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Research Article

Interval Oscillation Criteria for Second-Order Nonlinear Forced Dynamic Equations with Damping on Time Scales

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By using the Riccati transformation technique and constructing a class of Philos-type functions on time scales, we establish some new interval oscillation criteria for the second-order damped nonlinear dynamic equations with forced term of the form $(r(t)x^{\Delta}(t))^{\Delta} + p(t)x^{\Delta\sigma}(t) + q(t)(x^{\sigma}(t))^{\alpha} = F(t, x^{\sigma}(t))$ on a time scale $\mathbb{T}$ which is unbounded, where α is a quotient of odd positive integer. Our results in this paper extend and improve some known results. Some examples are given here to illustrate our main results.

1. Introduction

In this paper, we are concerned with the oscillation criteria for the following forced second-order nonlinear dynamic equations with damping:

$$\left(r(t)x^{\Delta}(t)\right)^{\Delta} + p(t)x^{\Delta\sigma}(t) + q(t)\left(x^{\sigma}(t)\right)^{\alpha} = F\left(t, x^{\sigma}(t)\right) \tag{1}$$

on a time scale $\mathbb{T}$, where α is a quotient of odd positive integer. Throughout this paper and without further mention, we assume that the functions $r, p, q \in C_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, $F \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$ with $r(t) > 0$, $p(t) \leq 0$, and $p/r^{\sigma} \in \mathscr{R}^{+}$.

The theory of time scales, which has recently received a lot of attention, was originally introduced by Stefan Hilger in his Ph.D. thesis in 1988 (see [1]). Since then a rapidly expanding body of the literature has sought to unify, extend, and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus, where a time scale is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications (see [2]). Not only does the new theory of the so-called dynamic equations unify the theories of differential equations and difference equations, but also it extends these classical cases to cases "in between", for example, to the so-called q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0, q > 1\}$ (which has important applications in quantum theory) and can be applied on different types of time scales like $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = \mathbb{N}^2$, and $\mathbb{T} = \mathbb{T}_n$, the space of the harmonic numbers. A book on the subject of time scales by Bohner and Peterson [2] summarizes and organizes much of the time scale calculus. For advances of dynamic equations on the time scales we refer the reader to the book [3].

Since we are interested in the oscillatory behavior of solutions near infinity, we make the assumption throughout this paper that the given time scale $\mathbb{T}$ is unbounded above. We assume $t_0 \in \mathbb{T}$ and it is convenient to assume $t_0 > 0$. We define the time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \bigcap \mathbb{T}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$.

By a solution of (1), we mean a nontrivial real-valued function x satisfying (1) on $[t_x, \infty)_{\mathbb{T}}$. A solution x of (1) is said to be oscillatory on $[t_x, \infty)_{\mathbb{T}}$ in case it is neither eventually positive nor eventually negative; otherwise, it is

nonoscillatory. Equation (1) is said to be oscillatory in case all its solutions are oscillatory. Our attention is restricted to those solutions of (1) which exist on some half line $[t_x, \infty)_{\mathbb{T}}$ and satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq t_x$.

In recent years, there has been much research activity concerning the interval oscillation criteria for various second order differential equations; see [4–9]. A great deal of effort has been spent in obtaining criteria for oscillation of dynamic equations on time scales without forcing terms and it is usually assumed that the potential function q is positive. We refer the reader to the papers [10–25] and the references cited therein. On the other hand, there has been an increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions of dynamic equations with forcing terms on time scales, and we refer the reader to the papers [26–35].

In 2004, by using two inequalities due to Hölder and Hardy and Littlewood and Polya as well as averaging functions, Li [4] established several interval oscillation criteria for the second order damped quasilinear differential equation with forced term of the following form:

$$\begin{aligned}&\left(r(t)\left|y'(t)\right|^{\alpha-1}y'(t)\right)' + p(t)\left|y'(t)\right|^{\alpha-1}y'(t)\\&\quad + q(t)\left|y(t)\right|^{\beta-1}y(t) = e(t),\end{aligned} \tag{2}$$

where $r \in C^1([t_0, \infty), \mathbb{R}^+)$, and $\beta > \alpha > 0$ are constants. The obtained results were based on the information only on a sequence of subintervals of $[t_0, \infty)$, rather than on the whole half line, made use of the oscillatory properties of the forcing term, and extended a known result which is obtained by means of a Picone identity.

Erbe et al. [26] studied the forced second-order nonlinear dynamic equation

$$\left(p(t)x^{\Delta}(t)\right)^{\Delta} + q(t)\left|x^{\sigma}(t)\right|^{\gamma}\operatorname{sgn} x^{\sigma}(t) = f(t) \tag{3}$$

on a time scale $\mathbb{T}$, where $\gamma \geq 1$. By using the Riccati substitution, the authors established some new interval oscillation criteria, that is, the criteria given by the behavior of q and f on a sequence of subintervals of $[a, \infty)_{\mathbb{T}}$.

In [31], by constructing a class of Philos-type functions on time scales, Li et al. established some oscillation criteria for the second order nonlinear dynamic equations with the forced term

$$x^{\Delta\Delta}(t) + a(t)f\left(x\left(q(t)\right)\right) = e(t) \tag{4}$$

on a time scale $\mathbb{T}$, where a, q, and e are real-valued rd-continuous functions defined on $\mathbb{T}$, with $q : T \to T, q(t) \to \infty$ as $t \to \infty$, and $f \to C(\mathbb{R}, \mathbb{R}), xf(x) > 0$ whenever $x \neq 0$. The obtained results unified the oscillation of the second order forced differential equation and the second order forced difference equation. An example was considered to illustrate the main results in the end.

Erbe et al. [32] were concerned with the oscillatory behavior of the forced second-order functional dynamic equation with mixed nonlinearities

$$\left(a(t)x^{\Delta}(t)\right)^{\Delta} + \sum_{i=0}^{n} p_i(t)\left|x\left(\tau_i(t)\right)\right|^{\alpha_i}\operatorname{sgn} x\left(\tau_i(t)\right) = e(t) \tag{5}$$

on an arbitrary time scale $\mathbb{T}$, where $\alpha_0 = 1$, $\alpha_1 > \alpha_2 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n$, and $\tau_i : \mathbb{T} \to \mathbb{T}$ are nondecreasing rd-continuous functions on $\mathbb{R}$, $\tau_i(t) \leq \sigma(t)$, and $\lim_{t\to\infty}\tau_i(t) = \infty$, for $i = 0, 1, \ldots, n$. Their results in a particular case solved a problem posed by Anderson, and their results in the special cases when the time scale is the set of real numbers and the set of integers involved and improved some oscillation results for second-order differential and difference equations, respectively.

In this paper, we intend to use the Riccati transformation technique to obtain some interval oscillation criteria for (1). Our results do not require that q and f be of definite sign and are based on the information only on a sequence of subintervals of $[t_0, \infty]_{\mathbb{T}}$ rather than the whole half line. To the best of our knowledge, nothing is known regarding the oscillation behavior of (1) on time scales until now, and there are few results regarding the interval oscillation criteria for (1) on time scales without the damping term when $\alpha < 1$, so our results expand the known scope of the study.

The paper is organized as follows. In Section 2, we present some basic definitions and useful results from the theory of calculus on time scales on which we rely in the later section. In Section 3, we intend to use the Riccati transformation technique, integral averaging technique, and inequalities to obtain some sufficient conditions for oscillation of every solution of (1). In Section 4, we give two examples to illustrate Theorems 3 and 7, respectively.

2. Some Preliminaries

On any time scale $\mathbb{T}$, we define the forward and the backward jump operators by

$$\begin{aligned}\sigma(t) &= \inf\{s \in \mathbb{T} : s > t\},\\ \rho(t) &= \sup\{s \in \mathbb{T} : s < t\},\end{aligned} \tag{6}$$

where $\inf\emptyset = \sup\mathbb{T}$ and $\sup\emptyset = \inf\mathbb{T}$. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}, \tag{7}$$

if f is continuous at t and t is right-scattered. If t is right-dense, then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s\to t^+}\frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s\to t^+}\frac{f(t) - f(s)}{t - s}, \tag{8}$$

provided this limit exists. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided f is continuous at right-dense points and there exists a finite left limit at all left-dense points in $\mathbb{T}$. The set of all such rd-continuous functions is denoted by $C_{\mathrm{rd}}(\mathbb{T})$. The derivative f^{Δ} of f and the forward jump operator σ are related by the formula

$$f^{\sigma}(t) = f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t). \tag{9}$$

Also, we will use $x^{\Delta\sigma}$ which is shorthand for $(x^{\Delta})^{\sigma}$ to denote $x^{\Delta}(t) + \mu(t)x^{\Delta\Delta}(t)$. We will make use of the following product and quotient rules for the derivative of two differentiable functions f and g:

$$\begin{aligned}(fg)^{\Delta}(t) &= f^{\Delta}(t) g(t) + f^{\sigma}(t) g^{\Delta}(t) \\ &= f(t) g^{\Delta}(t) + f^{\Delta}(t) g^{\sigma}(t),\end{aligned} \tag{10}$$

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t) g(t) - f(t) g^{\Delta}(t)}{g(t) g^{\sigma}(t)}, \quad \text{if } gg^{\sigma} \neq 0.$$

The integration by parts formula reads

$$\begin{aligned}\int_b^c f^{\Delta}(t) g(t) \Delta t &= f(c) g(c) - f(b) g(b) \\ &\quad - \int_b^c f^{\sigma}(t) g^{\Delta}(t) \Delta t.\end{aligned} \tag{11}$$

We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided

$$1 + \mu(t) p(t) \neq 0, \quad \forall t \in \mathbb{T}. \tag{12}$$

The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$\mathscr{R} = \mathscr{R}(\mathbb{T}) = \mathscr{R}(\mathbb{T}, \mathbb{R}). \tag{13}$$

If $p \in \mathscr{R}$, then we can define the exponential function by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta\tau\right) \quad \text{for } s, t \in \mathbb{T}, \tag{14}$$

where $\xi_h(z)$ is the cylinder transformation, which is defined by

$$\xi_h(z) = \begin{cases} \dfrac{\log(1 + hz)}{h}, & h \neq 0, \\ z, & h = 0. \end{cases} \tag{15}$$

Next, we give the following lemmas which will be used in the proof of our main results.

Lemma 1 (see [2, Chapter 2]). *If $g \in \mathscr{R}^{+}$; that is, $g : \mathbb{T} \to \mathbb{R}$ is rd-continuous and such that $1 + \mu(t)g(t) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$, then the initial value problem $y^{\Delta} = g(t)y$, $y(t_0) = y_0 \in \mathbb{R}$ has a unique and positive solution on $[t_0, \infty)_{\mathbb{T}}$, denoted by $e_g(t, t_0)y_0$. This "exponential function" $e_g(\cdot, t_0)$ satisfies the semigroup property $e_g(a, b)e_g(b, c) = e_g(a, c)$.*

Lemma 2 (see [36]). *If $\lambda > 1$ and $\rho > 1$ are conjugate numbers $(1/\lambda + 1/\rho = 1)$, then for any $X, Y \in \mathbb{R}$,*

$$\frac{|X|^{\lambda}}{\lambda} + \frac{|Y|^{\rho}}{\rho} \geq |XY|. \tag{16}$$

3. Main Results

Now, we are in a position to state and prove some new results which guarantee that every solution of (1) oscillates. In the sequel, we say that a function u belongs to a function class

$$\xi(a, b) := \left\{u \in C^1_{\mathrm{rd}}[a, b]_{\mathbb{T}} : u(a) = u(b) = 0, u(t) \not\equiv 0\right\}, \tag{17}$$

denoted by $u \in \xi(a, b)$.

Theorem 3. *Assume that $\alpha > 1$ and for any $T \in [t_0, \infty)_{\mathbb{T}}$, there exist constants a_k and $b_k \in [T, \infty)_{\mathbb{T}}$, such that $a_k < b_k$, $k = 1, 2$, with*

$$q(t) \geq 0, \quad \text{for } t \in [a_1, b_1]_{\mathbb{T}} \cup [a_2, b_2]_{\mathbb{T}}, \tag{18}$$

$$(-1)^k F(t, x^{\sigma}(t)) \geq (-1)^k f(t) \geq 0, \quad \text{for } t \in [a_k, b_k]_{\mathbb{T}}, \quad k = 1, 2, \tag{19}$$

where $f \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$. Furthermore, assume that there exist functions $\eta \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^{+})$, $\eta^{\Delta}(t) \geq 0$, and $u \in \xi(a_k, b_k)$, $k = 1, 2$, such that

$$\int_{a_k}^{b_k} \left(\eta(t) r(t) \left(u^{\Delta}(t)\right)^2 - P(t, a_k) \left(u^{\sigma}(t)\right)^2\right) \Delta t \leq 0, \quad k = 1, 2, \tag{20}$$

where

$$\begin{aligned} P(t, a_k) &= \delta_0(t) - \eta^{\Delta}(t) \delta_1(t, a_k) \\ &\quad + \frac{\eta^{\sigma}(t) p(t)}{r^{\sigma}(t)} \delta_1(\sigma(t), a_k), \\ \delta_0(t) &= \alpha^{1/\alpha} \left(\frac{\alpha}{\alpha - 1}\right)^{(\alpha-1)/\alpha} \eta^{\sigma}(t) q^{1/\alpha}(t) |f(t)|^{(\alpha-1)/\alpha}, \\ \delta_1(t, a_k) &= \frac{1}{e_{p/r^{\sigma}}(t, a_k)} \\ &\quad \times \left(\int_{a_k}^t \frac{1}{r(s) e_{p/r^{\sigma}}(s, a_k)} \Delta s\right)^{-1}. \end{aligned} \tag{21}$$

Then (1) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

Proof. Assume that x is a nonoscillatory solution of (1) on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$, such that $x(t) > 0$, $x^{\sigma}(t) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. By assumption, we can choose $b_1 > a_1 > t_1$, then $q(t) \geq 0$ and $F(t, x^{\sigma}(t)) \leq 0$ on the interval $[a_1, b_1]_{\mathbb{T}}$. From (1), we have

$$\left(r(t) x^{\Delta}(t)\right)^{\Delta} + p(t) x^{\Delta\sigma}(t) \leq 0. \tag{22}$$

Using Lemma 1 and the above inequality, we get

$$\left(r(t) x^{\Delta}(t) e_{p/r^{\sigma}}(t, a_1)\right)^{\Delta} \leq 0. \tag{23}$$

Hence $r(t)x^{\Delta}(t)e_{p/r^{\sigma}}(t,a_1)$ is nonincreasing on $[a_1,b_1]_{\mathbb{T}}$. So for $t\in[a_1,b_1]_{\mathbb{T}}$,

$$\begin{aligned} x(t) > x(t)-x(a_1) &= \int_{a_1}^{t}\frac{r(s)\,x^{\Delta}(s)\,e_{p/r^{\sigma}}(s,a_1)}{r(s)\,e_{p/r^{\sigma}}(s,a_1)}\Delta s \\ &\geq r(t)\,x^{\Delta}(t)\,e_{p/r^{\sigma}}(t,a_1)\int_{a_1}^{t}\frac{1}{r(s)\,e_{p/r^{\sigma}}(s,a_1)}\Delta s. \end{aligned} \tag{24}$$

Therefore,

$$\begin{aligned} \frac{r(t)\,x^{\Delta}(t)}{x(t)} &< \frac{1}{e_{p/r^{\sigma}}(t,a_1)}\left(\int_{a_1}^{t}\frac{1}{r(s)\,e_{p/r^{\sigma}}(s,a_1)}\Delta s\right)^{-1} \\ &= \delta_1(t,a_1). \end{aligned} \tag{25}$$

Define the function ω by

$$\omega(t)=\eta(t)\frac{r(t)\,x^{\Delta}(t)}{x(t)},\quad t\in[a_1,b_1]_{\mathbb{T}}. \tag{26}$$

Using the product rule and the quotient rule, we obtain

$$\begin{aligned} \omega^{\Delta}(t) &= \eta^{\sigma}(t)\frac{\left(r(t)\,x^{\Delta}(t)\right)^{\Delta}x(t)-r(t)\left(x^{\Delta}(t)\right)^{2}}{x(t)\,x^{\sigma}(t)} \\ &\quad+\eta^{\Delta}(t)\frac{r(t)\,x^{\Delta}(t)}{x(t)}. \end{aligned} \tag{27}$$

In view of (1), (26), and (27), we have

$$\begin{aligned} \omega^{\Delta}(t) &= -\eta^{\sigma}(t)\frac{p(t)\,x^{\Delta\sigma}(t)}{x^{\sigma}(t)}-\eta^{\sigma}(t)\,q(t)\left(x^{\sigma}(t)\right)^{\alpha-1} \\ &\quad+\eta^{\sigma}(t)\frac{F(t,x^{\sigma}(t))}{x^{\sigma}(t)}-\eta^{\sigma}(t)\frac{r(t)\left(x^{\Delta}(t)\right)^{2}}{x(t)\,x^{\sigma}(t)} \\ &\quad+\eta^{\Delta}(t)\frac{r(t)\,x^{\Delta}(t)}{x(t)} \\ &= -\frac{\eta^{\sigma}(t)\,p(t)}{r^{\sigma}(t)}\frac{r^{\sigma}(t)\,x^{\Delta\sigma}(t)}{x^{\sigma}(t)}-\eta^{\sigma}(t)\,q(t)\left(x^{\sigma}(t)\right)^{\alpha-1} \\ &\quad-\eta^{\sigma}(t)\frac{\left|F(t,x^{\sigma}(t))\right|}{x^{\sigma}(t)}-\frac{\eta^{\sigma}(t)\,x(t)}{\eta^{2}(t)\,r(t)\,x^{\sigma}(t)}\omega^{2}(t) \\ &\quad+\eta^{\Delta}(t)\frac{r(t)\,x^{\Delta}(t)}{x(t)}. \end{aligned} \tag{28}$$

From (19), (25), and (28), we get

$$\begin{aligned} \omega^{\Delta}(t) &\leq -\frac{\eta^{\sigma}(t)\,p(t)}{r^{\sigma}(t)}\delta_1(\sigma(t),a_1)-\eta^{\sigma}(t)\,q(t)\left(x^{\sigma}(t)\right)^{\alpha-1} \\ &\quad-\eta^{\sigma}(t)\frac{\left|f(t)\right|}{x^{\sigma}(t)}-\frac{\eta^{\sigma}(t)\,x(t)}{\eta^{2}(t)\,r(t)\,x^{\sigma}(t)}\omega^{2}(t) \\ &\quad+\eta^{\Delta}(t)\,\delta_1(t,a_1). \end{aligned} \tag{29}$$

Set

$$\begin{aligned} G(x) &= \eta^{\sigma}(t)\,q(t)\,x^{\alpha-1}+\eta^{\sigma}(t)\frac{\left|f(t)\right|}{x}, \\ \lambda &= \alpha,\qquad \rho=\frac{\alpha}{\alpha-1}. \end{aligned} \tag{30}$$

From Lemma 2, it is easy to see that

$$\begin{aligned} G(x^{\sigma}) &\geq \alpha^{1/\alpha}\left(\frac{\alpha}{\alpha-1}\right)^{(\alpha-1)/\alpha}\eta^{\sigma}(t)\,q^{1/\alpha}(t)\left|f(t)\right|^{(\alpha-1)/\alpha} \\ &= \delta_0(t). \end{aligned} \tag{31}$$

Since $x(t)>0$, we obtain

$$\begin{aligned} 0<\frac{x(t)}{r(t)\,x^{\sigma}(t)} &= \frac{1}{r(t)+\mu(t)\left(r(t)\,x^{\Delta}(t)/x(t)\right)} \\ &= \frac{\eta(t)}{\eta(t)\,r(t)+\mu(t)\,\omega(t)}. \end{aligned} \tag{32}$$

Thus, combining (29)–(32) and noticing that $\eta^{\Delta}(t)\geq 0$, we have

$$\omega^{\Delta}(t)\leq -P(t,a_1)-\frac{1}{\eta(t)\,r(t)+\mu(t)\,\omega(t)}\omega^{2}(t), \tag{33}$$

where P is defined as in Theorem 3. Multiplying (33) by $(u^{\sigma}(t))^2$ and integrating from a_1 to b_1, we get

$$\begin{aligned} \int_{a_1}^{b_1}\left(u^{\sigma}(t)\right)^{2}\omega^{\Delta}(t)\,\Delta t &\leq -\int_{a_1}^{b_1}P(t,a_1)\left(u^{\sigma}(t)\right)^{2}\Delta t \\ &\quad-\int_{a_1}^{b_1}\frac{\left(u^{\sigma}(t)\right)^{2}\omega^{2}(t)}{\eta(t)\,r(t)+\mu(t)\,\omega(t)}\Delta t. \end{aligned} \tag{34}$$

Using integration by parts on the first integral, we obtain

$$\begin{aligned} &u^{2}(t)\,\omega(t)\Big|_{a_1}^{b_1}-\int_{a_1}^{b_1}\left(u(t)+u^{\sigma}(t)\right)u^{\Delta}(t)\,\omega(t)\,\Delta t \\ &\quad\leq -\int_{a_1}^{b_1}P(t,a_1)\left(u^{\sigma}(t)\right)^{2}\Delta t-\int_{a_1}^{b_1}\frac{\left(u^{\sigma}(t)\right)^{2}\omega^{2}(t)}{\eta(t)\,r(t)+\mu(t)\,\omega(t)}\Delta t. \end{aligned} \tag{35}$$

Rearranging and using $u(a_1) = 0 = u(b_1)$, we have

$$\begin{aligned} 0 \geq & \int_{a_1}^{b_1} \frac{(u^\sigma(t))^2 \omega^2(t)}{\eta(t) r(t) + \mu(t)\omega(t)} \Delta t \\ & - \int_{a_1}^{b_1} (u(t) + u^\sigma(t)) u^\Delta(t) \omega(t) \Delta t \\ & + \int_{a_1}^{b_1} P(t, a_1) (u^\sigma(t))^2 \Delta t \\ = & \int_{a_1}^{b_1} \frac{(u^\sigma(t))^2 \omega^2(t)}{\eta(t) r(t) + \mu(t)\omega(t)} \Delta t \\ & - \int_{a_1}^{b_1} \Big(2u^\sigma(t) u^\Delta(t) \omega(t) \\ & \quad - \mu(t) (u^\Delta(t))^2 \omega(t)\Big) \Delta t \\ & + \int_{a_1}^{b_1} P(t, a_1) (u^\sigma(t))^2 \Delta t. \end{aligned} \tag{36}$$

Adding and subtracting the term $\int_{a_1}^{b_1} \eta(t) r(t) (u^\Delta(t))^2 \Delta t$ and using (20), we get

$$\begin{aligned} 0 \geq & \int_{a_1}^{b_1} \Bigg[\frac{(u^\sigma(t))^2 \omega^2(t)}{\eta(t) r(t) + \mu(t)\omega(t)} - 2u^\sigma(t) u^\Delta(t) \omega(t) \\ & + (\eta(t) r(t) + \mu(t)\omega(t)) (u^\Delta(t))^2 \Bigg] \Delta t \\ & - \int_{a_1}^{b_1} \Big(\eta(t) r(t) (u^\Delta(t))^2 - P(t, a_1) (u^\sigma(t))^2\Big) \Delta t \\ \geq & \int_{a_1}^{b_1} \Bigg[\frac{u^\sigma(t)\omega(t)}{\sqrt{\eta(t) r(t) + \mu(t)\omega(t)}} \\ & - \sqrt{\eta(t) r(t) + \mu(t)\omega(t)} u^\Delta(t) \Bigg]^2 \Delta t. \end{aligned} \tag{37}$$

It follows that

$$\begin{aligned} & \int_{a_1}^{b_1} \Bigg[\frac{u^\sigma(t)\omega(t)}{\sqrt{\eta(t) r(t) + \mu(t)\omega(t)}} \\ & \quad - \sqrt{\eta(t) r(t) + \mu(t)\omega(t)} u^\Delta(t) \Bigg]^2 \Delta t = 0. \end{aligned} \tag{38}$$

This implies that

$$\begin{aligned} & \frac{u^\sigma(t)\omega(t)}{\sqrt{\eta(t) r(t) + \mu(t)\omega(t)}} \\ & \quad - \sqrt{\eta(t) r(t) + \mu(t)\omega(t)} u^\Delta(t) = 0, \quad t \in [a_1, b_1]_{\mathbb{T}}. \end{aligned} \tag{39}$$

Solving for u^Δ, we get that u solves the IVP

$$\begin{aligned} u^\Delta(t) &= \frac{\omega(t)}{\eta(t) r(t) + \mu(t)\omega(t)} u^\sigma(t), \\ u(a_1) &= 0, \quad \text{for } t \in [a_1, b_1]_{\mathbb{T}}. \end{aligned} \tag{40}$$

Since $-\omega/(\eta r + \mu\omega) \in \mathscr{R}$, we obtain from [2, Theorem 2.7.1] that $u(t) \equiv 0$ on $[a_1, b_1]_{\mathbb{T}}$, which is a contradiction. The proof when x is eventually negative follows the same arguments using the interval $[a_2, b_2]_{\mathbb{T}}$ instead of $[a_1, b_1]_{\mathbb{T}}$, where we use $q(t) \geq 0, F(t, x^\sigma(t)) \geq 0$ on $[a_2, b_2]_{\mathbb{T}}$, and $\int_{a_2}^{b_2} (\eta(t) r(t) (u^\Delta(t))^2 - P(t)(u^\sigma(t))^2) \Delta t \leq 0$. The proof is complete. □

Remark 4. When $p(t) = 0$ and $F(t, x^\sigma(t)) = f(t)$, Theorem 3 contains Theorem 3.2 in [26].

Theorem 5. *Assume that $\alpha = 1$ and for any $T \in [t_0, \infty)_{\mathbb{T}}$, there exist constants a_k and $b_k \in [T, \infty)_{\mathbb{T}}$, such that $a_k < b_k$, $k = 1, 2$, with*

$$\begin{aligned} & q(t) \geq 0, \quad \text{for } t \in [a_1, b_1]_{\mathbb{T}} \cup [a_2, b_2]_{\mathbb{T}}, \\ & (-1)^k F(t, x^\sigma(t)) \geq 0, \quad \text{for } t \in [a_k, b_k]_{\mathbb{T}}, \ k = 1, 2. \end{aligned} \tag{41}$$

Furthermore, assume that there exist functions $\eta \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$, $\eta^\Delta(t) \geq 0$, and $u \in \xi(a_k, b_k)$, $k = 1, 2$, such that

$$\begin{aligned} & \int_{a_k}^{b_k} \Big(\eta(t) r(t) (u^\Delta(t))^2 - K(t, a_k) (u^\sigma(t))^2\Big) \Delta t \leq 0, \\ & \qquad k = 1, 2, \end{aligned} \tag{42}$$

where

$$\begin{aligned} K(t, a_k) = & \eta^\sigma(t) q(t) - \eta^\Delta(t) \delta_1(t, a_k) \\ & + \frac{\eta^\sigma(t) p(t)}{r^\sigma(t)} \delta_1(\sigma(t), a_k), \end{aligned} \tag{43}$$

and δ_1 is defined as in Theorem 3. Then (1) *is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.*

Proof. Assume that x is a nonoscillatory solution of (1) on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$, such that $x(t) > 0$, $x^\sigma(t) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. By assumption, we can choose $b_1 > a_1 > t_1$, then $q(t) \geq 0$ and $F(t, x^\sigma(t)) \leq 0$ on the interval $[a_1, b_1]_{\mathbb{T}}$. We define ω as in Theorem 3. Proceeding as in the proof of Theorem 3 and from (25) and (32), we get

$$\begin{aligned} \omega^\Delta(t) = & -\frac{\eta^\sigma(t) p(t)}{r^\sigma(t)} \frac{r^\sigma(t) x^{\Delta\sigma}(t)}{x^\sigma(t)} - \eta^\sigma(t) q(t) \\ & - \eta^\sigma(t) \frac{|F(t, x^\sigma(t))|}{x^\sigma(t)} \\ & - \frac{\eta^\sigma(t) x(t)}{\eta^2(t) r(t) x^\sigma(t)} \omega^2(t) + \eta^\Delta(t) \frac{r(t) x^\Delta(t)}{x(t)} \\ \leq & -K(t, a_1) - \frac{1}{\eta(t) r(t) + \mu(t)\omega(t)} \omega^2(t), \end{aligned} \tag{44}$$

where K is defined as in Theorem 5. Multiplying (44) by $(u^{\sigma}(t))^2$ and integrating from a_1 to b_1, we get

$$\int_{a_1}^{b_1} (u^{\sigma}(t))^2 \omega^{\Delta}(t)\,\Delta t \le -\int_{a_1}^{b_1} K(t,a_1)(u^{\sigma}(t))^2 \Delta t - \int_{a_1}^{b_1} \frac{(u^{\sigma}(t))^2 \omega^2(t)}{\eta(t) r(t) + \mu(t)\omega(t)} \Delta t. \tag{45}$$

The rest of the argument proceeds as in Theorem 3 to get a contradiction to (42). The proof is complete. □

Remark 6. When $p(t) = 0$ and $F(t, x^{\sigma}(t)) = f(t)$, Theorem 5 contains Theorem 2.1 in [26].

Theorem 7. *Assume that $\alpha < 1$ and for any $T \in [t_0, \infty)_{\mathbb{T}}$, there exist constants a_k and $b_k \in [T, \infty)_{\mathbb{T}}$, such that $a_k < b_k$, $k = 1, 2$, with*

$$q(t) \ge 0, \quad \text{for } t \in [a_1, b_1]_{\mathbb{T}} \cup [a_2, b_2]_{\mathbb{T}}, \tag{46}$$

$$(-1)^k F(t, x^{\sigma}(t)) \ge (-1)^k f(t)(x^{\sigma}(t))^{\alpha+1} \ge 0 \quad \text{for } t \in [a_k, b_k]_{\mathbb{T}}, \quad k = 1, 2, \tag{47}$$

where $f \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$. Furthermore, assume that there exist functions $\eta \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$, $\eta^{\Delta}(t) \ge 0$, and $u \in \xi(a_k, b_k)$, $k = 1, 2$, such that

$$\int_{a_k}^{b_k} \left(\eta(t) r(t) (u^{\Delta}(t))^2 - P_1(t, a_k)(u^{\sigma}(t))^2\right) \Delta t \le 0, \quad k = 1, 2, \tag{48}$$

where

$$P_1(t, a_k) = \delta_2(t) - \eta^{\Delta}(t)\delta_1(t, a_k) + \frac{\eta^{\sigma}(t) p(t)}{r^{\sigma}(t)} \delta_1(\sigma(t), a_k), \tag{49}$$

$$\delta_2(t) = \frac{1}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \eta^{\sigma}(t) q^{\alpha}(t) |f(t)|^{1-\alpha},$$

and δ_1 is defined as in Theorem 3. Then (1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Proof. Assume that x is a nonoscillatory solution of (1) on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$, such that $x(t) > 0$, $x^{\sigma}(t) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. By assumption, we can choose $b_1 > a_1 > t_1$, then $q(t) \ge 0$ and $F(t, x^{\sigma}(t)) \le 0$ on the interval $[a_1, b_1]_{\mathbb{T}}$. We define ω as in Theorem 3. Proceeding as in the proof of Theorem 3, we have (28). Hence, from (25), (28), and (47), we get

$$\omega^{\Delta}(t) \le -\frac{\eta^{\sigma}(t) p(t)}{r^{\sigma}(t)} \delta_1(\sigma(t), a_1) - \frac{\eta^{\sigma}(t) q(t)}{(x^{\sigma}(t))^{1-\alpha}} - \eta^{\sigma}(t) |f(t)| (x^{\sigma}(t))^{\alpha} - \frac{\eta^{\sigma}(t) x(t)}{\eta^2(t) r(t) x^{\sigma}(t)} \omega^2(t) + \eta^{\Delta}(t)\delta_1(t, a_1). \tag{50}$$

Set

$$G(x) = \frac{\eta^{\sigma}(t) q(t)}{x^{1-\alpha}} - \eta^{\sigma}(t)|f(t)| x^{\alpha}, \qquad \lambda = \frac{1}{\alpha}, \qquad \rho = \frac{1}{1-\alpha}. \tag{51}$$

From Lemma 2, it is easy to see that

$$G(x^{\sigma}) \ge \frac{1}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \eta^{\sigma}(t) q^{\alpha}(t) |f(t)|^{1-\alpha} = \delta_2(t). \tag{52}$$

Thus, combining (32), (50), and (52) and noticing that $\eta^{\Delta}(t) \ge 0$, we have

$$\omega^{\Delta}(t) \le -P_1(t, a_1) - \frac{1}{\eta(t) r(t) + \mu(t)\omega(t)} \omega^2(t), \tag{53}$$

where P_1 is defined as in Theorem 7. Multiplying (53) by $(u^{\sigma}(t))^2$ and integrating from a_1 to b_1, we get

$$\int_{a_1}^{b_1} (u^{\sigma}(t))^2 \omega^{\Delta}(t)\,\Delta t \le -\int_{a_1}^{b_1} P_1(t, a_1)(u^{\sigma}(t))^2 \Delta t - \int_{a_1}^{b_1} \frac{(u^{\sigma}(t))^2 \omega^2(t)}{\eta(t) r(t) + \mu(t)\omega(t)} \Delta t. \tag{54}$$

The rest of the argument proceeds as in Theorem 3 to get a contradiction to (47). The proof is complete. □

Next, let us introduce the class of functions Y, which will be extensively used in the sequel.

Let $\mathbb{D}_0 = \{(t, s) \in \mathbb{T}^2 : t > s \ge t_0\}$ and $\mathbb{D} = \{(t, s) \in \mathbb{T}^2 : t \ge s \ge t_0\}$. We say that the function $H \in C_{\mathrm{rd}}(\mathbb{D}, \mathbb{R})$ belongs to the class Y, if

(i) $H(t, t) = 0$, $t \ge t_0$, $H(t, s) > 0$ on $\mathbb{D}_0$;

(ii) H has continuous Δ-partial derivatives $H^{\Delta_t}(t, s)$ and $H^{\Delta_s}(t, s)$ on $\mathbb{D}$ such that

$$H^{\Delta_t}(t, \sigma(s)) = h_1(t, s)\sqrt{H(\sigma(t), \sigma(s))}, \qquad H^{\Delta_s}(\sigma(t), s) = -h_2(t, s)\sqrt{H(\sigma(t), \sigma(s))}, \tag{55}$$

where h_1 and $h_2 \in C_{\mathrm{rd}}(\mathbb{D}, \mathbb{R})$.

Theorem 8. *Assume that $\alpha > 1$ and for any $T \in [t_0, \infty)_{\mathbb{T}}$, there exist constants a_k and $b_k \in [T, \infty)_{\mathbb{T}}$, such that $a_k < b_k$, $k = 1, 2$, with*

$$q(t) \ge 0, \quad \text{for } t \in [a_1, b_1]_{\mathbb{T}} \cup [a_2, b_2]_{\mathbb{T}}, \qquad (-1)^k F(t, x^{\sigma}(t)) \ge (-1)^k f(t) \ge 0, \quad \text{for } t \in [a_k, b_k]_{\mathbb{T}}, \quad k = 1, 2, \tag{56}$$

where $f \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$. *Furthermore, assume that there exists a function* $\eta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ *such that for some* $H \in Y$ *and* $c_k \in (a_k,b_k)_{\mathbb{T}}$,

$$\begin{aligned}
&\frac{1}{H(\sigma(c_k),\sigma(a_k))}\int_{a_k}^{c_k}\left[H(\sigma(s),\sigma(a_k))Q(s,a_k)\right.\\
&\qquad\left.-\frac{\eta^2(s)r(s)}{4\eta^\sigma(s)\delta(s,a_k)}\phi_1^2(s,a_k)\right]\Delta s\\
&+\frac{1}{H(\sigma(b_k),\sigma(c_k))}\int_{c_k}^{b_k}\left[H(\sigma(b_k),\sigma(s))Q(s,a_k)\right.\\
&\qquad\left.-\frac{\eta^2(s)r(s)}{4\eta^\sigma(s)\delta(s,a_k)}\phi_2^2(b_k,s)\right]\\
&\times\Delta s>0,\quad k=1,2,
\end{aligned}\tag{57}$$

where

$$\begin{aligned}
\phi_1(s,a_k)&=h_1(s,a_k)+\sqrt{H(\sigma(s),\sigma(a_k))}\frac{\eta^\Delta(s)}{\eta(s)},\\
\phi_2(b_k,s)&=h_2(b_k,s)-\sqrt{H(\sigma(b_k),\sigma(s))}\frac{\eta^\Delta(s)}{\eta(s)},\\
Q(t,a_k)&=\delta_0(t)+\frac{\eta^\sigma(t)p(t)}{r^\sigma(t)}\delta_1(\sigma(t),a_k),\\
\delta(t,a_k)&=\int_{a_k}^{t}\frac{\Delta s}{r(s)e_{p/r^\sigma}(s,a_k)}\left(\int_{a_k}^{\sigma(t)}\frac{\Delta s}{r(s)e_{p/r^\sigma}(s,a_k)}\right)^{-1},
\end{aligned}\tag{58}$$

and δ_0 *and* δ_1 *are defined as in Theorem 3. Then* (1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume that x is a nonoscillatory solution of (1) on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, we may assume that there exists a $t_1 \in [t_0,\infty)_{\mathbb{T}}$, such that $x(t)>0$, $x^\sigma(t)>0$ for all $t\in[t_1,\infty)_{\mathbb{T}}$. By assumption, we can choose $b_1>a_1>t_1$, then $q(t)\geq 0$ and $F(t,x^\sigma(t))\leq 0$ on the interval $[a_1,b_1]_{\mathbb{T}}$. We define the function ω as in Theorem 3. Proceeding as in the proof of Theorem 3 and from (25) and (31), we get

$$\omega^\Delta(t)\leq -Q(t,a_1)+\frac{\eta^\Delta(t)}{\eta(t)}\omega(t)-\frac{\eta^\sigma(t)x(t)}{\eta^2(t)r(t)x^\sigma(t)}\omega^2(t),\tag{59}$$

where Q is defined as in Theorem 7. Since $r(t)x^\Delta(t)e_{p/r^\sigma}(t,a_1)$ is nonincreasing on $[a_1,b_1]_{\mathbb{T}}$, we obtain

$$\begin{aligned}
x^\sigma(t)-x(t)&=\int_t^{\sigma(t)}\frac{r(s)x^\Delta(s)e_{p/r^\sigma}(s,a_1)}{r(s)e_{p/r^\sigma}(s,a_1)}\Delta s\\
&\leq r(t)x^\Delta(t)e_{p/r^\sigma}(t,a_1)\int_t^{\sigma(t)}\frac{\Delta s}{r(s)e_{p/r^\sigma}(s,a_1)},
\end{aligned}\tag{60}$$

hence

$$\frac{x^\sigma(t)}{x(t)}\leq 1+\frac{r(t)x^\Delta(t)e_{p/r^\sigma}(t,a_1)}{x(t)}\int_t^{\sigma(t)}\frac{\Delta s}{r(s)e_{p/r^\sigma}(s,a_1)}.\tag{61}$$

From (25), we have

$$\frac{r(t)x^\Delta(t)e_{p/r^\sigma}(t,a_1)}{x(t)}<\left(\int_{a_1}^{t}\frac{\Delta s}{r(s)e_{p/r^\sigma}(s,a_1)}\right)^{-1}.\tag{62}$$

Therefore, from (61) and (62), we get

$$\begin{aligned}
\frac{x^\sigma(t)}{x(t)}&<\int_{a_1}^{\sigma(t)}\frac{\Delta s}{r(s)e_{p/r^\sigma}(s,a_1)}\left(\int_{a_1}^{t}\frac{\Delta s}{r(s)e_{p/r^\sigma}(s,a_1)}\right)^{-1}\\
&=\frac{1}{\delta(t,a_1)}.
\end{aligned}\tag{63}$$

Combining (59) and (63), we obtain

$$\begin{aligned}
\omega^\Delta(t)&\leq -Q(t,a_1)+\frac{\eta^\Delta(t)}{\eta(t)}\omega(t)\\
&\quad-\frac{\eta^\sigma(t)\delta(t,a_1)}{\eta^2(t)r(t)}\omega^2(t),\quad t\in[a_1,b_1]_{\mathbb{T}}.
\end{aligned}\tag{64}$$

Multiplying both sides of (64) by $H(\sigma(s),\sigma(t))$ and integrating with respect to s from t to c_1 for $t\in(a_1,c_1]_{\mathbb{T}}$, we have

$$\begin{aligned}
&\int_t^{c_1}H(\sigma(s),\sigma(t))Q(s,a_1)\Delta s\\
&\quad\leq-\int_t^{c_1}H(\sigma(s),\sigma(t))\omega^\Delta(s)\Delta s\\
&\qquad+\int_t^{c_1}H(\sigma(s),\sigma(t))\frac{\eta^\Delta(s)}{\eta(s)}\omega(s)\Delta s\\
&\qquad-\int_t^{c_1}H(\sigma(s),\sigma(t))\frac{\eta^\sigma(s)\delta(s,a_1)}{\eta^2(s)r(s)}\omega^2(s)\Delta s.
\end{aligned}\tag{65}$$

In view of (i) and (ii), we see that

$$\begin{aligned}
&\int_t^{c_1}H(\sigma(s),\sigma(t))\omega^\Delta(s)\Delta s\\
&\quad=H(\sigma(c_1),\sigma(t))\omega(c_1)\\
&\qquad-\int_t^{c_1}h_1(s,t)\sqrt{H(\sigma(s),\sigma(t))}\omega(s)\Delta s.
\end{aligned}\tag{66}$$

Using (66) in (65) leads to

$$\begin{aligned}
&\int_t^{c_1} H(\sigma(s),\sigma(t))\,Q(s,a_1)\,\Delta s \\
&\quad\le -H(\sigma(c_1),\sigma(t))\,\omega(c_1) \\
&\qquad - \int_t^{c_1} H(\sigma(s),\sigma(t))\,\frac{\eta^{\sigma}(s)\,\delta(s,a_1)}{\eta^2(s)\,r(s)}\,\omega^2(s)\,\Delta s \\
&\qquad + \int_t^{c_1}\Bigg(h_1(s,t)\sqrt{H(\sigma(s),\sigma(t))} \\
&\qquad\qquad + H(\sigma(s),\sigma(t))\,\frac{\eta^{\Delta}(s)}{\eta(s)}\Bigg)\omega(s)\,\Delta s \\
&\quad= -H(\sigma(c_1),\sigma(t))\,\omega(c_1) \\
&\qquad + \int_t^{c_1}\frac{\eta^2(s)\,r(s)}{4\eta^{\sigma}(s)\,\delta(s,a_1)}\,\phi_1^2(s,t)\,\Delta s \\
&\qquad - \int_t^{c_1}\Bigg(\sqrt{H(\sigma(s),\sigma(t))\,\frac{\eta^{\sigma}(s)\,\delta(s,a_1)}{r(s)}}\,\frac{\omega(s)}{\eta(s)} \\
&\qquad\qquad - \frac{\eta(s)\sqrt{r(s)}}{2\sqrt{\eta^{\sigma}(s)\,\delta(s,a_1)}}\,\phi_1(s,t)\Bigg)^2\Delta s \\
&\quad\le -H(\sigma(c_1),\sigma(t))\,\omega(c_1) \\
&\qquad + \int_t^{c_1}\frac{\eta^2(s)\,r(s)}{4\eta^{\sigma}(s)\,\delta(s,a_1)}\,\phi_1^2(s,t)\,\Delta s.
\end{aligned} \tag{67}$$

Letting $t \to a_1^+$ in the above inequality, we get

$$\begin{aligned}
&\frac{1}{H(\sigma(c_1),\sigma(a_1))}\int_{a_1}^{c_1}\Bigg[H(\sigma(s),\sigma(a_1))\,Q(s,a_1) \\
&\qquad - \frac{\eta^2(s)\,r(s)}{4\eta^{\sigma}(s)\,\delta(s,a_1)}\,\phi_1^2(s,a_1)\Bigg]\Delta s \\
&\quad\le -\omega(c_1).
\end{aligned} \tag{68}$$

Similarly, multiplying both sides of (64) by $H(\sigma(t),\sigma(s))$ and integrating with respect to s from c_1 to t for $t \in [c_1,b_1)_{\mathbb{T}}$, we obtain

$$\begin{aligned}
&\int_{c_1}^{t} H(\sigma(t),\sigma(s))\,Q(s,a_1)\,\Delta s \\
&\quad\le -\int_{c_1}^{t} H(\sigma(t),\sigma(s))\,\omega^{\Delta}(s)\,\Delta s \\
&\qquad + \int_{c_1}^{t} H(\sigma(t),\sigma(s))\,\frac{\eta^{\Delta}(s)}{\eta(s)}\,\omega(s)\,\Delta s \\
&\qquad - \int_{c_1}^{t} H(\sigma(t),\sigma(s))\,\frac{\eta^{\sigma}(s)\,\delta(s,a_1)}{\eta^2(s)\,r(s)}\,\omega^2(s)\,\Delta s \\
&\quad\le H(\sigma(t),\sigma(c_1))\,\omega(c_1) \\
&\qquad - \int_{c_1}^{t} H(\sigma(t),\sigma(s))\,\frac{\eta^{\sigma}(s)\,\delta(s,a_1)}{\eta^2(s)\,r(s)}\,\omega^2(s)\,\Delta s \\
&\qquad - \int_{c_1}^{t}\Bigg(h_2(t,s)\sqrt{H(\sigma(t),\sigma(s))} \\
&\qquad\qquad - H(\sigma(t),\sigma(s))\,\frac{\eta^{\Delta}(s)}{\eta(s)}\Bigg)\omega(s)\,\Delta s \\
&\quad= H(\sigma(t),\sigma(c_1))\,\omega(c_1) \\
&\qquad + \int_{c_1}^{t}\frac{\eta^2(s)\,r(s)}{4\eta^{\sigma}(s)\,\delta(s,a_1)}\,\phi_2^2(t,s)\,\Delta s \\
&\qquad - \int_{c_1}^{t}\Bigg(\sqrt{H(\sigma(t),\sigma(s))\,\frac{\eta^{\sigma}(s)\,\delta(s,a_1)}{r(s)}}\,\frac{\omega(s)}{\eta(s)} \\
&\qquad\qquad + \frac{\eta(s)\sqrt{r(s)}}{2\sqrt{\eta^{\sigma}(s)\,\delta(s,a_1)}}\,\phi_2(t,s)\Bigg)^2\Delta s \\
&\quad\le H(\sigma(t),\sigma(c_1))\,\omega(c_1) \\
&\qquad + \int_{c_1}^{t}\frac{\eta^2(s)\,r(s)}{4\eta^{\sigma}(s)\,\delta(s,a_1)}\,\phi_2^2(t,s)\,\Delta s.
\end{aligned} \tag{69}$$

Letting $t \to b_1^-$ in the above inequality, we get

$$\begin{aligned}
&\frac{1}{H(\sigma(b_1),\sigma(c_1))}\int_{c_1}^{b_1}\Bigg[H(\sigma(b_1),\sigma(s))\,Q(s,a_1) \\
&\qquad - \frac{\eta^2(s)\,r(s)}{4\eta^{\sigma}(s)\,\delta(s,a_1)}\,\phi_2^2(b_1,s)\Bigg]\Delta s \\
&\quad\le \omega(c_1).
\end{aligned} \tag{70}$$

Adding (68) and (70), we get a contradiction to (57). This completes the proof. □

Theorem 9. *Assume that $\alpha < 1$ and for any $T \in [t_0,\infty)_{\mathbb{T}}$, there exist constants a_k and $b_k \in [T,\infty)_{\mathbb{T}}$, such that $a_k < b_k$, $k = 1,2$, with*

$$\begin{gathered}
q(t) \ge 0 \quad \text{for } t \in [a_1,b_1]_{\mathbb{T}} \cup [a_2,b_2]_{\mathbb{T}}, \\
(-1)^k F(t,x^{\sigma}(t)) \ge (-1)^k f(t)\,(x^{\sigma}(t))^{\alpha+1} \ge 0, \\
\text{for } t \in [a_k,b_k]_{\mathbb{T}},\ k = 1,2,
\end{gathered} \tag{71}$$

where $f \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$. Furthermore, assume that there exists a function $\eta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ such that for some $H \in Y$ and $c_k \in (a_k,b_k)_{\mathbb{T}}$,

$$\begin{aligned}&\frac{1}{H(\sigma(c_k),\sigma(a_k))}\int_{a_k}^{c_k}\left[H(\sigma(s),\sigma(a_k))\widetilde{Q}(s,a_k)\right.\\&\qquad\left.-\frac{\eta^2(s)r(s)}{4\eta^\sigma(s)\delta(s,a_k)}\phi_1^2(s,a_k)\right]\Delta s\\&+\frac{1}{H(\sigma(b_k),\sigma(c_k))}\int_{c_k}^{b_k}\left[H(\sigma(b_k),\sigma(s))\widetilde{Q}(s,a_k)\right.\\&\qquad\left.-\frac{\eta^2(s)r(s)}{4\eta^\sigma(s)\delta(s,a_k)}\phi_2^2(b_k,s)\right]\\&\times\Delta s>0,\quad k=1,2,\end{aligned}\tag{72}$$

where

$$\widetilde{Q}(t,a_k)=\delta_2(t)+\frac{\eta^\sigma(t)p(t)}{r^\sigma(t)}\delta_1(\sigma(t),a_k).\tag{73}$$

δ_1 is defined as in Theorem 3, δ_2 is defined as in Theorem 7, and ϕ_1, ϕ_2, and δ are defined as in Theorem 8. Then (1) *is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.*

The proof of Theorem 9 is similar to that of Theorem 8, so we omit the proof.

Remark 10. The main results in this paper can also be extended to the following second order damped dynamic equations with mixed nonlinearities:

$$\begin{aligned}&\left(r(t)x^\Delta(t)\right)^\Delta+p(t)x^{\Delta\sigma}(t)+q_0(t)x(\tau_0(t))\\&\quad+q_1(t)\left|x(\tau_1(t))\right|^{\beta-1}x(\tau_1(t))\\&\quad+q_2(t)\left|x(\tau_2(t))\right|^{\gamma-1}x(\tau_2(t))=F(t,x^\sigma(t)),\end{aligned}\tag{74}$$

where $\gamma>1>\beta>0$, $\tau_i(t)\le\sigma(t)$, $i=0,1,2$, or the more general equation

$$\begin{aligned}&\left(r(t)x^\Delta(t)\right)^\Delta+p(t)x^{\Delta\sigma}(t)\\&\quad+\sum_{i=0}^{n}q_i(t)\left|x(\tau_i(t))\right|^{\alpha_i-1}x(\tau_i(t))=F(t,x^\sigma(t))\end{aligned}\tag{75}$$

on any arbitrary time scale $\mathbb{T}$, where $\alpha_0=1$, $\alpha_1>\alpha_2>\cdots>\alpha_m>1>\alpha_{m+1}>\cdots>\alpha_n>0$ and τ_i are nondecreasing rd-continuous functions on $\mathbb{R}$ with $\tau_i(t)\le\sigma(t)$, $i=0,1,\ldots,n$. Due to the limited space, we omit it here and leave it to the readers who are interested in this problem.

4. Examples

In this section, we will show the applications of our interval oscillation criteria in two examples. Firstly, we will give an example to illustrate Theorem 3.

Example 1. Consider the following second order forced difference equations with damping:

$$\begin{aligned}&\Delta\left(t\left(\sin\frac{\pi t}{4}+2\right)\Delta x(t)\right)\\&\quad-\frac{t^2-1}{t^2}\left(\sin\frac{\pi(t+1)}{4}+2\right)\Delta x(t)\\&\quad+\frac{c_0}{(t+1)^2}\left(\sin\frac{\pi t}{4}+2\right)x^2(t)=-\cos\frac{\pi t}{4},\end{aligned}\tag{76}$$

for $t\ge2$, where c_0 is a positive constant. Here

$$\begin{aligned}r(t)&=t\left(\sin\frac{\pi t}{4}+2\right),\\p(t)&=-\frac{t^2-1}{t^2}\left(\sin\frac{\pi(t+1)}{4}+2\right),\\q(t)&=\frac{c_0}{(t+1)^2}\left(\sin\frac{\pi t}{4}+2\right),\\F(t,x(t))&=f(t)=-\cos\frac{\pi t}{4},\qquad\alpha=2.\end{aligned}\tag{77}$$

Let

$$\begin{aligned}&a_1=8h,\qquad b_1=a_2=8h+2,\\&b_2=8h+4,\quad h=1,2,\ldots,\end{aligned}\tag{78}$$

such that

$$\begin{aligned}&q(t)\ge0,\quad(-1)^kf(t)\ge0,\\&t\in[8h,8h+2)\cup[8h+2,8h+4),\quad k=1,2.\end{aligned}\tag{79}$$

For $t\ge2$, we obtain

$$\begin{aligned}&\delta_1(\sigma(t),a_k)\\&\quad=\frac{1}{(1+\mu(t)(p(t)/r^\sigma(t)))e_{p/r^\sigma}(t,a_k)}\\&\qquad\times\left(\int_{a_k}^{\sigma(t)}\frac{1}{(1+\mu(s)(p(s)/r^\sigma(s)))r(s)e_{p/r^\sigma}(s,a_k)}\Delta s\right)^{-1}\\&\quad\le\frac{t}{t-1}\delta_1(t,a_k).\end{aligned}\tag{80}$$

Setting $\eta(t) = 1/t$ and $u(t) = \sin(\pi t/2)$, we have

$$\begin{aligned}
P(t,a_k) &\geq 2(t+1)\left(\frac{c_0}{(t+1)^2}\left(\sin\frac{\pi t}{4}+2\right)\right)^{1/2}\left|-\cos\frac{\pi t}{4}\right|^{1/2} \\
&= 2\left(c_0\left(\sin\frac{\pi t}{4}+2\right)\left|\cos\frac{\pi t}{4}\right|\right)^{1/2}, \\
&\sum_{j=a_1}^{b_1-1}\left(\eta(j)r(j)(\Delta u(j))^2 - P(j,a_1)u^2(j+1)\right) \\
&\leq \sum_{j=8h}^{8h+1}\left(\frac{1}{j}\cdot j\left(\sin\frac{\pi j}{4}+2\right)\left(\sin\frac{\pi(j+1)}{2}-\sin\frac{\pi j}{2}\right)^2\right. \\
&\quad \left.-2\left(c_0\left(\sin\frac{\pi j}{4}+2\right)\cos\frac{\pi j}{4}\right)^{1/2}\sin^2\frac{\pi(j+1)}{2}\right) \\
&= \frac{\sqrt{2}}{2}+4-2(2c_0)^{1/2}.
\end{aligned} \tag{81}$$

Then by Theorem 3, every solution of (76) is oscillatory if

$$c_0 \geq \frac{1}{2}\left(\frac{\sqrt{2}}{4}+2\right)^2. \tag{82}$$

Next, we will give an example to illustrate Theorem 7.

Example 2. Consider the following second order forced differential equations with damping:

$$\begin{aligned}
&\left(t(\sin 2t+2)x'(t)\right)' - (\sin 2t+2)x'(t) + \frac{c_0\cos^2 2t}{t^{1/\alpha}}x^\alpha(t) \\
&\quad = -\sin 2t, \quad t \geq 1,
\end{aligned} \tag{83}$$

where c_0 is a positive constant. Here,

$$\begin{aligned}
&r(t) = t(\sin 2t+2), \qquad p(t) = -\sin 2t - 2, \\
&q(t) = \frac{c_0\cos^2 2t}{t^{1/\alpha}}, \qquad F(t,x(t)) = f(t) = -\sin 2t, \\
&\qquad\qquad \alpha < 1.
\end{aligned} \tag{84}$$

Let

$$\begin{aligned}
&a_1 = 2h\pi, \qquad b_1 = a_2 = 2h\pi + \frac{\pi}{2}, \\
&b_2 = 2h\pi + \pi, \quad h = 1,2,\ldots,
\end{aligned} \tag{85}$$

such that

$$\begin{aligned}
&q(t) \geq 0, \quad (-1)^k f(t) \geq 0, \\
&t \in \left[2h\pi, 2h\pi+\frac{\pi}{2}\right) \cup \left[2h\pi+\frac{\pi}{2}, 2h\pi+\pi\right), \quad k = 1,2.
\end{aligned} \tag{86}$$

Setting $\eta(t) = 1/t$ and $u(t) = \sin 2t$, we obtain

$$\begin{aligned}
P_1(t,a_k) &= \frac{t}{\alpha^\alpha(1-\alpha)^{1-\alpha}}\left(\frac{c_0\cos^2 2t}{t^{1/\alpha}}\right)^\alpha |-\sin 2t|^{1-\alpha} \\
&= \frac{c_0^\alpha}{\alpha^\alpha(1-\alpha)^{1-\alpha}}\cos^{2\alpha}2t|-\sin 2t|^{1-\alpha}, \\
&\int_{a_1}^{b_1}\left(\eta(t)r(t)\left(u'(t)\right)^2 - P_1(t,a_1)u^2(t)\right)\mathrm{d}t \\
&= \int_0^{\pi/2}\left(\frac{1}{t}\cdot t(\sin 2t+2)(2\cos 2t)^2\right. \\
&\qquad \left.-\frac{c_0^\alpha}{\alpha^\alpha(1-\alpha)^{1-\alpha}}\cos^{2\alpha}2t\ \sin^{3-\alpha}2t\right)\mathrm{d}t \\
&= \frac{6}{5}\sqrt{\pi}+\pi - \frac{c_0^\alpha}{4\alpha^\alpha(1-\alpha)^{1-\alpha}} \\
&\quad \times \frac{\Gamma(2-\alpha/2)\Gamma(\alpha+1/2)}{\Gamma((\alpha+5)/2)},
\end{aligned} \tag{87}$$

where Γ is the gamma function. Then by Theorem 7, every solution of (83) is oscillatory if

$$\frac{6}{5}\sqrt{\pi}+\pi \leq \frac{c_0^\alpha}{4\alpha^\alpha(1-\alpha)^{1-\alpha}}\frac{\Gamma(2-\alpha/2)\Gamma(\alpha+1/2)}{\Gamma((\alpha+5)/2)}. \tag{88}$$

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