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Research Article

Mizoguchi-Takahashi's Fixed Point Theorem with α, η Functions

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We introduce the notion of generalized α_*-admissible mappings. By using this notion, we prove a fixed point theorem. Our result generalizes Mizoguchi-Takahashi's fixed point theorem. We also provide some examples to show the generality of our work.

1. Introduction and Preliminaries

Let (X, d) be a metric space. For each $x \in X$ and $A \subseteq X$, $d(x, A) := \{d(x, y) : y \in A\}$. We denote by $K(X)$ the class of all nonempty compact subset of X, by $CB(X)$ the class of all nonempty closed and bounded subsets of X, and by $CL(X)$ the class of all nonempty closed subsets of X. For every $A, B \in CL(X)$, let

$$H(A, B) = \begin{cases} \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}, & \text{if the maximum exists;} \\ \infty, & \text{otherwise.} \end{cases} \tag{1}$$

Such a map H is called generalized Hausdorff metric induced by d.

Nadler [1] extended the Banach contraction principle to multivalued mappings as follows.

Theorem 1 (see [1]). *Let (X, d) be a complete metric space and T is a mapping from X into $CB(X)$ such that*

$$H(Tx, Ty) \le rd(x, y), \quad \forall x, y \in X, \tag{2}$$

where $r \in [0, 1)$. Then T has a fixed point.

Reich [2] extended the above result in the following way.

Theorem 2 (see [2]). *Let (X, d) be a complete metric space and $T : X \to K(X)$ is a mapping satisfying*

$$H(Tx, Ty) \le \beta(d(x, y))\, d(x, y), \tag{3}$$

where $x, y \in X$ and β is a function from $(0, \infty)$ into $[0, 1)$ such that

$$\limsup_{r\to t^+} \beta(r) < 1, \quad \text{for each } t \in (0, \infty). \tag{4}$$

Then T has a fixed point.

Reich [2] raised the question: whether the range of T, $K(X)$ can be replaced by $CB(X)$ or $CL(X)$. Mizoguchi and Takahashi [3] gave a positive answer to the conjecture of Reich [2], when the inequality holds also for $t = 0$; in particular they proved the following.

Theorem 3 (see [3]). *Let (X, d) be a complete metric space and $T : X \to CB(X)$ is a mapping satisfying*

$$H(Tx, Ty) \le \beta(d(x, y))\, d(x, y), \tag{5}$$

where $x, y \in X$ and β is a function from $[0,\infty)$ into $[0,1)$ such that

$$\limsup_{r \to t^+} \beta(r) < 1, \quad \text{for each } t \in [0,\infty). \tag{6}$$

Then T has a fixed point.

The other proofs of Theorem 3 have been given by Daffer and Kaneko [4] and Chang [5]. Eldred et al. [6] claimed that Theorem 3 is equivalent to Theorem 1. Suzuki produced an example [7, page 753] to disprove their claim and showed that Mizoguchi-Takahashi's fixed point theorem is a real generalization of Nadler's fixed point theorem. Reader can find some more results related to Mizoguchi-Takahashi's fixed point theorem in [8–14].

Samet et al. [15] introduced the notion of α-ψ-contractive and α-admissible self-mappings and proved some fixed point results for such mappings in complete metric spaces. Karapinar and Samet [16] generalized these notions and obtained some fixed point results. Asl et al. [17] extended these notions to multifunctions by introducing the notions of α^*-ψ-contractive and α^*-admissible mappings and proved some fixed point results. Some results in this direction are also given by the authors [18, 19]. Ali and Kamran [20] further generalized the notion of α^*-ψ-contractive mappings and obtained some fixed point theorems for multivalued mappings.

Recently, Salimi et al. [21] modified the notions of α-ψ-contractive and α-admissible self-mappings by introducing another function η and established some fixed point theorems for such mappings in complete metric spaces. Hussain et al. [22] extended the result of Asl et al. and introduced the following definition.

Definition 4 (see [22]). Let $G : X \to CL(X)$ be a mapping on a metric space (X,d). Let $\alpha, \eta : X \times X \to [0,\infty)$ be two functions, where η is bounded. We say that G is an α_*-admissible mapping with respect to η if we have

$$x, y \in X, \quad \alpha(x,y) \geq \eta(x,y) \Longrightarrow \alpha_*(Gx, Gy) \geq \eta_*(Gx, Gy), \tag{7}$$

where $\alpha_*(Gx,Gy) = \inf\{\alpha(a,b) : a \in Gx, b \in Gy\}$ and $\eta_*(Gx,Gy) = \sup\{\eta(a,b) : a \in Gx, b \in Gy\}$. In case when $\alpha(x,y) = 1$ for all $x, y \in X$, then G is η_*-subadmissible mapping. In case when $\eta(x,y) = 1$ for all $x, y \in X$, then G is α_*-admissible.

Definition 5 (see [17]). Let (X,d) be a metric space and let $\alpha : X \times X \to [0,\infty)$ be a mapping. A mapping $G : X \to CL(X)$ is α_*-admissible if $\alpha(x,y) \geq 1 \Rightarrow \alpha^*(Gx,Gy) \geq 1$, where $\alpha^*(Gx,Gy) = \inf\{\alpha(a,b) : a \in Gx, b \in Gy\}$.

In this paper, we generalize Definition 4 and provide some examples to show generality of such concept. We also establish a fixed point theorem which generalizes Mizoguchi-Takahashi's fixed point theorem. Some illustrative examples to claim that our results properly generalize some results in the literature are presented. Furthermore, at the end of this paper, we give an open problem for further investigation.

2. Main Results

We begin this section by generalizing Definition 4.

Definition 6. Let $G : X \to CL(X)$ be a mapping on a metric space (X,d). Let $\alpha, \eta : X \times X \to [0,\infty)$ be two functions. We say that G is generalized α_*-admissible mapping with respect to η if we have

$$x, y \in X, \quad \alpha(x,y) \geq \eta(x,y) \Longrightarrow \alpha(u,v) \geq \eta(u,v), \quad \forall u \in Gx,\ v \in Gy. \tag{8}$$

When $\alpha(x,y) = 1$ for all $x, y \in X$, then G is a generalized η_*-subadmissible mapping. When $\eta(x,y) = 1$ for all $x, y \in X$, then G is α_*-admissible.

Remark 7. Note that inequality (8) is weaker than (7). Moreover, η involved in inequality (8) is not necessarily bounded.

Example 8. Let $X = \{1/n : n \in \mathbb{N}\} \cup \{0\} \cup \{n+1 : n \in \mathbb{N}\}$ be endowed with the usual metric d. Define $G : X \to CL(X)$ by $Gx = \{0, x^2\}$ for all $x, y \in X$, $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x, y \in \left\{\dfrac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\} \\ 0, & \text{otherwise,} \end{cases} \tag{9}$$

and $\eta : X \times X \to [0,\infty)$ by $\eta(x,y) = x + y$ for each $x, y \in X$. Then for $x, y \in X$ with $\alpha(x,y) \geq \eta(x,y)$, we have $\alpha(u,v) \geq \eta(u,v)$ for all $u \in Gx$ and $v \in Gy$. Therefore G is generalized α_*-admissible mapping with respect to η but it is not α_*-admissible mapping with respect to η.

Theorem 9. *Let (X,d) be a complete metric space and let $G : X \to CL(X)$ be a generalized α_*-admissible mapping with respect to η such that*

$$x, y \in X, \quad \alpha(x,y) \geq \eta(x,y) \Longrightarrow H(Gx,Gy) \leq \beta(d(x,y))\, d(x,y), \tag{10}$$

where $\beta : [0,\infty) \to [0,1)$ satisfying $\limsup_{r \to t^+} \beta(r) < 1$ for all $t \in [0,\infty)$. Assume that the following conditions hold:

(i) *there exist $x_0 \in X$ and $x_1 \in Gx_0$ such that $\alpha(x_0,x_1) \geq \eta(x_0,x_1)$;*

(ii) *either*

(1) *G is continuous*

or

(2) *if $\{x_n\}$ is a sequence in X with $x_n \to x$ as $n \to \infty$ and $\alpha(x_{n-1},x_n) \geq \eta(x_{n-1},x_n)$ for each $n \in \mathbb{N}$, then one has $\alpha(x_{n-1},x) \geq \eta(x_{n-1},x)$ for each $n \in \mathbb{N}$.*

Then G has a fixed point.

Proof. By hypothesis, there exist $x_0 \in X$ and $x_1 \in Gx_0$ such that $\alpha(x_0,x_1) \geq \eta(x_0,x_1)$. If $x_0 = x_1$, then we have nothing to prove. Let $x_0 \neq x_1$. Then from (10), we have

$$d(x_1, Gx_1) \leq H(Gx_0, Gx_1) \leq \beta(d(x_0,x_1))\, d(x_0,x_1). \tag{11}$$

There exists $x_2 \in Gx_1$ such that

$$\begin{aligned} d(x_1, x_2) &\le H(Gx_0, Gx_1) + \frac{1-\beta(d(x_0,x_1))}{2} d(x_0, x_1) \\ &\le \beta(d(x_0,x_1)) d(x_0,x_1) \\ &\quad + \frac{1-\beta(d(x_0,x_1))}{2} d(x_0,x_1). \end{aligned} \tag{12}$$

Consider $\gamma(t) = (\beta(t)+1)/2$ for all $t \in [0,\infty)$. Then $\limsup_{r\to t^+}\gamma(r) < 1$ for all $t \in [0,\infty)$. From (12), we have

$$d(x_1,x_2) \le \gamma(d(x_0,x_1)) d(x_0,x_1). \tag{13}$$

Since G is an α_*-admissible mapping with respect to η, then $\alpha(x_1,x_2) \ge \eta(x_1,x_2)$. If $x_1 = x_2$, then we have nothing to prove. Let $x_1 \ne x_2$. Then from (10), we have

$$d(x_2, Gx_2) \le H(Gx_1, Gx_2) \le \beta(d(x_1,x_2)) d(x_1,x_2). \tag{14}$$

There exists $x_3 \in Gx_2$ such that

$$\begin{aligned} d(x_2,x_3) &\le H(Gx_1,Gx_2) \\ &\quad + \frac{1-\beta(d(x_1,x_2))}{2} d(x_1,x_2) \\ &\le \beta(d(x_1,x_2)) d(x_1,x_2) \\ &\quad + \frac{1-\beta(d(x_1,x_2))}{2} d(x_1,x_2) \\ &= \gamma(d(x_1,x_2)) d(x_1,x_2). \end{aligned} \tag{15}$$

Continuing the same process, we get a sequence $\{x_n\}$ in X such that $x_n \in Gx_{n-1}$, $x_n \ne x_{n-1}$, $\alpha(x_n,x_{n-1}) \ge \eta(x_n,x_{n-1})$, and

$$d(x_n,x_{n+1}) \le \gamma(d(x_{n-1},x_n)) d(x_{n-1},x_n) \quad \text{for each } n \in \mathbb{N}. \tag{16}$$

It follows from $\gamma(t) < 1$ for all $t \in [0,\infty)$ that $\{d(x_n,x_{n+1})\}$ is a strictly decreasing sequence in $\mathbb{R}$. Hence it converges to some nonnegative real number v. Since $\limsup_{r\to v^+}\gamma(r) < 1$ and $\gamma(v) < 1$, there exists $s \in [0,1)$ and $\epsilon > 0$ such that $\gamma(t) \le s$ for all $t \in [v, v+\epsilon]$. We can find $w \in \mathbb{N}$ such that $v \le d(x_n,x_{n+1}) \le v+\epsilon$ for all $n \in \mathbb{N}$ with $n \ge w$. Then

$$d(x_n,x_{n+1}) \le \gamma(d(x_{n-1},x_n)) d(x_{n-1},x_n) \le s d(x_{n-1},x_n), \tag{17}$$

for each $n \ge w$. Also, we have

$$\begin{aligned} \sum_{n=1}^{\infty} d(x_{n-1},x_n) &\le \sum_{n=1}^{w} d(x_{n-1},x_n) \\ &\quad + \sum_{n=1}^{\infty} s^n d(x_{w-1},x_w) < \infty. \end{aligned} \tag{18}$$

Hence $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, then there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. If we suppose that G is continuous, then

$$d(x^*, Gx^*) = \lim_{n\to\infty} d(x_n, Gx^*) \le \lim_{n\to\infty} H(Gx_{n-1}, Gx^*) = 0. \tag{19}$$

On the other hand, since

$$\alpha(x_{n-1},x_n) \ge \eta(x_{n-1},x_n), \tag{20}$$

for each $n \in \mathbb{N}$ and $x_n \to x^*$ as $n \to \infty$, then we have

$$\alpha(x_{n-1},x^*) \ge \eta(x_{n-1},x^*), \tag{21}$$

for each $n \in \mathbb{N}$. Then from (10), we have

$$\begin{aligned} d(x^*, Gx^*) &= \lim_{n\to\infty} d(x_n, Gx^*) \\ &\le \lim_{n\to\infty} H(Gx_{n-1}, Gx^*) \\ &\le \lim_{n\to\infty} \beta(d(x_n,x^*)) d(x_n,x^*) \\ &\le \lim_{n\to\infty} d(x_n,x^*) = 0. \end{aligned} \tag{22}$$

Therefore $x^* \in Gx^*$. This completes the proof. □

The following example shows that Theorem 9 properly generalizes Theorem 3, in Section 1.

Example 10. Let $X = \mathbb{R}$ be endowed with the usual metric d. Define $G : X \to CL(X)$ by

$$Gx = \begin{cases} [x,\infty) & \text{if } x > 1 \\ \left[0, \dfrac{x}{2}\right] & \text{if } 0 \le x \le 1 \\ \left(-\infty, x^2\right] & \text{if } x < 0, \end{cases} \tag{23}$$

$\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in [0,\infty) \\ 0 & \text{otherwise,} \end{cases} \tag{24}$$

and $\eta : X \times X \to [0,\infty)$ by $\eta(x,y) = |x| + |y|$ for each $x, y \in X$. Take $\beta(t) = 1/2$ for all $t \ge 0$. Then for $x, y \in X$ with $\alpha(x,y) \ge \eta(x,y)$, we have

$$H(Gx,Gy) = \frac{|x-y|}{2} = \beta(d(x,y)) d(x,y). \tag{25}$$

Also, G is generalized α_*-admissible mapping with respect to η. For $x_0 = 1$ and $x_1 = 0 \in Gx_0$ we have $\alpha(x_0,x_1) \ge \eta(x_0,x_1)$. Further, for any sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ and $\alpha(x_{n-1},x_n) \ge \eta(x_{n-1},x_n)$ for each $n \in \mathbb{N}$, we have $\alpha(x_{n-1},x) \ge \eta(x_{n-1},x)$ for each $n \in \mathbb{N}$. Therefore, all conditions of Theorem 9 are satisfied and G has infinitely many fixed points.

Corollary 11. *Let (X, d) be a complete metric space and let $G : X \to CL(X)$ be an α_*-admissible mapping with respect to η such that*

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \Longrightarrow H(Gx, Gy) \leq \beta(d(x, y))\, d(x, y), \tag{26}$$

where $\beta : [0, \infty) \to [0, 1)$ satisfying $\limsup_{r \to t^+} \beta(r) < 1$ for all $t \in [0, \infty)$. Assume that the following conditions hold:

(i) *there exist $x_0 \in X$ and $x_1 \in Gx_0$ such that $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$;*

(ii) *either*

(1) *G is continuous*

or

(2) *if $\{x_n\}$ is a sequence in X with $x_n \to x$ as $n \to \infty$ and $\alpha(x_{n-1}, x_n) \geq \eta(x_{n-1}, x_n)$ for each $n \in \mathbb{N}$, then we have $\alpha(x_{n-1}, x) \geq \eta(x_{n-1}, x)$ for each $n \in \mathbb{N}$.*

Then G has a fixed point.

Proof. We can prove this result by using Theorem 9 and the fact that inequality (8) is weaker than (7). □

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