tuberculosis (a disease spread from animal to animal mainly by direct contact), may take months to develop to the infectious stage and also can relapse. Since the time it takes from the moment of new infection to the moment of becoming infectious may differ from disease to disease, even for the same disease, it differs from individual to individual, and it is indeed a random variable. It is thus of interest from both mathematical and biological viewpoints to investigate whether sustained oscillations are the result of general exposed distribution.

Following the method of [6], we also assume that the disease does not cause deaths during the latent period, taking the natural death rate into consideration. Let $P(t)$ denote the probability that an exposed individual remains in the time t after entering the exposed class. For $1 \leq i, j \leq m$, $\beta_{ij} \geq 0$ denotes the coefficient of transmission between compartments S_i and I_j. It is assumed that m-square matrix $(\beta_{ij})_{1\leq i,j\leq m}$ is irreducible [7]. So the proportion of exposed individuals can be expressed by the integral

$$E_i(t) = \sum_{j=1}^{m}\beta_{ij}\int_0^t f_{ij}\left(S_i(u), I_j(u)\right)e^{-\delta_j(t-u)}P_j(t-u)\,du, \quad (1)$$

where the sum takes into account cross-infections from all groups. Integrals in (1) are in the Riemann-Stieltjes sense. $P_j(t)$ satisfies the following reasonable properties:

(A) $P_j : [0,\infty) \to [0,1]$ is nonincreasing, piecewise continuous with possibly finitely many jumps and satisfies $P_j(0^+) = 1$, and $\lim_{t\to\infty}P_j(t) = 0$ with $\int_0^\infty P_j(t)dt$ is positive and finite.

Differentiating (1) gives

$$\begin{aligned} E_i'(t) &= \sum_{j=1}^{m}\beta_{ij}f_{ij}\left(S_i(t), I_j(t)\right) \\ &\quad + \sum_{j=1}^{m}\beta_{ij}\int_0^t f_{ij}\left(S_i(u), I_j(u)\right)e^{-\delta_j(t-u)} \\ &\quad \times P_j'(t-u)\,du - \delta_i E_i(t). \end{aligned} \quad (2)$$

The first term on the right hand side in (2) is the rate at which new infected individuals come into the exposed class, and the last term explains the natural deaths. The second term accounts for the rate at which the individuals move to the infectious class (noting that $P_j'(t-u) \leq 0$ due to the aformentioned property) from the exposed class; hence

$$\begin{aligned} I_i'(t) = &-\sum_{j=1}^{m}\beta_{ij}\int_0^t f_{ij}\left(S_i(u), I_j(u)\right)e^{-\delta_j(t-u)} \\ &\times P_j'(t-u)\,du - \left(\delta_i + \varepsilon_i + \gamma_i\right)I_i(t). \end{aligned} \quad (3)$$

Let $h_j(t) = -P_j'(t)$ be the probability density function for the time (a random variable) it takes for an infected individual in the ith group to become infectious. Then (4) becomes

$$\begin{aligned} I_i'(t) = &\sum_{j=1}^{m}\beta_{ij}\int_0^t f_{ij}\left(S_i(u), I_j(u)\right)e^{-\delta_j(t-u)} \\ &\times h_j(t-u)\,du - \left(\delta_i + \varepsilon_i + \gamma_i\right)I_i(t). \end{aligned} \quad (4)$$

Within the ith group, $\varphi_i(S_i)$ denotes the growth rate of S_i, which includes both the production and the natural death of susceptible individuals. Therefore, under the assumptions, the model to be studied takes the following differential and integral equations form:

$$\begin{aligned} S_i'(t) &= \varphi_i\left(S_i(t)\right) - \sum_{j=1}^{m}\beta_{ij}f_{ij}\left(S_i(t), I_j(t)\right), \\ E_i'(t) &= \sum_{j=1}^{m}\beta_{ij}f_{ij}\left(S_i(t), I_j(t)\right) \\ &\quad - \sum_{j=1}^{m}\beta_{ij}\int_0^t f_{ij}\left(S_i(u), I_j(u)\right)e^{-\delta_j(t-u)} \\ &\quad \times h_j(t-u)\,du - \delta_i E_i(t), \\ I_i'(t) &= \sum_{j=1}^{m}\beta_{ij}\int_0^t f_{ij}\left(S_i(u), I_j(u)\right)e^{-\delta_j(t-u)} \\ &\quad \times h_j(t-u)\,du - \left(\delta_i + \varepsilon_i + \gamma_i\right)I_i(t), \\ R_i'(t) &= \gamma_i I_i(t) - \delta_i R_i(t). \end{aligned} \quad (5)$$

Since the variables E_i and R_i do not appear in the first and third equations of model (5), $E_i(t)$ and $R_i(t)$, $i = 1, \ldots, m$, can be decoupled from the $S_i(t)$ and $I_i(t)$ equations; we only need to consider the subsystem of (5) consisting of only the S_i and I_i equations:

$$\begin{aligned} S_i'(t) &= \varphi_i\left(S_i(t)\right) - \sum_{j=1}^{m}\beta_{ij}f_{ij}\left(S_i(t), I_j(t)\right), \\ I_i'(t) &= \sum_{j=1}^{m}\beta_{ij}\int_0^t f_{ij}\left(S_i(u), I_j(u)\right)e^{-\delta_j(t-u)} \\ &\quad \times h_j(t-u)\,du - \left(\delta_i + \varepsilon_i + \gamma_i\right)I_i(t), \end{aligned} \quad (6)$$

where δ_i denotes the natural death rates of I_i compartments in the ith group, ε_i is the death rate caused by disease in the ith group, and γ_i is the rate of recovery of infectious individuals in the ith group. In what follows we investigate the global stability of system (5).

When $m = 1$, $P(t) = e^{\varepsilon t}$, and with bilinear incidence rate, system (5) will reduce to the standard SEIR ordinary differential equation (ODE) model studied in [8, 9], and if we further assume that $P(t)$ is a step function, system (5) becomes the SEIR model with a discrete delay studied in [10]. Recently, a model of this type, but including the possibility of disease relapse, has been proposed in [11, 12] to investigate the transmission of herpes, and its global dynamics have been completely investigated in [5, 13].

To express the main idea and the approaches more clearly, we consider a simpler case in which all groups share the same natural death rate: $\delta_j = \delta$ for $j = 1, 2, \ldots, m$. Further, we assume that the functions $h_j(u)$ are disease specific only, implying that $h_j(u) = h(u)$ for $j = 1, 2, \ldots, m$. We choose the gamma distribution:

$$h(u) = h_{n,b}(u) = \frac{u^{n-1}}{(n-1)!b^n} e^{-u/b}, \tag{7}$$

where $b > 0$ is a real number and $n > 1$ is an integer, which is widely used and can approximate several frequently used distributions. For example, when $b \to 0^+$, $h_{n,b}(s)$ will approach the Dirac delta function, and when $n = 1$, $h_{n,b}(s)$ is an exponentially decaying function.

The main object of this paper is to carry out the well-known "linear chain trick" to system (6), transfer system into an equivalent ordinary differential equations system, and establish its global dynamics. We derive the basic reproductive number R_0 and show that R_0 completely determines the global dynamics of system (6). More specifically, if $R_0 \le 1$, the disease-free equilibrium is globally asymptotically stable and the disease dies out; if $R_0 > 1$, a unique endemic equilibrium exists and is globally asymptotically stable, and the disease persists at the endemic equilibrium. The global stability of P^* rules out any possibility for Hopf bifurcations and the existence of sustained oscillations. We should point out here that this work is motivated by Yuan and Zou [11, 12, 14]. In the proof we demonstrate that the graph-theoretic approach developed in [2, 3] can be successfully applied to construct suitable Lyapunov functionals and thus prove the global stability of the endemic equilibrium for model (6) with general exposed distribution and nonlinear incidence rate. Our work is also based on a recent work by Sun and Shi [15], which resolved the dynamics of multigroup SEIR epidemic models with nonlinear incidence of infection and nonlinear removal functions between compartments.

In Section 2, we first give the model, preliminaries and the basic reproduction number R_0. The global stability of the corresponding equilibria for $R_0 \le 1$ and $R_0 > 1$ is shown, respectively, in Section 3—the key results of this paper. And in Section 4, some numerical simulations are shown to illustrate the effectiveness of the proposed result.

2. Preliminaries

We make the following basic assumptions for the intrinsic growth rate of susceptible individuals in the ith group $\varphi_i(S_i)$ and the transmission functions $f_{ij}(S_i, I_j)$.

(A_1) φ_i are C^1 non-increasing functions on $[0,\infty)$ with $\varphi_i(0) > 0$, and there is a unique positive solution $\xi = S_i^0$ for the equation $\varphi_i(\xi) = 0$. $\varphi_i(S) > 0$ for $0 \le S < S_i^0$, and $\varphi_i(S) < 0$ for $S > S_i^0$; that is

$$\begin{gathered} \left[\varphi_i(S_i) - \varphi_i\left(S_i^0\right)\right]\left(S_i - S_i^0\right) < 0, \\ \text{for } S_i \ne S_i^0,\ i = 1, 2, \ldots, m. \end{gathered} \tag{8}$$

(A_2) $f_{ij}(S_i, I_j) \le c_{ij}(S_i) I_j$ for all $I_j > 0$.

(A_3) $c_{ij}(S_i) \le c_{ij}(S_i^0)$, $0 < S_i < S_i^0$, $i, j = 1, \ldots, m$.

Following the technique and method in [14], define

$$\hat{b} \equiv \frac{b}{1 + \delta b}, \tag{9}$$

which can absorb the exponential term $e^{-\delta u}$ into the delay kernel. The second equation in (6) can be rewritten as

$$\begin{aligned} I_i'(t) = {} & \sum_{j=1}^m \frac{\beta_{ij}}{(1+\delta b)^n} \int_0^t f_{ij}\left(S_i, I_j\right) h_{n,\hat{b}}(t-u)\, du \\ & - \left(\delta + \varepsilon_i + \gamma_i\right) I_i. \end{aligned} \tag{10}$$

For $l = 1, \ldots, n$, let

$$\begin{aligned} y_{i,l}(t) = \sum_{j=1}^m \frac{\beta_{ij}\hat{b}}{(1+\delta b)^n} \int_0^t f_{ij}\left(S_i, I_j\right) h_{l,\hat{b}}(t-u)\, du, \\ i = 1, 2, \ldots, m. \end{aligned} \tag{11}$$

Thus, for $l \in \{2, \ldots, n\}$, we obtain

$$\begin{aligned} \dot{y}_{i,l} = {} & h_{l,\hat{b}}(0) \sum_{j=1}^m \frac{\beta_{ij}\hat{b}}{(1+\delta b)^n} f_{ij}\left(S_i, I_j\right) \\ & + \sum_{j=1}^m \frac{\beta_{ij}\hat{b}}{(1+\delta b)^n} \int_{-\infty}^t \frac{(l-1)(t-u)^{l-2}}{(l-1)!\hat{b}^l} e^{-(t-u)/\hat{b}} f_{ij}\left(S_i, I_j\right) du \\ & - \sum_{j=1}^m \frac{\beta_{ij}\hat{b}}{(1+\delta b)^n} \int_{-\infty}^t \frac{(t-u)^{l-1}}{(l-1)!\hat{b}^{l+1}} e^{-(t-u)/\hat{b}} f_{ij}\left(S_i, I_j\right) du \\ = {} & \frac{\left[y_{i,l-1} - y_{i,l}\right]}{\hat{b}}. \end{aligned} \tag{12}$$

For $l = 1$, we have

$$\begin{aligned} y_{i,1} = \sum_{j=1}^m \frac{\beta_{ij}\hat{b}}{(1+\delta b)^n} \int_{-\infty}^t \frac{e^{-(t-u)/\hat{b}}}{\hat{b}} f_{ij}\left(S_i, I_j\right) du, \\ i = 1, \ldots, m. \end{aligned} \tag{13}$$

It follows that

$$\begin{aligned} \dot{y}_{i,1} = {} & \sum_{j=1}^m \frac{\beta_{ij}}{(1+\delta b)^n} f_{ij}\left(S_i, I_j\right) \\ & - \sum_{j=1}^m \frac{\beta_{ij}}{(1+\delta b)^n} \int_{-\infty}^t \frac{e^{-(t-u)/\hat{b}}}{\hat{b}} f_{ij}\left(S_i, I_j\right) du \\ = {} & \sum_{j=1}^m \frac{\beta_{ij}}{(1+\delta b)^n} f_{ij}\left(S_i, I_j\right) - \frac{1}{\hat{b}} y_{i,1}, \quad i = 1, \ldots, m. \end{aligned} \tag{14}$$

Thus the integro-differential system (6) is equivalent to the ordinary differential equations

$$
\begin{aligned}
S_i'(t) &= \varphi_i(S_i(t)) - \sum_{j=1}^{m}\beta_{ij}f_{ij}(S_i(t), I_j(t)),\\
y_{i,1}'(t) &= \frac{1}{(1+\delta b)^n}\sum_{j=1}^{m}\beta_{ij}f_{ij}(S_i(t), I_j(t)) - \frac{1}{\hat{b}}y_{i,1}(t),\\
y_{i,2}'(t) &= \frac{1}{\hat{b}}(y_{i,1}(t) - y_{i,2}(t)), \quad i = 1,2,\ldots,m,\\
&\vdots\\
y_{i,n}'(t) &= \frac{1}{\hat{b}}(y_{i,n-1}(t) - y_{i,n}(t)),\\
I_i'(t) &= \frac{1}{\hat{b}}y_{i,n}(t) - (\delta + \varepsilon_i + \gamma_i)I_i(t).
\end{aligned}
\tag{15}
$$

For initial condition

$$
\begin{aligned}
&(S_1(0), y_{1,1}(0), \ldots, y_{1,n}(0), I_1(0),\\
&\quad S_2(0), y_{2,1}(0), \ldots, y_{1,n}(0), I_2(0), \ldots,\\
&\quad S_m(0), y_{m,1}(0), \ldots, y_{m,n}(0), I_m(0)) \in \mathrm{R}^{m(n+2)},
\end{aligned}
\tag{16}
$$

the existence, uniqueness, and continuity of the solution $(S_i, y_{i,1}, y_{i,2}, \ldots, y_{i,n}, I_i)$ of system (15) follow from the standard theory of Volterra integro-differential equation [16]. It can also be verified that every solution of (15) with nonnegative initial condition remains nonnegative.

It follows from (A_1) and the first equation of (15) that $\limsup_{t\to\infty}S_i(t) \le S_i^0$ for all $i = 1,2,\ldots,m$. Let N_{φ_i} be the maximum of the function φ_i on R_+ and let q be a positive real number such that $q > \hat{b}N_{\varphi_i}$. Denote by Υ_i the ith tube for system (15); that is,

$$
\Upsilon_i = (S_i, y_{i,1}, y_{i,2}, \ldots, y_{i,n}, I_i).
\tag{17}
$$

It follows from a similar argument to that in [14] that we can show that the set D_ϵ defined by

$$
\begin{aligned}
\Gamma_\epsilon = \Big\{&(S_i, y_{i,1}, y_{i,2}, \ldots, y_{i,n}, I_i) \in \mathrm{R}_+^{m(n+2)} \mid\\
&S_i \le S_i^0 + \epsilon, S_i + (1+\delta b)^n y_{i,1} \le q + S_i^0,\\
&y_{i,l} \le \frac{q + S_i^0 + l\epsilon}{(1+\delta b)^n},\\
&I_i \le \frac{q + S_i^0 + (n+1)\epsilon}{\hat{b}(1+\delta b)^n(\delta + \epsilon_i + \gamma_i)},\\
&i = 1,2,\ldots,m,\ l = 2,3,\ldots,n\Big\}
\end{aligned}
\tag{18}
$$

is a forward invariant compact absorbing set for system (15) for $\epsilon > 0$ and that the set Γ_0 (i.e., when $\epsilon = 0$) is a forward invariant compact set.

Under the assumption (A_1), we know that system (15) always has the disease-free equilibrium

$$
\begin{aligned}
P_0 = &\left(S_1^0, 0, \ldots, 0, I_1^0, S_2^0, 0, \ldots, 0, I_2^0, \ldots, S_m^0, 0, \ldots, 0, I_m^0\right)\\
&\in \mathrm{R}^{m(n+2)}.
\end{aligned}
\tag{19}
$$

An equilibrium P^* of (6) has the form $P^* = (S_1^*, I_1^*, S_2^*, I_2^*, \ldots, S_m^*, I_m^*) \in \mathrm{R}^{2m}$ with $S_i^* > 0$, $I_i^* > 0$, $i = 1,\ldots,m$. Translating to the equivalent system (15), P^* is corresponding to

$$
\begin{aligned}
\overline{P}^* = &\left(S_1^*, y_{1,1}^*, \ldots, y_{1,n}^*, I_1^*, S_2^*, y_{2,1}^*, \ldots, y_{2,n}^*, I_2^*, \ldots,\right.\\
&\left. S_m^*, y_{m,1}^*, \ldots, y_{m,n}^*, I_m^*\right) \in \mathrm{R}^{m(n+2)}.
\end{aligned}
\tag{20}
$$

$\overline{P}^*$ in the interior of Γ_0 is called an endemic equilibrium, and it satisfies the following equilibrium equations:

$$
\begin{aligned}
0 &= \varphi_i(S_i^*) - \sum_{j=1}^{m}\beta_{ij}f_{ij}(S_i^*, I_j^*),\\
0 &= \frac{1}{(1+\delta b)^n}\sum_{j=1}^{m}\beta_{ij}f_{ij}(S_i^*, I_j^*) - \frac{1}{\hat{b}}y_{i,1}^*,\\
0 &= \frac{1}{\hat{b}}(y_{i,1}^* - y_{i,2}^*),\\
&\vdots\\
0 &= \frac{1}{\hat{b}}(y_{i,n-1}^* - y_{i,n}^*),\\
0 &= \frac{1}{\hat{b}}y_{i,n}^* - (\delta + \varepsilon_i + \gamma_i)I_i^*.
\end{aligned}
\tag{21}
$$

The basic reproduction number R_0 is defined as the expected number of secondary cases produced by single infectious individual during its entire period of infectiousness in a completely susceptible population. For system (15), we can calculate it as the spectral radius of a matrix called the next generation matrix. Let

$$
\begin{aligned}
\mathscr{F} &= \begin{pmatrix}
\dfrac{c_{11}(S_1^0)\beta_{11}}{(1+\delta b)^n} & \cdots & \dfrac{c_{1m}(S_1^0)\beta_{1m}}{(1+\delta b)^n}\\
\vdots & \ddots & \vdots\\
\dfrac{c_{m1}(S_m^0)\beta_{m1}}{(1+\delta b)^n} & \cdots & \dfrac{c_{mm}(S_m^0)\beta_{mm}}{(1+\delta b)^n}
\end{pmatrix},\\
\mathscr{V} &= \operatorname{diag}(\delta + \varepsilon_i + \gamma_i)\\
&= \begin{pmatrix}
\delta + \varepsilon_1 + \gamma_1 & 0 & \cdots & 0\\
0 & \delta + \varepsilon_2 + \gamma_2 & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots\\
0 & 0 & \cdots & \delta + \varepsilon_m + \gamma_m
\end{pmatrix}.
\end{aligned}
\tag{22}
$$

Then the next generation matrix is

$$\mathscr{F}\mathscr{V}^{-1} = \begin{pmatrix} \frac{c_{11}(S_1^0)\beta_{11}}{(1+\delta b)^n(\delta+\varepsilon_1+\gamma_1)} & \cdots & \frac{c_{1m}(S_1^0)\beta_{1m}}{(1+\delta b)^n(\delta+\varepsilon_m+\gamma_m)} \\ \vdots & \ddots & \vdots \\ \frac{c_{m1}(S_1^0)\beta_{m1}}{(1+\delta b)^n(\delta+\varepsilon_1+\gamma_1)} & \cdots & \frac{c_{mm}(S_1^0)\beta_{mm}}{(1+\delta b)^n(\delta+\varepsilon_m+\gamma_m)} \end{pmatrix}, \tag{23}$$

and hence, the basic reproduction number R_0 is

$$R_0 = \rho\left(\mathscr{F}\mathscr{V}^{-1}\right) = \max\left\{|\lambda| ; \lambda \in \sigma\left(\mathscr{F}\mathscr{V}^{-1}\right)\right\}, \tag{24}$$

where $\rho(\cdot)$ and $\sigma(\cdot)$ denote the spectral radius and the set of eigenvalues of a matrix, respectively. Since it can be verified that system (15) satisfies conditions (A_1)–(A_5) of Theorem 2 of [17], we have the following proposition.

Lemma 1. *For system* (15)*, the disease-free equilibrium P_0 is locally asymptotically stable if $R_0 < 1$, while it is unstable if $R_0 > 1$.*

Following the method of [2], one defines a matrix

$$\begin{aligned} M^0 &= \mathscr{V}^{-1}\mathscr{F} \\ &= \begin{pmatrix} \frac{c_{11}(S_1^0)\beta_{11}}{(1+\delta b)^n(\delta+\varepsilon_1+\gamma_1)} & \cdots & \frac{c_{1m}(S_1^0)\beta_{1m}}{(1+\delta b)^n(\delta+\varepsilon_1+\gamma_1)} \\ \vdots & \ddots & \vdots \\ \frac{c_{m1}(S_1^0)\beta_{m1}}{(1+\delta b)^n(\delta+\varepsilon_m+\gamma_m)} & \cdots & \frac{c_{mm}(S_1^0)\beta_{mm}}{(1+\delta b)^n(\delta+\varepsilon_m+\gamma_m)} \end{pmatrix}, \end{aligned} \tag{25}$$

whose spectral radius has a similar threshold property to that of R_0, since both of the nonnegative matrices $\mathscr{F}\mathscr{V}^{-1}$ and M^0 are irreducible, and hence from the Perron-Frobenius theorem [7] that their spectral radii are given by each of their simple eigenvalues. Thus, we obtain $R_0 = \rho(\mathscr{F}\mathscr{V}^{-1}) = \rho(\mathscr{V}^{-1}\mathscr{F}) = \rho(M^0)$. Then the following lemma immediately follows.

Lemma 2. $\rho(M^0) \le 1$ *if and only if* $R_0 \le 1$.

3. Main Results

The following main theorems are summarized in terms of system (15).

Theorem 3. *Assume that the functions φ_i and f_{ij} satisfy assumptions (A_1)–(A_3), and the matrix $B = (\beta_{ij})_{m\times m}$ is irreducible and R_0 is defined by* (24)*.*

(i) *If $R_0 \le 1$, then P_0 is the unique equilibrium of system* (15)*, and P_0 is globally asymptotically stable in Γ_0.*

(ii) *If $R_0 > 1$, then P_0 is unstable, and system* (15) *is uniformly persistent in Γ_0.*

Proof. Let us define $M(S) = (\beta_{ij}c_{ij}(S_i)/(1+\delta b)^n(\delta+\varepsilon_i+\gamma_i))_{m\times m}$, where $S = (S_1, S_2, \ldots, S_m)^T$. Note that $M(S_0) = M^0$. Since $B = (\beta_{ij})_{m\times m}$ is irreducible, the matrix M^0 is also irreducible.

First we claim that there does not exist any endemic equilibrium $\overline{P}^*$ in Ω. Suppose that $S \neq S_0$. Then we have $0 < M(S) < M^0$. Since nonnegative matrix $M(S) + M^0$ is irreducible, it follows from the Perron-Frobenius theorem (see Corollary 2.1.5 of [7]) that $\rho(M(S)) < \rho(M^0) \le 1$. This implies that $M(S)I = I$ has only the trivial solution $I = 0$, where $I = (I_1, \ldots, I_m)^T$. Hence the claim is true. Next we claim that the disease-free equilibrium P_0 is globally asymptotically stable in Γ_0. From the Perron-Frobenius (see Theorem 2.1.4 of [7]), the nonnegative irreducible matrix M^0 has a strictly positive left eigenvector $(\omega_1, \omega_2, \ldots, \omega_m)$ associated with the eigenvalue $\rho(M^0)$ such that

$$(\omega_1, \omega_2, \ldots, \omega_m)\,\rho\left(M^0\right) = (\omega_1, \omega_2, \ldots, \omega_m)\, M^0. \tag{26}$$

Using this positive eigenvector, we construct the following Lyapunov function:

$$V_{\mathrm{DFE}} = \sum_{i=1}^{m} \frac{\omega_i}{\delta+\varepsilon_i+\gamma_i}\left(\sum_{j=1}^{n} y_{i,j} + I_i\right). \tag{27}$$

Computing the derivative of V_{DFE} along the solutions of (15) in Γ_0, we get

$$\begin{aligned} V'_{\mathrm{DFE}} &= \sum_{i=1}^{m}\left[\sum_{j=1}^{m} \frac{\omega_i\beta_{ij}}{(1+\delta b)^n(\delta+\varepsilon_i+\gamma_i)} f_{ij}\left(S_i, I_j\right) - \omega_i I_i\right] \\ &\le \sum_{i=1}^{m}\left[\sum_{j=1}^{m} \frac{\omega_i\beta_{ij}c_{ij}(S_i)}{(1+\delta b)^n(\delta+\varepsilon_i+\gamma_i)} I_j - \omega_i I_i\right] \\ &\le \sum_{i=1}^{m}\left[\sum_{j=1}^{m} \frac{\omega_i\beta_{ij}c_{ij}(S_i^0)}{(1+\delta b)^n(\delta+\varepsilon_i+\gamma_i)} I_j - \omega_i I_i\right] \\ &= (\omega_1, \ldots, \omega_m)\left[M^0 I - I\right] \\ &= \left[\rho\left(M^0\right) - 1\right](\omega_1, \ldots, \omega_m)\, I. \end{aligned} \tag{28}$$

Thus, under the assumption $R_0 = \rho(M^0) < 1$, $V'_{\mathrm{DFE}} \le 0$, and $V'_{\mathrm{DFE}} = 0$ if and only if $I = 0$ and $S = S^0 = (S_1^0, S_2^0, \ldots, S_m^0)$. Suppose that $\rho(M^0) = 1$. Then it follows from the previous that $V'_{\mathrm{DFE}} = 0$ implies

$$(\omega_1, \ldots, \omega_m)\, M^0 I = (\omega_1, \ldots, \omega_m)\, I. \tag{29}$$

Hence, if $S \neq S_0$, then $(\omega_1, \ldots, \omega_m)M(S) < (\omega_1, \ldots, \omega_m)M^0 = \rho(M^0)(\omega_1, \ldots, \omega_m) = (\omega_1, \ldots, \omega_m)$ and thus $I = 0$ is the only solution of (29). Summarizing the statements, we see that $V'_{\mathrm{DFE}} = 0$ if and only if $I = 0$ or $S = S_0$, which implies that the compact invariant subset of the set where $V'_{\mathrm{DFE}} = 0$ is only the singleton P_0. Thus, by LaSalle's invariance principle

[18], it follows that the disease-free equilibrium E^0 is globally asymptotically stable in Γ_0.

If $R_0 = \rho(M^0) > 1$, then

$$(\omega_1, \omega_2, \ldots, \omega_m) M^0 - (\omega_1, \omega_2, \ldots, \omega_m) = \left[\rho\left(M^0\right) - 1\right](\omega_1, \omega_2, \ldots, \omega_m) > 0, \tag{30}$$

and then, by continuity, we can obtain

$$V'_{\mathrm{DFE}} = (\omega_1, \ldots, \omega_m)\left[M^0 I - I\right] > 0, \tag{31}$$

in a neighborhood of P_0 in Γ_0; then P_0 is unstable.

Assume $R_0 = \rho(M^0) > 1$. By the uniform persistence result from [19] and a similar argument as in the proof of [2], the instability of P_0 implies the uniform persistence of (15). This together with the dissipativity of (15) resulted from the forward invariant and compact property of Γ_0 stated previously, implies which that (15) has an equilibrium in Γ_0, denoted by $\overline{P}^*$ (see, e.g., Theorem D.3 in [20]). □

In what follows we prove that the endemic equilibrium $\overline{P}^*$ of system (15) is globally asymptotically stable when $R_0 > 1$.

Throughout the paper, we denote

$$H(z) = z - 1 - \ln z. \tag{32}$$

Then $H(z) \geq 0$ for $z > 0$ and has global minimum at $z = 1$.

For convenience of notations, set $\overline{\beta}_{ij} = \beta_{ij} f_{ij}(S_i^*, I_j^*)$, $1 \leq i, j \leq m$, and

$$\overline{B} = \begin{bmatrix} \sum\limits_{l\neq 1}\overline{\beta}_{1l} & -\overline{\beta}_{21} & \cdots & -\overline{\beta}_{ml} \\ -\overline{\beta}_{12} & \sum\limits_{l\neq 2}\overline{\beta}_{2l} & \cdots & -\overline{\beta}_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ -\overline{\beta}_{1m} & -\overline{\beta}_{2m} & \cdots & \sum\limits_{l\neq m}\overline{\beta}_{ml} \end{bmatrix}. \tag{33}$$

Then, $\overline{B}$ is also irreducible. One knows that the solution space of the linear system

$$\overline{B}v = 0 \tag{34}$$

has dimension 1 and

$$(v_1, v_2, \ldots v_m) = (C_{11}, \ldots, C_{mm}) \tag{35}$$

gives a base of this space, where $C_{kk} > 0$, $k = 1, 2, \ldots, m$, is the cofactor of the kth diagonal entry of $\overline{B}$. To get the global stability of $\overline{P}^*$, the following assumptions in [15] are proposed:

(A_4) : $[\varphi_i(S_i) - \varphi_i(S_i^*)](S_i - S_i^*) < 0$ for $S_i \neq S_i^*$, $S_i \in [0, S_i^0]$,

(A_5) : For $S_i \neq S_i^*$, $[\varphi_i(S_i) - \varphi_i(S_i^*)] \cdot [f_{ii}(S_i, I_i^*) - f_{ii}(S_i^*, I_i^*)] < 0$.

(A_6) : For $S_i, I_j > 0$,

$$\left(f_{ii}\left(S_i^*, I_i^*\right) f_{ij}\left(S_i, I_j\right) - f_{ii}\left(S_i, I_i^*\right) f_{ij}\left(S_i^*, I_j^*\right)\right) \cdot \left(\frac{f_{ii}\left(S_i^*, I_i^*\right) f_{ij}\left(S_i, I_j\right)}{I_j} - \frac{f_{ii}\left(S_i, I_i^*\right) f_{ij}\left(S_i^*, I_j^*\right)}{I_j^*}\right) \leq 0. \tag{36}$$

A difficult mathematical question for system (15) is that of whether the endemic equilibrium $\overline{P}^*$ is unique when $R_0 > 1$ and whether $\overline{P}^*$ is globally asymptotically stable when it is unique. Our main global stability result is given.

Theorem 4. *Consider system* (15). *Assume that* (A_4)–(A_6) *hold and the matrix* $B = (\beta_{ij})_{m\times m}$ *is irreducible. If* $R_0 > 1$, *then there is a unique endemic equilibrium* $\overline{P}^*$ *for system* (15), *and* $\overline{P}^*$ *is globally asymptotically stable in* Γ_0.

Proof. We show that $\overline{P}^*$ is globally asymptotically stable in Γ_0, which implies that there exists a unique endemic equilibrium.

Consider a Lyapunov function as

$$V_{\mathrm{EE}} = S_i - f_{ii}\left(S_i^*, I_i^*\right)\int_{S_i^*}^{S_i}\frac{d\xi}{f_{ii}\left(\xi, I_i^*\right)} + (1+\delta b)^n\left[\sum_{j=1}^{n}\left(y_{i,j} - y_{i,j}^* - y_{i,j}^*\ln\frac{y_{i,j}}{y_{i,j}^*}\right) + I_i - I_i^* - I_i^*\ln\frac{I_i}{I_i^*}\right]. \tag{37}$$

This function has a linear part V_{EE} expressed by

$$L_{\mathrm{EE}} = S_i + (1+\delta b)^n\left[\sum_{j=1}^{n}\left(y_{i,j} - y_{i,j}^*\right) + I_i - I_i^*\right]. \tag{38}$$

First, calculating the derivatives of L_{EE}, we obtain

$$L'_{\mathrm{EE}} = \varphi_i\left(S_i\right) - (1+\delta b)^n\left(\delta + \varepsilon_i + \gamma_i\right) I_i. \tag{39}$$

Calculating the time derivative of V_{EE} along the solutions of system (15) and using equilibrium equation (21), we have

$$\begin{aligned} V'_{\mathrm{EE}} &= L'_{\mathrm{EE}} - \frac{f_{ii}\left(S_i^*, I_i^*\right)}{f_{ii}\left(S_i, I_i^*\right)}\dot{S}_i + (1+\delta b)^n\left[\sum_{j=1}^{n}\frac{y_{i,j}^*}{y_{i,j}}\dot{y}_{i,j} + \frac{I_i^*}{I_i}\dot{I}_i\right] \\ &= \varphi_i\left(S_i\right) - (1+\delta b)^n\left(\delta + \varepsilon_i + \gamma_i\right) I_i \\ &\quad - \left\{\left(\varphi_i\left(S_i\right) - \sum_{j=1}^{m}\beta_{ij} f_{ij}\left(S_i, I_j\right)\right)\frac{f_{ii}\left(S_i^*, I_i^*\right)}{f_{ii}\left(S_i, I_i^*\right)}\right. \end{aligned}$$

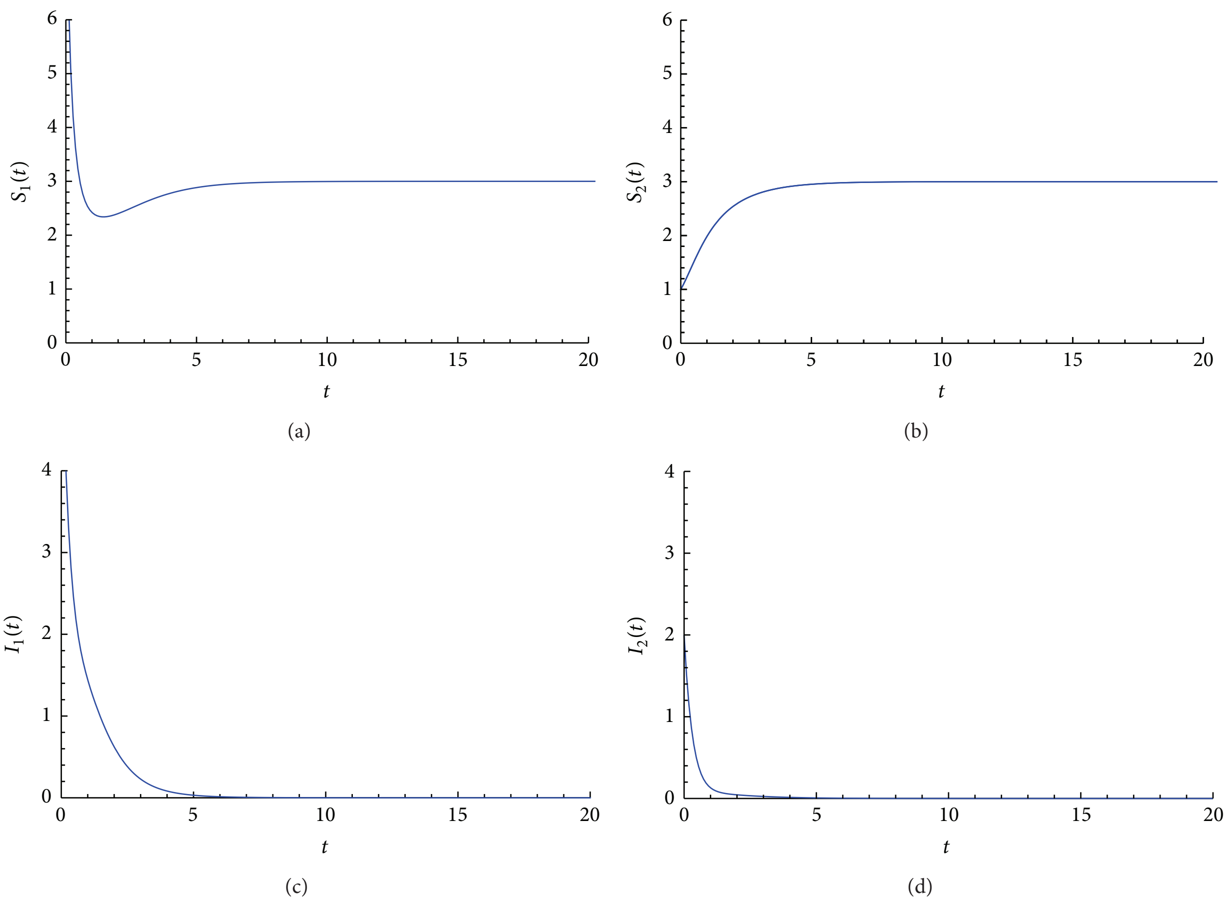


FIGURE 1: Trajectories of $S_1(t), I_1(t), S_2(t)$, and $I_2(t)$ for $R_0 = 0.051 < 1$, and $P_0 = (3, 0, 0, 0, 3, 0, 0, 0)$ is globally stable. $S_1(t)$, $S_2(t)$, $I_1(t)$, and $I_2(t)$ versus t are illustrated by (a), (b), (c), and (d). Initial values are $S_1(0) = 9, S_2(0) = 1, y_{1,1}(0) = 2, y_{1,2}(0) = 2, y_{2,1}(0) = 0, y_{2,2}(0) = 0, I_1(0) = 6$, and $I_2(0) = 2$.

$$
\begin{aligned}
&+ (1+\delta b)^n \Bigg[\sum_{j=1}^{m} \frac{\beta_{ij} f_{ij}\left(S_i, I_j\right) y_{i,1}^*}{(1+\delta b)^n y_{i,1}} - \frac{y_{i,1}^*}{\widehat{b}} + \frac{1}{\widehat{b}} \sum_{k=2}^{n} y_{i,k}^* \left(\frac{y_{i,k-1}}{y_{i,k}} - 1 \right) + \frac{y_{i,n} I_i^*}{\widehat{b} I_i} - \left(\delta + \varepsilon_i + \gamma_i\right) I_i^* \Bigg] \Bigg\} \\
&= \varphi_i\left(S_i\right) \left(1 - \frac{f_{ii}\left(S_i^*, I_i^*\right)}{f_{ii}\left(S_i, I_i^*\right)} \right) - \sum_{j=1}^{m} \beta_{ij} f_{ij}\left(S_i^*, I_j^*\right) \frac{y_{i,1}^* f_{ij}\left(S_i, I_j\right)}{y_{i,1} f_{ij}\left(S_i^*, I_j^*\right)} - \frac{(1+\delta b)^n}{\widehat{b}} \sum_{k=2}^{n} \frac{y_{i,k}^* y_{i,k-1}}{y_{i,k}} \\
&\quad + \frac{(1+\delta b)^n}{\widehat{b}} n y_i^* - \frac{(1+\delta b)^n}{\widehat{b}} y_{i,n}^* \frac{y_{i,n} I_i^*}{y_{i,n}^* I_i} + (1+\delta b)^n \left(\delta + \varepsilon_i + \gamma_i\right) I_i^* \\
&\quad + \sum_{j=1}^{m} \beta_{ij} f_{ij}\left(S_i^*, I_j^*\right) \frac{f_{ij}\left(S_i, I_j\right) f_{ii}\left(S_i^*, I_i^*\right)}{f_{ij}\left(S_i^*, I_j^*\right) f_{ii}\left(S_i, I_i^*\right)} - (1+\delta b)^n \left(\delta + \varepsilon_i + \gamma_i\right) I_i \\
&= \left(\varphi_i\left(S_i\right) - \varphi_i\left(S_i^*\right)\right) \left(1 - \frac{f_{ii}\left(S_i^*, I_i^*\right)}{f_{ii}\left(S_i, I_i^*\right)} \right) + \sum_{j=1}^{m} \overline{\beta}_{ij} \Bigg\{ n + 2 - \frac{f_{ii}\left(S_i^*, I_i^*\right)}{f_{ii}\left(S_i, I_i^*\right)} - \sum_{k=2}^{n} \frac{y_{i,k}^* y_{i,k-1}}{y_{i,k} y_{i,k-1}^*} \\
&\quad - \frac{y_{i,n} I_i^*}{y_{i,n}^* I_i} - \frac{I_i}{I_i^*} - \frac{f_{ij}\left(S_i, I_j\right) y_{i,1}^*}{f_{ij}\left(S_i^*, I_j^*\right) y_{i,1}} + \frac{f_{ij}\left(S_i, I_j\right) f_{ii}\left(S_i^*, I_i^*\right)}{f_{ij}\left(S_i^*, I_j^*\right) f_{ii}\left(S_i, I_i^*\right)} \Bigg\}.
\end{aligned}
\tag{40}
$$

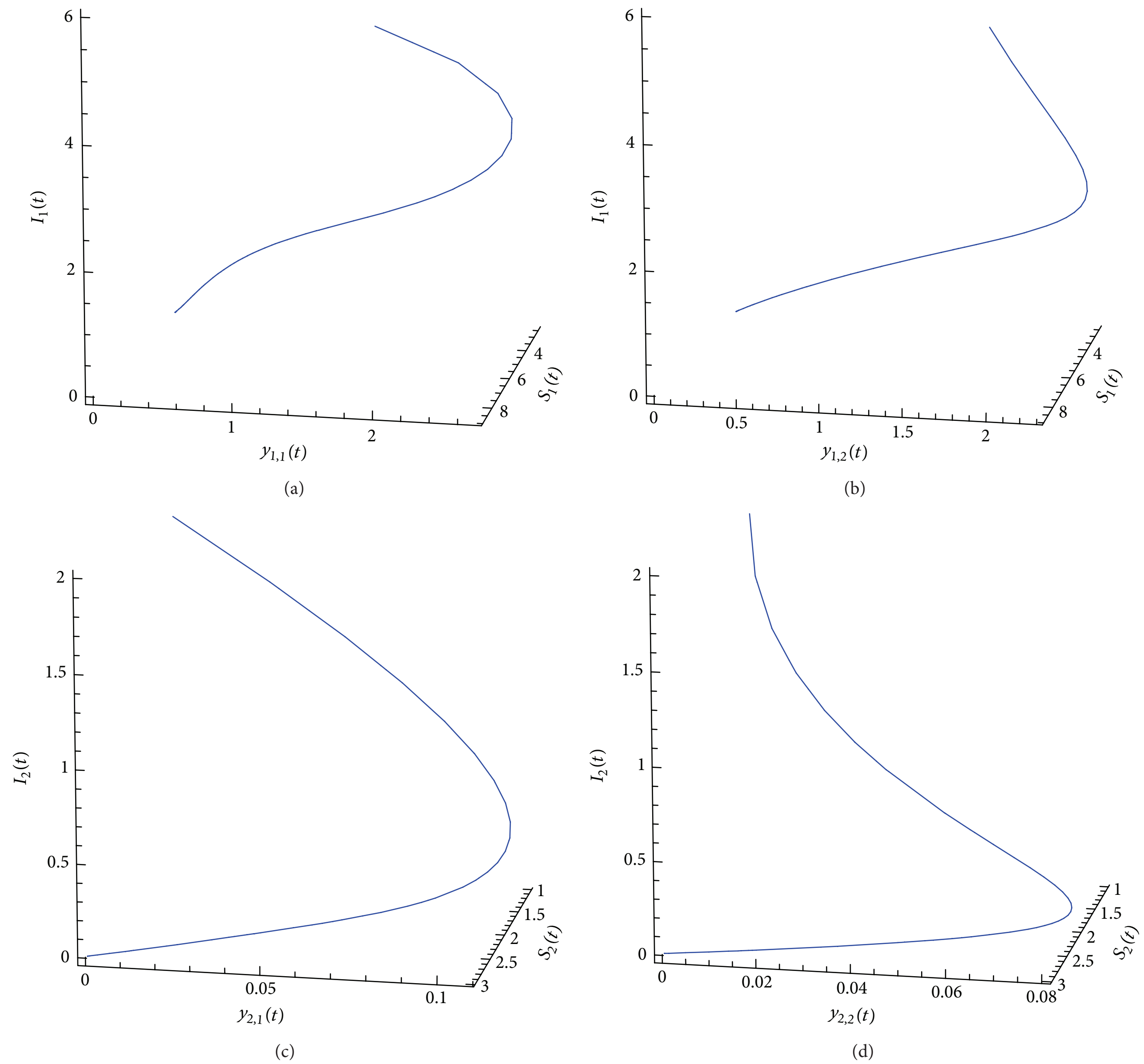


FIGURE 2: Numerical simulation of (45) with $R_0 = 0.051 < 1$; hence $P_0 = (3, 0, 0, 0, 3, 0, 0, 0)$ is globally stable. Graphs (a) and (b) illustrate that $S_1(\mathrm{t})$, $y_{1,1}(t)$, $y_{1,2}(t)$ and $I_1(t)$ will eventually towards to steady state. Graphs (c) and (d) illustrate that $S_2(t)$, $y_{2,1}(t)$, $y_{2,2}(t)$, and $I_2(t)$ will eventually towards to steady state. Initial values are $S_1(0) = 9$, $S_2(0) = 1$, $y_{1,1}(0) = 2$, $y_{1,2}(0) = 2$, $y_{2,1}(0) = 0$, $y_{2,2}(0) = 0$, $I_1(0) = 6$, and $I_2(0) = 2$.

It follows from the assumptions (A_4)-(A_5) that V_{EE}' can be estimated by

$$
\begin{aligned}
V_{\mathrm{EE}}' \le \sum_{i,j=1}^{m} \overline{\beta}_{ij} \Bigg\{ & G_i(I_i) - G_j(I_j) + H\left(\frac{f_{ii}(S_i^*, I_i^*)}{f_{ii}(S_i, I_i^*)}\right) \\
& + H\left(\frac{f_{ij}(S_i, I_j)\, y_{i,1}^*}{f_{ij}(S_i^*, I_j^*)\, y_{i,1}}\right) \\
& + \sum_{k=2}^{n} H\left(\frac{y_{i,k}^* y_{i,k-1}}{y_{i,k} y_{i,k-1}^*}\right) \\
& + H\left(\frac{y_{i,n} I_i^*}{y_{i,n}^* I_i}\right) \\
& + H\left(\frac{I_j f_{ii}(S_i, I_i^*)\, f_{ij}(S_i^*, I_j^*)}{I_j^* f_{ii}(S_i^*, I_i^*)\, f_{ij}(S_i, I_j)}\right)
\end{aligned}
$$

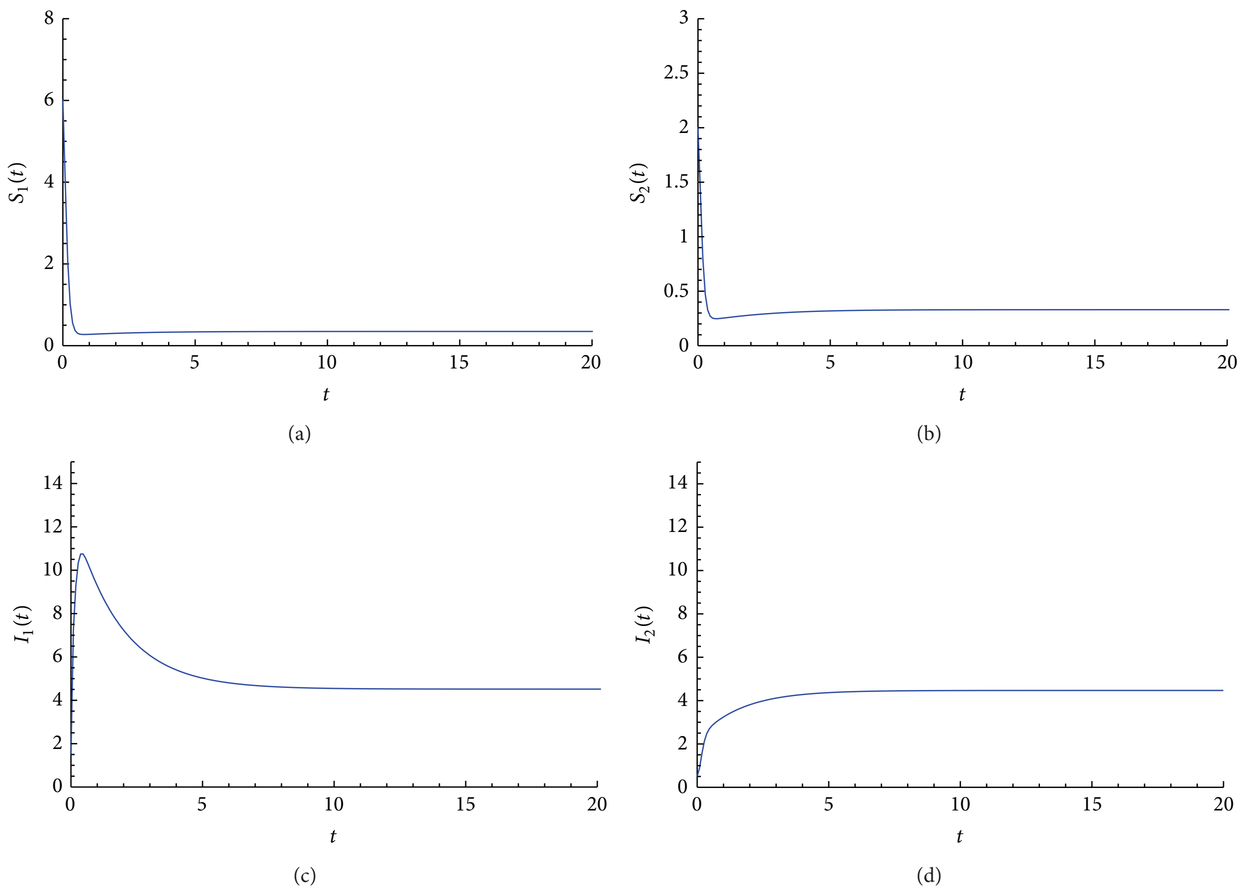


FIGURE 3: Trajectories of $S_1(t), I_1(t), S_2(t)$, and $I_2(t)$ for $R_0 = 1.67355 > 1$, and $\overline{P}^* = (0.347644, 0.0760948, 0.0760948, 4.51674, 0.330353, 0.0765909, 0.0765909, 4.4678)$ is globally stable. $S_1(t)$, $S_2(t)$, $I_1(t)$, and $I_2(t)$ versus t are illustrated by (a), (b), (c), and (d). Initial values are $S_1(0) = 6, S_2(0) = 2, y_{1,1}(0) = 3, y_{1,2}(0) = 3, y_{2,1}(0) = 0.1, y_{2,2}(0) = 0.1, I_1(0) = 1.5$, and $I_2(0) = 0.5$.

$$+\left[\frac{f_{ii}(S_i^*, I_i^*) f_{ij}(S_i, I_j)}{f_{ii}(S_i, I_i^*) f_{ij}(S_i^*, I_j^*)} - 1\right] \cdot \left[1 - \frac{I_j f_{ii}(S_i, I_i^*) f_{ij}(S_i^*, I_j^*)}{I_j^* f_{ii}(S_i^*, I_i^*) f_{ij}(S_i, I_j)}\right]\Bigg\}. \tag{41}$$

From the assumption (A_6) and (32), we know that

$$V_{\mathrm{EE}}' \leq \sum_{i,j=1}^{m} \overline{\beta}_{ij} \left\{G_i(I_i) - G_j(I_j)\right\}, \tag{42}$$

where $G_i(I_i) = -I_i/I_i^* + \ln(I_i/I_i^*)$.

Obviously, the equalities in (41) and (42) hold if and only if

$$\frac{f_{ii}(S_i^*, I_i^*)}{f_{ii}(S_i, I_i^*)} = 1,$$
$$\left(1 - \frac{f_{ii}(S_i^*, I_i^*)}{f_{ii}(S_i, I_i^*)}\right)[\varphi_i(S_i) - \varphi_i(S_i^*)] = 0,$$
$$\left[\frac{f_{ij}(S_i, I_j) f_{ii}(S_i^*, I_i^*)}{f_{ij}(S_i^*, I_j^*) f_{ii}(S_i, I_i^*)} - 1\right] \times \left[1 - \frac{f_{ii}(S_i, I_i^*) f_{ij}(S_i^*, I_j^*) I_j}{f_{ii}(S_i, I_i^*) f_{ij}(S_i, I_j) I_j^*}\right] = 0. \tag{43}$$

That is, $S_i = S_i^*$, $I_i = I_i^*$, $i = 1, 2, \ldots, m$. We can show that V_{EE} and $\overline{\beta}_{ij}$ satisfy the assumptions of Theorem 3.1 and Corollary 3.3 in [21]. Therefore, the function

$$L = \sum_{i=1}^{n} v_i V_{\mathrm{EE}} \tag{44}$$

is a Lyapunov function for system (15); namely, $L'|_{(15)} \leq 0$ for $\overline{P}^* \in \Gamma_0$. One can only show that the largest invariant subset, where $L'|_{(15)} = 0$, is the singleton $\{\overline{P}^*\}$ by the same

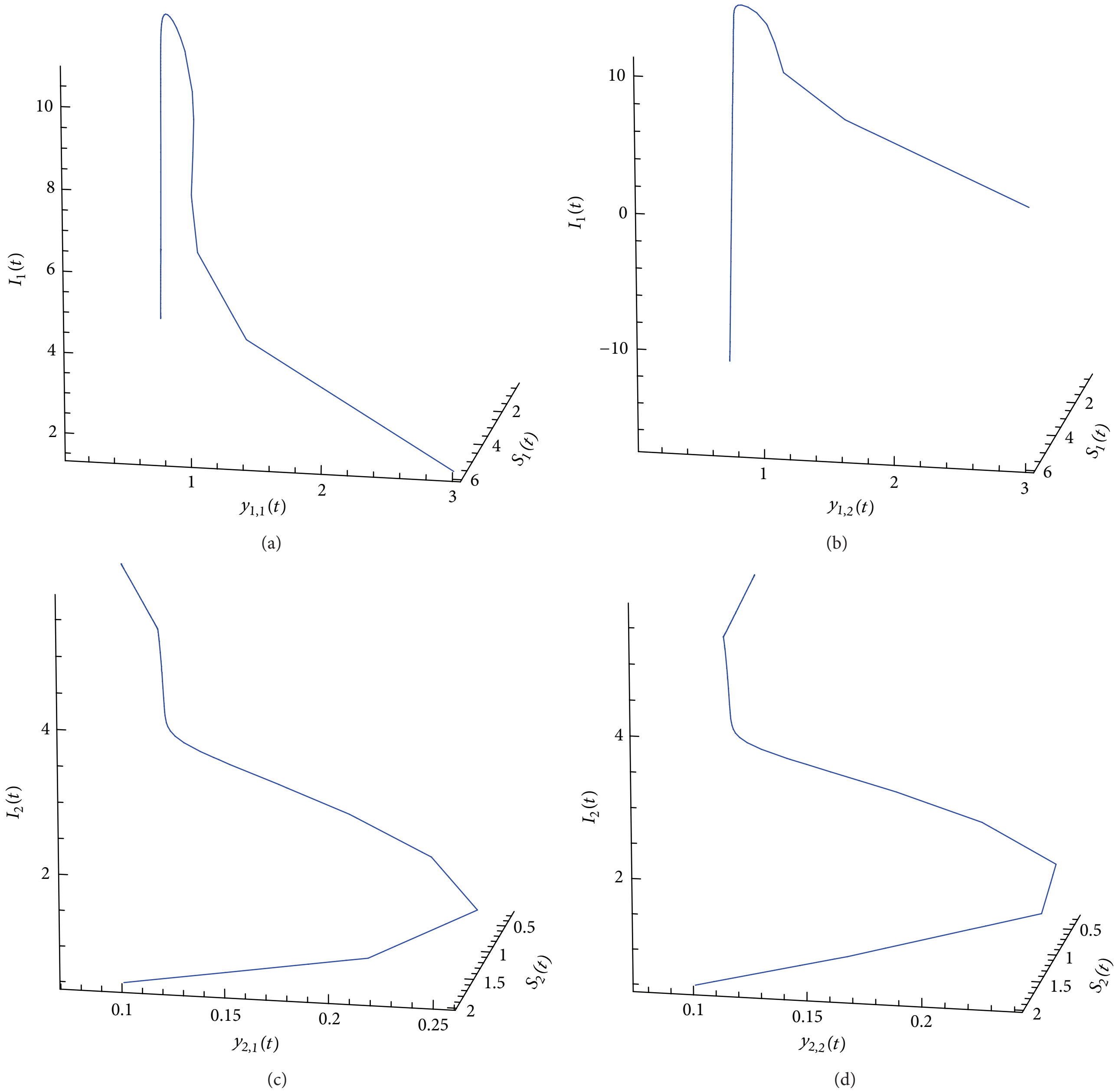


FIGURE 4: Numerical simulation of (45) with $R_0 = 1.67355 > 1$; hence $P^* = (0.347644,\ 0.0760948,\ 0.0760948,\ 4.51674,\ 0.330353,\ 0.0765909,\ 0.0765909,\ 4.4678)$ is globally stable. Graphs (a) and (b) illustrate that $S_1(t)$, $y_{1,1}(t)$, $y_{1,2}(t)$, and $I_1(t)$ will eventually towards to steady state. Graphs (c) and (d) illustrate that $S_2(t)$, $y_{2,1}(t)$, $y_{2,2}(t)$, and $I_2(t)$ will eventually towards to steady state. Initial values are $S_1(0) = 6, S_2(0) = 2$, $y_{1,1}(0) = 3$, $y_{1,2}(0) = 3$, $y_{2,1}(0) = 0.1$, $y_{2,2}(0) = 0.1$, $I_1(0) = 1.5$, and $I_2(0) = 0.5$.

argument as in [2–5, 13, 21]. By LaSalle's invariance principle, $\overline{P}^*$ is globally asymptotically stable in Γ_0. This completes the proof of Theorem 4. □

Remark 5. We show a complete proof for global asymptotic stability of unique endemic equilibrium of system (15). In the case of $f_{ij}(S_i, I_j) = S_i I_j$, system (15) will reduce to the system studied in [14, 22]. Here Theorem 4 extends related results in [14, 22] to a result to a more general case allowing a nonlinear incidence rate. Our result also cover the related results of single group model in [13] for the case of $f(S, I) = f(S)I$.

4. Numerical Example

Consider the system (15) when $m = 2$, $n = 2$, $\varphi_i(S_i(t)) = 3 - S_i$, and $f_{ij}(S_i, I_j) = S_i I_j$, $i, j = 1, 2$. One then has a two-group model as follows:

$$S_1'(t) = 3 - S_1 - [\beta_{11} S_1(t) I_1(t) + \beta_{12} S_1(t) I_2(t)],$$

$$y_{1,1}'(t) = \frac{1}{(1+\delta b)^n} [\beta_{11} S_1(t) I_1(t) + \beta_{12} S_1(t) I_2(t)] - \frac{1}{\overline{b}} y_{1,1}(t),$$

$$y_{1,2}'(t) = \frac{1}{\hat{b}}\left(y_{1,1}(t) - y_{1,2}(t)\right),$$

$$I_1'(t) = \frac{1}{\hat{b}}y_{1,2}(t) - \left(\delta + \varepsilon_1 + \gamma_1\right)I_1(t),$$

$$S_2'(t) = 3 - S_2 - \left[\beta_{21}S_2(t)I_1(t) + \beta_{22}S_2(t)I_2(t)\right],$$

$$y_{2,1}'(t) = \frac{1}{(1+\delta b)^n}\left[\beta_{21}S_2(t)I_1(t) + \beta_{22}S_2(t)I_2(t)\right] - \frac{1}{\hat{b}}y_{2,1}(t),$$

$$y_{2,2}'(t) = \frac{1}{\hat{b}}\left(y_{2,1}(t) - y_{2,2}(t)\right),$$

$$I_2'(t) = \frac{1}{\hat{b}}y_{2,2}(t) - \left(\delta + \varepsilon_2 + \gamma_2\right)I_2(t). \tag{45}$$

If we choose parameters as $\beta_{11} = 5/24$, $\beta_{12} = 1$, $\beta_{21} = 1/36$, $\beta_{22} = 1/2$, $\delta = 0.8$, $\varepsilon_1 = 2$, $\varepsilon_2 = 2$, $\gamma_1 = 1/4$, and $\gamma_2 = 1/4$, we can compute $R_0 = 0.051 < 1$, and hence $P_0 = (3, 0, 0, 0, 3, 0, 0, 0)$ is the unique equilibrium of system (45) and it is globally stable from Theorem 4 (see Figures 1 and 2).

On the other hand, if β_{ij} are chosen as $\beta_{11} = 0.7$, $\beta_{12} = 1$, $\beta_{21} = 0.8$, $\beta_{22} = 1$, $\delta = 0.5$, $\varepsilon_1 = 0.02$, $\varepsilon_2 = 0.03$, $\gamma_1 = 0.05$, and $\gamma_2 = 0, 05$, we can compute $R_0 = 1.67355 > 1$, and hence $\overline{P}^* = (0.347644, 0.0760948, 0.0760948, 4.51674, 0.330353, 0.0765909, 0.0765909, 4.4678)$ is the unique equilibrium of system (45) and it is globally stable from Theorem 4 (see Figures 3 and 4).

Acknowledgments

The authors wish to thank the reviewers for their valuable comments and suggestions that led to truly significant improvement of the paper. J. Wang is supported by the Science and Technology Research Project of the Department of Education of Heilongjiang Province (no. 12531495).

References

[1] H. R. Thieme, *Mathematics in Population Biology*, Princeton University Press, Princeton, NJ, USA, 2003.

[2] H. Guo, M. Y. Li, and Z. Shuai, "Global stability of the endemic equilibrium of multigroup SIR epidemic models," *Canadian Applied Mathematics Quarterly*, vol. 14, no. 3, pp. 259–284, 2006.

[3] H. Guo, M. Y. Li, and Z. Shuai, "A graph-theoretic approach to the method of global Lyapunov functions," *Proceedings of the American Mathematical Society*, vol. 136, no. 8, pp. 2793–2802, 2008.

[4] H. Shu, D. Fan, and J. Wei, "Global stability of multi-group SEIR epidemic models with distributed delays and nonlinear transmission," *Nonlinear Analysis: Real World Applications*, vol. 13, no. 4, pp. 1581–1592, 2012.

[5] J. Wang, Y. Takeuchi, and S. Liu, "A multi-group SVEIR epidemic model with distributed delay and vaccination," *International Journal of Biomathematics*, vol. 5, no. 3, Article ID 1260001, 18 pages, 2012.

[6] P. van den Driessche, L. Wang, and X. Zou, "Modeling diseases with latency and relapse," *Mathematical Biosciences and Engineering*, vol. 4, no. 2, pp. 205–219, 2007.

[7] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, NY, USA, 1979.

[8] A. Korobeinikov and P. K. Maini, "A Lyapunov function and global properties for SIR and SEIR epidemiological models with nonlinear incidence," *Mathematical Biosciences and Engineering*, vol. 1, no. 1, pp. 57–60, 2004.

[9] M. Y. Li and J. S. Muldowney, "Global stability for the SEIR model in epidemiology," *Mathematical Biosciences*, vol. 125, no. 2, pp. 155–164, 1995.

[10] G. Huang and Y. Takeuchi, "Global analysis on delay epidemiological dynamic models with nonlinear incidence," *Journal of Mathematical Biology*, vol. 63, no. 1, pp. 125–139, 2011.

[11] P. van den Driessche and X. Zou, "Modeling relapse in infectious diseases," *Mathematical Biosciences*, vol. 207, no. 1, pp. 89–103, 2007.

[12] P. van den Driessche, L. Wang, and X. Zou, "Impact of group mixing on disease dynamics," *Mathematical Biosciences*, vol. 228, no. 1, pp. 71–77, 2010.

[13] S. Liu, S. Wang, and L. Wang, "Global dynamics of delay epidemic models with nonlinear incidence rate and relapse," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 1, pp. 119–127, 2011.

[14] Z. Yuan and X. Zou, "Global threshold property in an epidemic model for disease with latency spreading in a heterogeneous host population," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 5, pp. 3479–3490, 2010.

[15] R. Sun and J. Shi, "Global stability of multigroup epidemic model with group mixing and nonlinear incidence rates," *Applied Mathematics and Computation*, vol. 218, no. 2, pp. 280–286, 2011.

[16] R. K. Miller, *Nonlinear Volterra Integral Equations*, W. A. Benjamin, New York, NY, USA, 1971.

[17] P. van den Driessche and J. Watmough, "Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission," *Mathematical Biosciences*, vol. 180, no. 1-2, pp. 29–48, 2002.

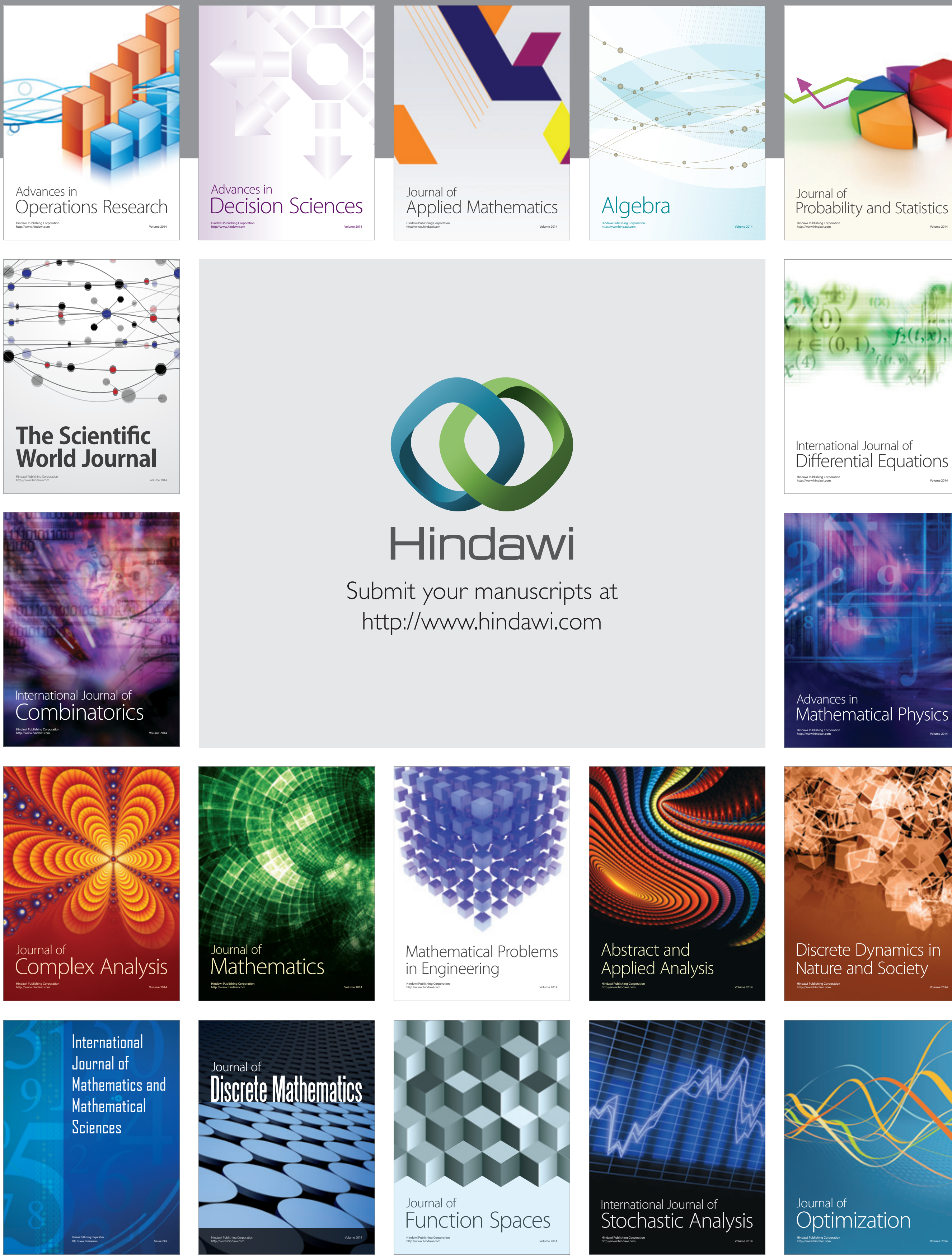
[18] J. P. Lasalle, *The Stability of Dynamical Systems*, Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, Pa, USA, 1976.

[19] H. I. Freedman, M. X. Tang, and S. G. Ruan, "Uniform persistence and flows near a closed positively invariant set," *Journal of Dynamics and Differential Equations*, vol. 6, no. 4, pp. 583–600, 1994.

[20] H. L. Smith and P. Waltman, *The Theory of the Chemostat: Dynamics of Microbial Competition*, Cambridge University Press, Cambridge, UK, 1995.

[21] M. Y. Li and Z. S. Shuai, "Global-stability problem for coupled systems of differential equations on networks," *Journal of Differential Equations*, vol. 248, no. 1, pp. 1–20, 2010.

[22] Z. Shuai and P. van den Driessche, "Impact of heterogeneity on the dynamics of an SEIR epidemic model," *Mathematical Biosciences and Engineering*, vol. 9, no. 2, pp. 393–411, 2012.


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
Algebra
Journal of
Probability and Statistics
The Scientific World Journal
International Journal of
Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
Mathematical Problems
in Engineering
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
International Journal of
Stochastic Analysis
Journal of
Optimization