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Research Article

Some Convergence Theorems for Contractive Type Mappings in $CAT(0)$ Spaces

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We establish theorems of strong convergence, for the Ishikawa-type (or two step; *cf.* Ishikawa, 1974) iteration scheme, to a fixed point of a uniformly L-Lipschitzian asymptotically demicontractive mapping and a uniformly L-Lipschitzian hemicontractive mapping in $CAT(0)$ space. Moreover, we will propose some open problems.

1. Introduction

Let (X, d) be a metric space. One of the most interesting aspects of metric fixed point theory is to extend a linear version of known result to the nonlinear case in metric spaces. To achieve this, Takahashi [1] introduced a convex structure in a metric space (X, d). A mapping $W : X \times X \times [0, 1] \to X$ is a *convex structure* in X if

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y) \tag{1}$$

for all $x, y \in X$ and $\lambda \in [0, 1]$. A metric space together with a convex structure W is known as a convex metric space. A nonempty subset K of a convex metric space is said to be *convex* if

$$W(x, y, \lambda) \in K \tag{2}$$

for all $x, y \in K$ and $\lambda \in [0, 1]$. In fact, every normed space and its convex subsets are convex metric spaces but the converse is not true, in general (see, [1]).

Example 1 (see [2]). Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$, for all $x = (x_1, x_2), y = (y_1, y_2) \in X$, and $\lambda \in [0, 1]$. We define a mapping $W : X \times X \times [0, 1] \to X$ by

$$W(x, y, \lambda) = \left(\lambda x_1 + (1 - \lambda) y_1, \frac{\lambda x_1 x_2 + (1 - \lambda) y_1 y_2}{\lambda x_1 + (1 - \lambda) y_1}\right), \tag{3}$$

and define a metric $d : X \times X \to [0, \infty)$ by

$$d(x, y) = |x_1 - y_1| + |x_1 x_2 - y_1 y_2|. \tag{4}$$

Then we can show that (X, d, W) is a convex metric space, but it is not a normed linear space.

A metric space X is a $CAT(0)$ space (the term is due to Gromov [3] and it is an acronym for E. Cartan, A. D. Aleksandrov, and V. A. Toponogov) if it is geodesically connected and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane (see, e.g., [4], page 159). It is well known that any complete, simply connected Riemannian manifold nonpositive sectional curvature is a $CAT(0)$ space. The precise definition is given below. For a thorough discussion of these spaces and of the fundamental role they play in various branches of mathematics, see Bridson and Haefliger [4] or Burago et al. [5].

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a mapping c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$, for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A

subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ is a geodesic metric space (X, d) that consists of three points $x_1, x_2, x_3 \in X$ (the *vertices* of Δ) and is a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) = \Delta(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see, [4]).

A geodesic metric space is said to be a $CAT(0)$ *space* if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

Let Δ be a geodesic triangle in X and let $\overline{\Delta} \subset \mathbb{R}^2$ be a comparison triangle for Δ. Then Δ is said to satisfy the $CAT(0)$ *inequality* if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,

$$d(x, y) \le d(\overline{x}, \overline{y}). \tag{5}$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see, [6]). If x, y_1, y_2 are points of a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denote by $y_1 \oplus y_2/2$, then the $CAT(0)$ inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \le \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \tag{6}$$

This inequality is the (CN) inequality of Bruhat and Tits [7]. In fact, a geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (*cf.* [4], page 163). The previous inequality has been extended by Khamsi and Kirk [8] as

$$\begin{aligned} &d^2(z, \alpha x \oplus (1-\alpha) y) \\ &\quad \le \alpha d^2(z, x) + (1-\alpha) d^2(z, y) - \alpha(1-\alpha) d^2(x, y), \end{aligned} \tag{CN*}$$

for any $\alpha \in [0, 1]$ and $x, y, z \in X$. The inequality (CN^*) also appeared in [9].

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (see, [4], page 163). Moreover, if X is a $CAT(0)$ metric space and $x, y \in X$, then for any $\alpha \in [0, 1]$, there exists a unique point $\alpha x \oplus (1-\alpha) y \in [x, y]$ such that

$$d(z, \alpha x \oplus (1-\alpha) y) \le \alpha d(z, x) + (1-\alpha) d(z, y) \tag{7}$$

for any $z \in X$ and $[x, y] = \{\alpha x \oplus (1-\alpha) y : \alpha \in [0, 1]\}$. In view of the previous inequality, $CAT(0)$ space has Takahashi's convex structure $W(x, y, \alpha) = \alpha x \oplus (1-\alpha) y$. It is easy to see that for any $x, y \in X$ and $\lambda \in [0, 1]$,

$$\begin{aligned} d(x, (1-\lambda) x \oplus \lambda y) &= \lambda d(x, y), \\ d(y, (1-\lambda) x \oplus \lambda y) &= (1-\lambda) d(x, y). \end{aligned} \tag{8}$$

As a consequence,

$$\begin{aligned} 1 \cdot x \oplus 0 \cdot y &= x, \\ (1-\lambda) x \oplus \lambda x = \lambda x \oplus (1-\lambda) x &= x. \end{aligned} \tag{9}$$

Moreover, a subset K of $CAT(0)$ space X is convex if for any $x, y \in K$, we have $[x, y] \subset K$.

Definition 2. Let C be a nonempty subset of a metric space (X, d). Let $F(T)$ denote the fixed point set of T. Let $F(T) \ne \emptyset$.

(1) A mapping $T : C \to C$ is said to be *k-strict asymptotically pseudocontractive* with sequence $\{a_n\}$ if $\lim_{n\to\infty} a_n = 1$ for some constant k, $0 \le k < 1$ and

$$d^2(T^n x, T^n y) \le a_n^2 d^2(x, y) + k(d(x, T^n x) - d(y, T^n y))^2, \tag{10}$$

for all $x, y \in C$, $n \in \mathbb{N}$.
If $k = 0$, then T is said to be *asymptotically nonexpansive* with sequence $\{a_n\}$, that is,

$$d(T^n x, T^n y) \le a_n d(x, y), \quad \forall x, y \in C. \tag{11}$$

(2) A mapping $T : C \to C$ is said to be *asymptotically demicontractive* with sequence $\{a_n\}$ if $\lim_{n\to\infty} a_n = 1$ for some constant k, $0 \le k < 1$, and

$$\begin{aligned} d^2(T^n x, p) \le a_n^2 d^2(x, p) + k \cdot d^2(x, T^n x), \\ \forall p \in F(T), \end{aligned} \tag{12}$$

for all $x \in C$, $n \in \mathbb{N}$.
If $k = 0$, then T is said to be *asymptotically quasi-nonexpansive* with sequence $\{a_n\}$, that is,

$$d(T^n x, p) \le a_n d(x, p), \quad \forall x \in C, \ \forall p \in F(T). \tag{13}$$

(3) A mapping $T : C \to C$ is said to be *asymptotically pseudocontractive* with sequence $\{a_n\}$ if $\lim_{n\to\infty} a_n = 1$ and

$$d^2(T^n x, T^n y) \le a_n d^2(x, y) + [d(x, T^n x) - d(y, T^n y)]^2, \tag{14}$$

for all $x, y \in C$, $n \in \mathbb{N}$.

(4) A mapping $T : C \to C$ is said to be *asymptotically hemicontractive* with sequence $\{a_n\}$ if $\lim_{n\to\infty} a_n = 1$ and

$$d^2(T^n x, p) \le a_n d^2(x, p) + d^2(x, T^n x), \quad \forall p \in F(T), \tag{15}$$

for all $x \in C$, $n \in \mathbb{N}$.

(5) A mapping $T : C \to C$ is said to be *uniformly L-Lipschitzian* if for some constant $L > 0$,

$$d(T^n x, T^n y) \le L \cdot d(x, y), \quad \forall x, y \in C, \tag{16}$$

for all $n \in \mathbb{N}$.

Liu [10] has proved the convergence of Mann and Ishikawa iterative sequence for uniformly L-Lipschitzian asymptotically demicontractive and hemicontractive mappings in Hilbert space (*cf.* [11]). The existence of (common)

fixed points of one mapping (or two mappings or family of mappings) is not known in many situations. So the approximation of fixed points of one or more nonexpansive, asymptotically nonexpansive, or asymptotically quasi-nonexpansive mappings by various iterations have been extensively studied in Banach spaces, convex metric spaces, $CAT(0)$ spaces, and so on (see, [2, 6, 8, 9, 12–27]).

In this paper, we establish theorems of strong convergence for the Ishikawa-type (or two step, *cf.* [28]) iteration scheme to a fixed point of a uniformly L-Lipschitzian asymptotically demicontractive mapping and a uniformly L-Lipschitzian asymptotically hemicontractive mapping in $CAT(0)$ space. Moreover, we will propose some open problems.

2. Preliminaries

We introduce the following iteration process.

Let C be a nonempty convex subset of a $CAT(0)$ space (X, d) and let $T : C \to C$ be a given mapping. Let $x_1 \in C$ be a given point.

Algorithm 3. The sequences $\{x_n\}$ and $\{y_n\}$ defined by the iterative process

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)\, x_n \oplus \alpha_n T^n y_n, \\ y_n &= (1-\beta_n)\, x_n \oplus \beta_n T^n x_n, \quad n \geq 1, \end{aligned} \tag{17}$$

is called an *Ishikawa-type* iterative sequence (*cf.* [28]).

If $\beta_n \equiv 0$, then Algorithm 3 reduces to the following.

Algorithm 4. The sequence $\{x_n\}$ defined by the iterative process

$$x_{n+1} = (1-\alpha_n)\, x_n \oplus \alpha_n T^n x_n, \quad n \geq 1, \tag{18}$$

is called a *Mann-type* iterative sequence (*cf.* [29]).

Lemma 5 (see [10]). *Let sequences $\{a_n\}$, $\{b_n\}$ satisfy that*

$$a_{n+1} \leq a_n + b_n, \tag{19}$$

$a_n \geq 0$, for all $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} b_n$ is convergent, and $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ converging to 0. Then, we must have

$$\lim_{n\to\infty} a_n = 0. \tag{20}$$

3. Convergence Theorems

Lemma 6. *Let (X, d) be a $CAT(0)$ space and let C be a nonempty convex subset of X. Let $T : C \to C$ be an uniformly L-Lipschitzian mapping and let $\{\alpha_n\}$, $\{\beta_n\}$ be sequence in $[0, 1]$. Define the iteration scheme $\{x_n\}$ as Algorithm 3. Then*

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, T^n x_n) \\ &\quad + L\left(1 + 2L + L^2\right) d\left(x_{n-1}, T^{n-1} x_{n-1}\right), \end{aligned} \tag{21}$$

for all $n \geq 1$.

Proof. Let $C_n = d(x_n, T^n x_n)$. We have

$$\begin{aligned} d(x_{n-1}, y_{n-1}) &= d\left(x_{n-1}, (1-\beta_{n-1})\, x_{n-1} \oplus \beta_{n-1} T^{n-1} x_{n-1}\right) \\ &\leq \beta_{n-1} \cdot d\left(x_{n-1}, T^{n-1} x_{n-1}\right) \\ &= \beta_{n-1} C_{n-1}. \end{aligned} \tag{22}$$

From (22), we get

$$\begin{aligned} d\left(x_{n-1}, T^{n-1} y_{n-1}\right) &\leq d\left(x_{n-1}, T^{n-1} x_{n-1}\right) \\ &\quad + d\left(T^{n-1} x_{n-1}, T^{n-1} y_{n-1}\right) \\ &\leq C_{n-1} + L \cdot d(x_{n-1}, y_{n-1}) \\ &\leq C_{n-1} + \beta_{n-1} \cdot L \cdot C_{n-1}. \end{aligned} \tag{23}$$

From (22) and (23), we get

$$\begin{aligned} & d(x_n, Tx_n) \\ &\leq d(x_n, T^n x_n) + d(T^n x_n, Tx_n) \\ &\leq C_n + L \cdot d\left(T^{n-1} x_n, x_n\right) \\ &\leq C_n + L\left\{d\left(T^{n-1} x_n, T^{n-1} x_{n-1}\right) + d\left(T^{n-1} x_{n-1}, x_n\right)\right\} \\ &\leq C_n + L^2 \cdot d(x_n, x_{n-1}) + L \cdot d\left(T^{n-1} x_{n-1}, x_n\right) \\ &\leq C_n + L^2 \cdot d\left((1-\alpha_{n-1})\, x_{n-1} \oplus \alpha_{n-1} T^{n-1} y_{n-1}, x_{n-1}\right) \\ &\quad + L \cdot d\left(T^{n-1} x_{n-1}, (1-\alpha_{n-1})\, x_{n-1} \oplus \alpha_{n-1} T^{n-1} y_{n-1}\right) \\ &\leq C_n + L^2 \cdot \alpha_{n-1} \cdot d\left(T^{n-1} y_{n-1}, x_{n-1}\right) \\ &\quad + L\left\{(1-\alpha_{n-1})\, d\left(T^{n-1} x_{n-1}, x_{n-1}\right)\right. \\ &\qquad \left. + \alpha_{n-1} \cdot d\left(T^{n-1} x_{n-1}, T^{n-1} y_{n-1}\right)\right\} \\ &\leq C_n + L^2 \cdot \alpha_{n-1} \left(C_{n-1} + \beta_{n-1} \cdot L \cdot C_{n-1}\right) \\ &\quad + L(1-\alpha_{n-1})\, C_{n-1} + L^2 \cdot \alpha_{n-1} \cdot \beta_{n-1} \cdot C_{n-1} \\ &\leq C_n + L\left(1 + 2L + L^2\right) C_{n-1}, \quad n \geq 1. \end{aligned} \tag{24}$$

This completes the proof of Lemma 6. □

Theorem 7. *Let (X, d) be a complete $CAT(0)$ space, let C be a nonempty bounded closed convex subset of X, and let $T : C \to C$ be a completely continuous and uniformly L-Lipschitzian and asymptotically demicontractive with sequence $\{a_n\}$, $a_n \in [1, \infty)$, $\sum_{n=1}^{\infty}(a_n^2 - 1) < \infty$, $\varepsilon \leq \alpha_n \leq 1 - k - \varepsilon$, for all $n \in \mathbb{N}$ and some $\varepsilon > 0$. Given $x_0 \in C$, define the iteration scheme $\{x_n\}$ by*

$$x_{n+1} = (1-\alpha_n)\, x_n \oplus \alpha_n T^n x_n, \quad n \geq 1. \tag{25}$$

Then $\{x_n\}$ converges strongly to some fixed point of T.

Proof. Since T is a completely continuous mapping in a bounded closed convex subset C of complete metric space, from Schauder's theorem, $F(T)$ is nonempty. It follows from (CN^*) inequality that

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2((1-\alpha_n)x_n \oplus \alpha_n T^n x_n, p) \\ &\le (1-\alpha_n) d^2(x_n, p) + \alpha_n d^2(T^n x_n, p) \\ &\quad - (1-\alpha_n)\alpha_n d^2(x_n, T^n x_n), \end{aligned} \tag{26}$$

for all $p \in F(T)$. Since T is a asymptotically demicontractive, we get

$$\begin{aligned} d^2(x_{n+1}, p) &\le (1-\alpha_n) d^2(x_n, p) \\ &\quad + \alpha_n \left\{a_n^2 d^2(x_n, p) + k \cdot d^2(x_n, T^n x_n)\right\} \\ &\quad - \alpha_n (1-\alpha_n) d^2(x_n, T^n x_n) \\ &= d^2(x_n, p) + \alpha_n (a_n^2 - 1) d^2(x_n, p) \\ &\quad - \alpha_n (1-\alpha_n - k) d^2(x_n, T^n x_n), \\ &\qquad \forall p \in F(T). \end{aligned} \tag{27}$$

Since $0 < \varepsilon \le \alpha_n \le 1 - k - \varepsilon$, we have $1 - k - \alpha_n \ge \varepsilon$. Thus,

$$\alpha_n (1 - k - \alpha_n) \ge \varepsilon^2. \tag{28}$$

From (27), we have

$$\begin{aligned} d^2(x_{n+1}, p) &\le d^2(x_n, p) + \alpha_n (a_n^2 - 1) d^2(x_n, p) \\ &\quad - \varepsilon^2 \cdot d^2(x_n, T^n x_n), \end{aligned} \tag{29}$$

for all $p \in F(T)$. Since C is bounded and T is self-mapping in C, there exist some $M > 0$ so that $d^2(x_n, p) \le M$, for all $n \in \mathbb{N}$. Since $0 \le \alpha_n \le 1$, it follows from (29) that

$$\begin{aligned} d^2(x_{n+1}, p) &\le d^2(x_n, p) + (a_n^2 - 1) M - \varepsilon^2 \cdot d^2(x_n, T^n x_n), \\ &\qquad \forall p \in F(T). \end{aligned} \tag{30}$$

Therefore,

$$\varepsilon^2 \cdot d^2(x_n, T^n x_n) \le d^2(x_n, p) - d^2(x_{n+1}, p) + (a_n^2 - 1) M. \tag{31}$$

So

$$\begin{aligned} &\sum_{n=1}^{m} \varepsilon^2 \cdot d^2(x_n, T^n x_n) \\ &\quad \le d^2(x_1, p) - d^2(x_{m+1}, p) + \sum_{n=1}^{m} (a_n^2 - 1) M \\ &\quad \le 2M + \sum_{n=1}^{\infty} (a_n^2 - 1) M, \end{aligned} \tag{32}$$

for all $m \in \mathbb{N}$. Since $\sum_{n=1}^{\infty}(a_n^2 - 1) < \infty$, we get

$$\sum_{n=1}^{\infty} \varepsilon^2 \cdot d^2(x_n, T^n x_n) < \infty. \tag{33}$$

Therefore,

$$\lim_{n\to\infty} d^2(x_n, T^n x_n) = 0, \qquad \lim_{n\to\infty} d(x_n, T^n x_n) = 0. \tag{34}$$

Since T is a uniformly L-Lipschitzian, it follows from Lemma 6 that

$$\lim_{n\to\infty} d(x_n, T x_n) = 0. \tag{35}$$

Since $\{x_n\}$ is a bounded sequence and T is completely continuous, there exist a convergent subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$. Therefore, from (35), $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Let $\lim_{k\to\infty} x_{n_k} = q$. It follows from the continuity of T and (35), we have $q = Tq$. Therefore, $\{x_n\}$ has a subsequence which converges to the fixed point q of T. Let $p = q$ in the inequality (30). Since $\sum_{n=1}^{\infty}(a_n^2 - 1) < \infty$ and $\sum_{n=1}^{\infty} \varepsilon^2 \cdot d^2(x_n, T^n x_n) < \infty$, from (30) and Lemma 5, we have

$$\lim_{n\to\infty} d^2(x_n, q) = 0. \tag{36}$$

Therefore,

$$\lim_{n\to\infty} x_n = q. \tag{37}$$

This completes the proof of Theorem 7. □

Corollary 8. *Let (X, d) be a complete $CAT(0)$ space, let C be a nonempty bounded closed convex subset of X, and let $T : C \to C$ be a completely continuous and uniformly L-Lipschitzian and k-strict asymptotically pseudocontractive with sequence $\{a_n\}$, $a_n \in [1, \infty)$, $\sum_{n=1}^{\infty}(a_n^2 - 1) < \infty$, and $\varepsilon \le \alpha_n \le 1 - k - \varepsilon$, for all $n \in \mathbb{N}$ and some $\varepsilon > 0$. Given $x_0 \in C$, define the iteration scheme $\{x_n\}$ by*

$$x_{n+1} = (1-\alpha_n) x_n \oplus \alpha_n T^n x_n, \quad n \ge 1. \tag{38}$$

Then $\{x_n\}$ converges strongly to some fixed point of T.

Proof. By Definition 2, T is k-strict asymptotically pseudocontractive; then T must be asymptotically demicontractive. Therefore, Corollary 8 can be proved by using Theorem 7. □

Lemma 9. *Let (X, d) be a $CAT(0)$ space and let C be a nonempty convex subset of X. Let $T : C \to C$ be an uniformly L-Lipschitzian and asymptotically hemicontractive with sequence $\{a_n\} \subset [1, \infty)$, for all $n \in \mathbb{N}$, and $F(T)$ is nonempty. Define the iteration scheme $\{x_n\}$ as follows:*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n) x_n \oplus \alpha_n T^n y_n, \\ y_n &= (1-\beta_n) x_n \oplus \beta_n T^n x_n, \quad n \ge 1. \end{aligned} \tag{39}$$

Then the following inequality holds:

$$\begin{aligned} d^2(x_{n+1}, p) &\le [1 + \alpha_n (a_n - 1)(a_n \beta_n + 1)] d^2(x_n, p) \\ &\quad - \alpha_n \beta_n (1 - \beta_n - a_n \beta_n - L^2 \beta_n^2) d^2(x_n, T^n x_n) \\ &\quad - \alpha_n (\beta_n - \alpha_n) d^2(x_n, T^n y_n), \end{aligned} \tag{40}$$

for all $p \in F(T)$.

Proof. It follows from (CN^*) inequality that

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2((1-\alpha_n)x_n \oplus \alpha_n T^n y_n, p) \\ &\le (1-\alpha_n) d^2(x_n, p) + \alpha_n d^2(T^n y_n, p) \\ &\quad - (1-\alpha_n)\alpha_n d^2(x_n, T^n y_n), \end{aligned} \tag{41}$$

$$\begin{aligned} d^2(y_n, p) &= d^2((1-\beta_n)x_n \oplus \beta_n T^n x_n, p) \\ &\le (1-\beta_n) d^2(x_n, p) + \beta_n d^2(T^n x_n, p) \\ &\quad - (1-\beta_n)\beta_n d^2(x_n, T^n x_n), \end{aligned} \tag{42}$$

for all $p \in F(T)$. Since T is asymptotically hemicontractive, we get

$$d^2(T^n y_n, p) \le a_n d^2(y_n, p) + d^2(y_n, T^n y_n), \tag{43}$$

$$d^2(T^n x_n, p) \le a_n d^2(x_n, p) + d^2(x_n, T^n x_n). \tag{44}$$

From (42) and (44), we have

$$\begin{aligned} d^2(y_n, p) &\le (1-\beta_n) d^2(x_n, p) \\ &\quad + \beta_n \left[a_n d^2(x_n, p) + d^2(x_n, T^n x_n)\right] \\ &\quad - (1-\beta_n)\beta_n d^2(x_n, T^n x_n) \\ &= [1 + (a_n - 1)\beta_n] d^2(x_n, p) + \beta_n^2 d^2(x_n, T^n x_n). \end{aligned} \tag{45}$$

From (CN^*) inequality, we have

$$\begin{aligned} d^2(y_n, T^n y_n) &= d^2((1-\beta_n)x_n \oplus \beta_n T^n x_n, T^n y_n) \\ &\le (1-\beta_n) d^2(x_n, T^n y_n) + \beta_n d^2(T^n x_n, T^n y_n) \\ &\quad - (1-\beta_n)\beta_n d^2(x_n, T^n x_n). \end{aligned} \tag{46}$$

Substituting (45) and (46) into (43), we get

$$\begin{aligned} &d^2(T^n y_n, p) \\ &\quad \le a_n [1 + (a_n - 1)\beta_n] d^2(x_n, p) + a_n \beta_n^2 d^2(x_n, T^n x_n) \\ &\qquad + (1-\beta_n) d^2(x_n, T^n y_n) + \beta_n d^2(T^n x_n, T^n y_n) \\ &\qquad - (1-\beta_n)\beta_n d^2(x_n, T^n x_n). \end{aligned} \tag{47}$$

From (41) and (47), we obtain

$$\begin{aligned} &d^2(x_{n+1}, p) \\ &\quad \le (1-\alpha_n) d^2(x_n, p) + \alpha_n a_n (1 + (a_n - 1)\beta_n) d^2(x_n, p) \\ &\qquad + \alpha_n a_n \beta_n^2 d^2(x_n, T^n x_n) + \alpha_n (1-\beta_n) d^2(x_n, T^n y_n) \\ &\qquad + \alpha_n \beta_n d^2(T^n x_n, T^n y_n) - \alpha_n (1-\beta_n)\beta_n d^2(x_n, T^n x_n) \\ &\qquad - (1-\alpha_n)\alpha_n d^2(x_n, T^n y_n) \\ &\quad = [1 + \alpha_n \{(a_n - 1) + a_n (a_n - 1)\beta_n\}] d^2(x_n, p) \\ &\qquad - \alpha_n \beta_n (1 - \beta_n - a_n \beta_n) d^2(x_n, T^n x_n) \\ &\qquad - \alpha_n (\beta_n - \alpha_n) d^2(x_n, T^n y_n) + \alpha_n \beta_n d^2(T^n x_n, T^n y_n). \end{aligned} \tag{48}$$

Since T is uniformly L-Lipschitzian, we have

$$\begin{aligned} d(T^n x_n, T^n y_n) &\le L \cdot d(x_n, y_n) \\ &= L \cdot d((1-\beta_n)x_n \oplus \beta_n T^n x_n) \\ &\le L\beta_n \cdot d(x_n, T^n x_n). \end{aligned} \tag{49}$$

Substituting (49) into (48), we obtain

$$\begin{aligned} d^2(x_{n+1}, p) &\le [1 + \alpha_n (a_n - 1)(1 + a_n \beta_n)] d^2(x_n, p) \\ &\quad - \alpha_n \beta_n \left(1 - \beta_n - a_n \beta_n - L^2 \beta_n^2\right) d^2(x_n, T^n x_n) \\ &\quad - \alpha_n (\beta_n - \alpha_n) d^2(x_n, T^n y_n), \quad \forall p \in F(T). \end{aligned} \tag{50}$$

This completes the proof of Lemma 9. □

Lemma 10. *Let (X, d) be a $CAT(0)$ space and let C be a nonempty bounded convex subset of X. Let $T : C \to C$ be a uniformly L-Lipschitzian and asymptotically hemicontractive with sequence $\{a_n\} \subset [1, \infty)$, for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty}(a_n - 1) < \infty$. Let $F(T)$ be nonempty. Given $x_1 \in C$, define the iteration scheme $\{x_n\}$ by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n T^n y_n, \\ y_n &= (1-\beta_n)x_n \oplus \beta_n T^n x_n, \quad n \ge 1. \end{aligned} \tag{51}$$

If $\varepsilon \le \alpha_n \le \beta_n \le b$ for some $\varepsilon > 0$ and $b \in (0, (\sqrt{1+L^2} - 1)/L^2)$, then

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{52}$$

Proof. First, we will prove $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$. From Lemma 9 and $0 \le \alpha_n \le \beta_n$, we have

$$\begin{aligned} d^2(x_{n+1}, p) &\le [1 + \alpha_n (a_n - 1)(1 + a_n \beta_n)] d^2(x_n, p) \\ &\quad - \alpha_n \beta_n \left(1 - \beta_n - a_n \beta_n - L^2 \beta_n^2\right) d^2(x_n, T^n x_n). \end{aligned} \tag{53}$$

Thus

$$\begin{aligned} & d^2(x_{n+1}, p) - d^2(x_n, p) \\ & \quad \le \alpha_n (a_n - 1)(1 + a_n\beta_n) d^2(x_n, p) \\ & \quad - \alpha_n\beta_n \left(1 - \beta_n - a_n\beta_n - L^2\beta_n^2\right) d^2(x_n, T^n x_n). \end{aligned} \tag{54}$$

Since $\sum_{n=1}^{\infty}(a_n - 1) < \infty$, we have $\lim_{n\to\infty}(a_n - 1) = 0$. Hence, $\{a_n\}$ is bounded. By boundedness of C and $0 \le \alpha_n \le \beta_n \le 1$, we obtain that $\{\alpha_n(1 + a_n\beta_n)d^2(x_n, p)\}$ is bounded. Therefore, there exists a constant $M > 0$ such that

$$0 \le \alpha_n (1 + a_n\beta_n) d^2(x_n, p) \le M. \tag{55}$$

From (54) and (55), we get

$$\begin{aligned} & d^2(x_{n+1}, p) - d^2(x_n, p) \\ & \quad \le (a_n - 1) M - \alpha_n\beta_n \left(1 - \beta_n - a_n\beta_n \right. \\ & \qquad \left. - L^2\beta_n^2\right) d^2(x_n, T^n x_n). \end{aligned} \tag{56}$$

Let $D = 1 - 2b - L^2b^2 > 0$. Since $\lim_{n\to\infty} a_n = 1$, there exists $N \in \mathbb{N}$ such that

$$1 - \beta_n - a_n\beta_n - L^2\beta_n^2 \ge 1 - b - a_n b - L^2b^2 \ge \frac{D}{2} > 0, \tag{57}$$

for all $n \ge N$. Suppose that $\lim_{n\to\infty} d(x_n, T^n x_n) \ne 0$, then there exist a $\varepsilon_0 > 0$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$d^2\left(x_{n_i}, T^{n_i} x_{n_i}\right) \ge \varepsilon_0. \tag{58}$$

Without loss of generality, we let $n_1 \ge N$. From (56), we have

$$\begin{aligned} & \alpha_n\beta_n \left(1 - \beta_n - a_n\beta_n - L^2\beta_n^2\right) d^2(x_n, T^n x_n) \\ & \quad \le (a_n - 1) M + d^2(x_n, p) - d^2(x_{n+1}, p), \end{aligned} \tag{59}$$

so

$$\begin{aligned} & \sum_{l=1}^{i} \alpha_{n_l}\beta_{n_l} \left(1 - \beta_{n_l} - a_{n_l}\beta_{n_l} - L^2\beta_{n_l}^2\right) d^2\left(x_{n_l}, T^{n_l} x_{n_l}\right) \\ & \quad = \sum_{m=n_1}^{n_i} \alpha_m\beta_m \left(1 - \beta_m - a_m\beta_m - L^2\beta_m^2\right) d^2(x_m, T^m x_m) \\ & \quad \le \sum_{m=n_1}^{n_i} (a_m - 1) M + d^2\left(x_{n_1}, p\right) - d^2\left(x_{n_i+1}, p\right). \end{aligned} \tag{60}$$

From (57)–(60) and $\varepsilon \le \alpha_n \le \beta_n$, we obtain

$$i \cdot \varepsilon^2 \cdot \frac{D}{2} \cdot \varepsilon_0 \le \sum_{m=n_1}^{n_i} (a_m - 1) M + d^2\left(x_{n_1}, p\right) - d^2\left(x_{n_i+1}, p\right). \tag{61}$$

Since $\sum_{n=1}^{\infty}(a_n - 1) < \infty$ and the boundedness of C, the right side of (61) is bounded. However, if we have $i \to \infty$, then the left side of (61) is unbounded. This is a contradiction. Therefore,

$$\lim_{n\to\infty} d\left(x_n, T^n x_n\right) = 0. \tag{62}$$

Since T is a uniformly L-Lipschitzian, from Lemma 6, we get

$$\lim_{n\to\infty} d\left(x_n, T x_n\right) = 0. \tag{63}$$

This completes the proof of Lemma 10. □

Theorem 11. *Let (X, d) be a complete $CAT(0)$ space, let C be a nonempty bounded closed convex subset of X, and let $T : C \to C$ be a completely continuous and uniformly L-Lipschitzian and asymptotically hemicontractive with sequence $\{a_n\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(a_n - 1) < \infty$, for all $n \in \mathbb{N}$. Given $x_1 \in C$, define the iterative scheme $\{x_n\}$ by*

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n) x_n \oplus \alpha_n T^n y_n, \\ y_n &= (1 - \beta_n) x_n \oplus \beta_n T^n x_n, \quad n \ge 1. \end{aligned} \tag{64}$$

If $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ with $\varepsilon \le \alpha_n \le \beta_n \le b$ for some $\varepsilon > 0$ and $b \in (0, (\sqrt{1 + L^2} - 1)/L^2)$, then $\{x_n\}$ converges strongly to some fixed point of T.

Proof. Since T is a completely continuous mapping in a bounded closed convex subset C of complete metric space, from Schauder's theorem, $F(T)$ is nonempty. Since T is completely continuous, there exist a convergent subset $\{Tx_{n_i}\}$ of $\{Tx_n\}$. Let

$$\lim_{i\to\infty} Tx_{n_i} = p. \tag{65}$$

Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, from Lemma 10, we have

$$\lim_{i\to\infty} x_{n_i} = p. \tag{66}$$

On the other hand, from the continuity of T, (66), and Lemma 10, we have

$$d(p, Tp) = \lim_{i\to\infty} d\left(x_{n_i}, Tx_{n_i}\right) = 0. \tag{67}$$

This means that p is a fixed point of T. From (55), (57), and $\alpha_n \le \beta_n$, we obtain Lemma 9 that

$$d^2(x_{n+1}, p) \le d^2(x_n, p) + (a_n - 1) M. \tag{68}$$

From (66), there exists a subsequence $\{d^2(x_{n_i}, p)\}$ of $\{d^2(x_n, p)\}$ which converges to 0. Therefore, from Lemma 5 and (68),

$$\lim_{n\to\infty} d^2(x_n, p) = 0. \tag{69}$$

Hence,

$$\lim_{n\to\infty} x_n = p. \tag{70}$$

This completes the proof of Theorem 11. □

Corollary 12. *Let (X,d) be a complete $CAT(0)$ space, let C be a nonempty bounded closed convex subset of X, and let $T : C \to C$ be a completely continuous and uniformly L-Lipschitzian and asymptotically pseudocontractive with sequence $\{a_n\} \subset [1,\infty)$ satisfying $\sum_{n=1}^{\infty}(a_n^2 - 1) < \infty$, for all $n \in \mathbb{N}$. Given $x_1 \in C$, define the iterative scheme $\{x_n\}$ by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n T^n y_n, \\ y_n &= (1-\beta_n)x_n \oplus \beta_n T^n x_n, \quad n \geq 1. \end{aligned} \tag{71}$$

If $\{\alpha_n\},\{\beta_n\} \subset [0,1]$ with $\varepsilon \leq \alpha_n \leq \beta_n \leq b$ for some $\varepsilon > 0$ and $b \in (0,(\sqrt{1+L^2}-1)/L^2)$, then $\{x_n\}$ converges strongly to some fixed point of T.

Proof. By Definition 2, T is an asymptotically pseudocontractive mapping, then T is an asymptotically hemicontractive mapping. Since $a_n \in [1,\infty)$, we have $a_n^2 - 1 \geq a_n - 1 \geq 0$. Obviously, $\sum_{n=1}^{\infty}(a_n - 1) \leq \sum_{n=1}^{\infty}(a_n^2 - 1) < \infty$. Therefore, Corollary 12 can be proved by using Theorem 11. □

4. Some Remarks and Open Problems

Let S be a semigroup. We denote by $B(S)$ the space of all bounded real-valued functions defined on S with supremum norm. For each $s \in S$, we define the left and right translation operators l_s and r_s on $B(S)$ by

$$(l_s f)(t) = f(st), \qquad (r_s f)(t) = f(ts), \tag{72}$$

for each $t \in S$ and $f \in B(S)$, respectively. Let X be a subspace of $B(S)$ containing 1. An element μ in the dual space X^* of X is said to be a *mean* on X if $\|\mu\| = \mu(1) = 1$. For $s \in S$, we can define a point evaluation δ_s by $\delta_s(f) = f(s)$ for each $f \in X$. It is well known that μ is mean on X if and only if

$$\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s) \tag{73}$$

for each $f \in X$. Each mean on X is the weak*-limit of convex combination of point evaluations.

Let X be a translation invariant subspace of $B(S)$ (i.e., $l_s X \subset X$ and $r_s X \subset X$ for each $s \in S$) containing 1. Then a mean μ on X is said to be *left invariant* (resp., *right invariant*) if

$$\mu(l_s f) = \mu(f) \quad (\text{resp., } \mu(r_s f) = \mu(f)), \tag{74}$$

for each $s \in S$ and $f \in X$. A mean μ on X is said to be *invariant* if μ is both left and right invariant ([30–34]). X is said to be *left* (resp., *right*) *amenable* if X has a left (resp., right) invariant mean. X is amenable if X is left and right amenable. In this case, we say that the semigroup S is an amenable semigroup (see [35, 36]). Moreover, $B(S)$ is amenable when S is a commutative semigroup or a solvable group. However, the free group or semigroup of two generators is not left or right amenable.

A net $\{\mu_\alpha\}$ of means on X is said to be *asymptotically left* (resp., *right*) *invariant* if

$$\begin{aligned} &\lim_\alpha (\mu_\alpha(l_s f) - \mu_\alpha(f)) = 0 \\ &\left(\text{resp., } \lim_\alpha (\mu_\alpha(r_s f) - \mu_\alpha(f)) = 0\right), \end{aligned} \tag{75}$$

for each $f \in X$ and $s \in S$, and it is said to be *left* (resp., *right*) *strongly asymptotically invariant* (or *strong regular*) if

$$\lim_\alpha \|l_s^* \mu_\alpha - \mu_\alpha\| = 0 \quad \left(\text{resp., } \lim_\alpha \|r_s^* \mu_\alpha - \mu_\alpha\| = 0\right), \tag{76}$$

for each $s \in S$, where l_s^* and r_s^* are the adjoint operators of l_s and r_s, respectively. Such nets were first studied by Day in [35] where they were called *weak* invariant* and *norm invariant*, respectively.

It is easy to see that if a semigroup S is left (resp., right) amenable, then the semigroup $S' = S \cup \{e\}$, where $es' = s'e = s'$ for all $s' \in S$ is also left (resp., right) amenable and conversely.

A semigroup S is called *left reversible* if any two right ideals of S have nonvoid intersection, that is, $aS \cap bS \neq \emptyset$ for $a,b \in S$. In this case, $(S,\preceq)$ is a directed system when the binary relation "$\preceq$" on S is defined by $a \preceq b$ if and only if $\{a\} \cup aS \supseteq \{b\} \cup bS$ for $a,b \in S$. It is easy to see that $t \preceq ts$ for all $t,s \in S$. Further, if $t \preceq s$, then $pt \preceq ps$ for all $p \in S$. The class of left reversible semigroup includes all groups and commutative semigroups. If a semigroup S is left amenable, then S is left reversible. But the converse is not true ([31, 37–41]).

Let S be a semigroup and $F(T)$ denote the fixed point set of T. Then $\mathfrak{T} = \{T_s : s \in S\}$ is called a *representation of* S if $T_e = I$ and $T_{st} = T_s T_t$ for each $s,t \in S$. We denote by $F(\mathfrak{T})$ the set of common fixed points of $\{T_s : s \in S\}$, that is,

$$F(\mathfrak{T}) = \bigcap_{s\in S} F(T_s) = \bigcap_{s\in S} \{x \in C : T_s x = x\}. \tag{77}$$

Open Problem 1. It will be interesting to obtain a generalization of both Theorems 7 and 11 to commutative, amenable, and reversible semigroups as in the case of Hilbert spaces or some Banach spaces (*cf.* [8, 30, 32, 42–45]).

For a real number κ, a $CAT(\kappa)$ space is defined by a geodesic metric space whose geodesic triangle is sufficiently thinner than the corresponding triangle in a model space with curvature κ.

For $\kappa = 0$, the 2-dimensional model space $M_\kappa^2 = M_0^2$ is the Euclidean space $\mathbb{R}^2$ with the metric induced from the Euclidean norm. For $\kappa > 0$, M_κ^2 is the 2-dimensional sphere $(1/\sqrt{\kappa})\mathbb{S}^2$ whose metric is length of a minimal great arc joining each two points. For $\kappa < 0$, M_κ^2 is the 2-dimensional hyperbolic space $(1/\sqrt{-\kappa})\mathbb{H}^2$ with the metric defined by a usual hyperbolic distance. For more details about the properties of $CAT(\kappa)$ spaces, see [4, 46–48].

Open Problem 2. It will be interesting to obtain a generalization of both Theorems 7 and 11 to $CAT(\kappa)$ space.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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